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Maximal Subgroups of Compact Lie Groups

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Abstract. This report aims at giving a general overview on the classification of the maximal subgroups of compact Lie groups (not necessarily connected). In the first part, it is shown that these fall naturally into three types: (1) those of trivial type, which are simply defined as inverse images of maximal subgroups of the corresponding component group under the canonical projection and whose classification constitutes a problem in finite group theory, (2) those of normal type, whose connected one-component is a normal subgroup, and (3) those of normalizer type, which are the normalizers of their own connected one-component. It is also shown how to reduce the classification of maximal subgroups of the last two types to: (2) the classification of the finite maximal Σ-invariant subgroups of centerfree connected compact simple Lie groups and (3) the classification of the Σ-primitive subalgebras of compact simple Lie algebras, where Σ is a subgroup of the corresponding outer automorphism group. In the second part, we explicitly compute the normalizers of the primitive subalgebras of the compact classical Lie algebras (in the corresponding classical groups), thus arriving at the complete classification of all (non-discrete) maximal subgroups of the compact classical Lie groups.
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1. Introduction

In this paper, we address an important problem from the theory of Lie groups, namely that of classifying their maximal subgroups.[1] This is closely related (though not completely equivalent) to another classical problem from the theory of Lie groups, namely that of classifying their primitive actions, which goes all the way back to Sophus Lie and has over decades received a great deal of attention in the literature: see Dynkin [8, 9, 39], Golubitsky et al. [12, 13], Chekalov [4] and Komrakov [28–30].

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[1] In this paper, the term "maximal ... subgroup" will always mean "maximal ... closed subgroup", where ... stands for a string of other possible adjectives. A more detailed discussion, with precise definitions, can be found in Section 2, for the case where this string is empty, and in Section 3, for the case where this string represents the expression "Γ-invariant" or "Σ-invariant".

Of course, the same two problems and, in particular, the notion of a primitive action can also be formulated in the context of abstract groups. A transitive action of an abstract group G on a set X is said to be *primitive* if the only G-invariant equivalence relations $R \subset X \times X$ are the trivial ones: $R = X \times X$ and $R = \{(x,x) \mid x \in X\}$.[2] This is equivalent to X being a homogeneous space G/H where the stability group H is a maximal subgroup of G, so in this context the two problems are completely equivalent. For Lie groups, the corresponding concept is defined somewhat differently. A transitive action of a Lie group G on a manifold M is said to be *(Lie) primitive* if M admits no G-invariant foliation with leaves of positive dimension smaller than $\dim M$.[2] This is equivalent to M being a homogeneous space G/H where the stability group H satisfies the following weaker maximality condition: for any Lie subgroup $\tilde{H}$ of G such that $H \subset \tilde{H} \subset G$, $\tilde{H}_0 = H_0$ or $\tilde{H}_0 = G_0$, where the index 0 denotes taking the connected one-component. Clearly every action of a Lie group which is primitive as the action of an abstract group is also Lie primitive, but not conversely.

The infinitesimal version of both problems leads to the quest for a classification of the maximal subalgebras of complex semisimple Lie algebras. This was completely solved in the early 1950s by Morozov [33, 34], Karpelevich [25], Borel and de Siebenthal [3] and Dynkin [8, 9]; see also the Bourbaki review by Tits [39]. However, the classification of maximal subalgebras of Lie algebras (or at least of certain classes of Lie algebras such as compact or semisimple or, more generally, reductive Lie algebras) only provides a classification of maximal *connected* subgroups of *connected* Lie groups (within the corresponding class): dealing with the discrete parts is quite another story. In fact, although maximal subalgebras of a Lie algebra do give rise to maximal subgroups of any corresponding connected Lie group (namely, as we shall see, by taking their normalizer under the adjoint representation), it is *not* true that maximal subgroups are necessarily associated with maximal subalgebras: an extreme counterexample is provided by the trivial subalgebra $\{0\}$, corresponding to discrete maximal subgroups. Thus we must face the question as to what is the class of subalgebras corresponding to the maximal subgroups.

The study of the global version including non-connected groups was initiated in the 1970s by Golubitsky [12], who determined the maximal rank primitive subgroups of the classical complex Lie groups. Soon after, his work was extended to the exceptional complex Lie groups by Golubitsky and Rothschild [13]. Finally, this classification was completed by including primitive subgroups of any rank, first for the classical complex Lie groups by Chekalov [4] and later for the exceptional complex Lie groups by Komrakov [30].

It is clear from this brief summary that most of the efforts were centered around the notion of primitive actions, leaving that of maximal subgroups as an auxiliary concept. It is not difficult to see that when the ambient group is a simple Lie group, the non-discrete maximal subgroups can be obtained from the primitive subgroups: the former are precisely the normalizers of the connected one-components of the latter. However, this correspondence is less clear when the ambient group is not simple. In fact, it is shown in [12, p. 179] that if a semisimple

[2] Often it is also assumed that the action is effective, but this is not essential.

Lie group has more than three simple factors, then it does not admit any primitive subgroups, while it is obvious that it does admit non-discrete maximal subgroups. Thus although the results mentioned above constitute important steps towards a full classification of maximal subgroups, some ingredients are still missing.

The main purpose of the present paper is to fill these gaps and to provide a comprehensive treatment of the subject, which does not seem to be available in the literature. In particular, our approach includes the study of maximal subgroups of Lie groups which are not necessarily connected: a problem which apparently has never before been investigated systematically and which cannot be neglected if one wants to arrive at a conceptually consistent picture. Indeed, such a picture can only emerge if one requires the maximal subgroups considered to belong to the same category as their ambient groups: either they should both be assumed to be connected or else they should both be allowed to have a non-trivial component group. But apart from logical coherence and completeness, such an approach is also of practical use; in fact, it is imperative if one wants to iterate the process in order to construct descending chains of subgroups in which each subgroup is maximal in the previous one: this is an important ingredient in studies of symmetry breaking [24]. On the other hand, one cannot of course hope for a completely general solution: some restriction on the type of Lie groups involved will certainly have to be imposed. Not surprisingly, an adequate category for which a complete theory can be developed is that of compact Lie groups, but it can be verified with little effort that practically all results obtained in this context continue to hold within the larger category of reductive Lie groups, which contains that of semisimple Lie groups – compact as well as non-compact.

The paper is divided into two parts: the first (Sections 2-5) is devoted to the general theory whereas the second (Sections 6-8) deals with the classification problem for the classical groups. In Section 2, we begin by specifying the precise mathematical setup and, in particular, by giving an exact definition of the term "maximal subgroup". We also collect some elementary facts about the action of a Lie group on its connected one-component and on its Lie algebra by automorphisms and about normalizers. In Section 3, we analyze the relation between subgroups of a Lie group G and subgroups of its connected one-component G_0 (see Proposition 3.1), which leads us to introduce the important and useful concept of a maximal Γ-invariant subgroup of G_0, where $\Gamma = G/G_0$ is the component group of G. Next, we prove as our first main theorem that maximal subgroups of Lie groups fall into three distinct types which we shall refer to as the trivial type, the normal type and the normalizer type (see Theorem 3.8). Briefly, for any Lie group G with connected one-component G_0 and component group $\Gamma = G/G_0$, a maximal subgroup of G of trivial type[3] is obtained as the union of the connected components of G labelled by the elements of a maximal subgroup of Γ (so this type can only exist if G is not connected), whereas any other maximal subgroup M of G must meet every connected component of G and is classified according to what is the normalizer $N_G(M_0)$ in G of its connected one-component M_0, since maximality of M leaves only two options:

[3]This terminology, borrowed from the theory of finite groups (see Remark 3.10), is perhaps unfortunate, and the authors are open to suggestions for a better one.

- $N_G(M_0) = G$: this is called the normal type since M_0 is a normal subgroup of G,
- $N_G(M_0) = M$: this is called the normalizer type since M itself is the normalizer of its own connected one-component, or equivalently, of its own Lie algebra.

In Section 4, we introduce the concept of a Σ-quasiprimitive subalgebra of a Lie algebra $\mathfrak{g}$, containing as a special case that of a Σ-primitive subalgebra, where Σ is any given subgroup of the outer automorphism group $\mathrm{Out}(\mathfrak{g})$ of $\mathfrak{g}$. In Section 5, we turn to reductive Lie groups and prove first that, at least in this context, the Σ-quasiprimitive subalgebras of $\mathfrak{g}$ correspond precisely to the the maximal subgroups of G of normal type or of normalizer type, where G is any Lie group with Lie algebra $\mathfrak{g}$ whose component group Γ projects to Σ under the natural homomorphism from Γ to $\mathrm{Out}(\mathfrak{g})$ induced by the adjoint representation of G (see Theorem 5.3).[4] Subsequently, we show how the classification of the maximal subgroups of reductive Lie groups can, in a sequence of steps, be reduced to that of the maximal Σ-invariant subgroups of centerfree connected simple Lie groups, where Σ runs through the subgroups of the respective outer automorphism group. Together with the results obtained in the preceding sections, this procedure reduces the classification of the maximal subgroups of a general compact Lie group G with connected one-component G_0, component group $\Gamma = G/G_0$ and Lie algebra $\mathfrak{g}$ to

(1) the classification of the maximal subgroups of Γ, for the trivial type,

(2) the classification of the discrete (hence finite) maximal Σ_s-invariant subgroups of G_s, for the normal type,

(3) the classification of the Σ_s-primitive subalgebras of $\mathfrak{g}_s$, whose normalizers provide the non-discrete maximal Σ_s-invariant subgroups of G_s, for the normalizer type,

where G_s is any one of the simple factors of the quotient of G_0 by its center $Z(G_0)$, $\mathfrak{g}_s$ is its Lie algebra and Σ_s is any subgroup of the outer automorphism group $\mathrm{Out}(\mathfrak{g}_s)$ of $\mathfrak{g}_s$. Note that the first item is part of the problem of classifying the maximal subgroups of finite groups, which constitutes an important issue in the theory of finite groups that has been vigorously investigated in the last decade; see, for example, [2, 26, 31, 32, 35, 36]. On the other hand, the second item (even with the simplifying assumption that G is connected and hence $\Gamma = \{1\}$, or more generally, that G acts on G_0 by inner automorphisms and hence $\Sigma = \{1\}$) leads directly to the problem of classifying the finite maximal subgroups of compact connected simple Lie groups. For the classical Lie groups, it is known that this may be reduced to a problem in the representation theory of finite groups [18], whose solution in the simplest case A_1 is stated in Example 2.4 below, whereas for the exceptional Lie groups, it constitutes a highly non-trivial problem which has been the subject of recent (and still ongoing) research by experts in the theory of finite groups [17–19].

[4] In this situation, we shall use the terms "Σ-(quasi)primitive" and "Γ-(quasi)primitive" interchangeably. Of course, the prefix will be omitted as soon as Σ is trivial.

In view of this situation, we shall in the second part of this paper deal with the third of the above items: the classification of the maximal subgroups of compact simple Lie groups of normalizer type, or equivalently, of the Σ-primitive subalgebras of compact simple Lie algebras $\mathfrak{g}$, where Σ is a subgroup of the (finite) outer automorphism group $\mathrm{Out}(\mathfrak{g})$ of $\mathfrak{g}$. Note that with only one exception, $\mathrm{Out}(\mathfrak{g})$ (and hence Σ) is either trivial or equal to $\mathbb{Z}_2$; in fact,

$$\mathrm{Out}(\mathfrak{g}) \;=\; \left\{ \begin{array}{lll} \{1\} & \text{for} & A_1,\, B_n\ (n \geqslant 2),\, C_n\ (n \geqslant 3),\, E_7,\, E_8,\, F_4,\, G_2 \\ \mathbb{Z}_2 & \text{for} & A_n\ (n \geqslant 2),\, D_n\ (n \geqslant 5),\, E_6 \\ S_3 & \text{for} & D_4 \end{array} \right\}\,.$$

Therefore, our strategy will be to first deal with the situation where Σ is trivial and leave the determination of Σ-primitive subalgebras for non-trivial Σ to a second step; this will be greatly facilitated by the fact that any primitive subalgebra which is also Σ-invariant is automatically Σ-primitive. In Section 6, we discuss how the classification problem for primitive subalgebras (and analogously, for maximal subalgebras) of compact simple Lie algebras can be translated to the complex setting, where it becomes a classification problem for primitive reductive subalgebras (and analogously, for maximal reductive subalgebras) of complex simple Lie algebras. In Section 7, we use this method to translate the existing classification of primitive reductive subalgebras of the complex classical Lie algebras [4, 8, 9, 12] to the compact setting and, in a second step, extend it to a classification of Σ-primitive subalgebras of all compact classical Lie algebras except $\mathfrak{so}(8)$. Finally, in Section 8, we address the task of computing the corresponding normalizers in the respective classical groups – a task that, strangely enough, is largely neglected in the existing literature. We begin by assembling a few general tools that are needed in these computations and then proceed to a case by case treatment of each of the classical groups $SU(n)$, $SO(n)$ and $Sp(n)$. The results are collected in Tables 5–8. It should be pointed out that we restrict ourselves to listing maximal subgroups whose connected one-component is not simple since the others are explicitly given in terms of irreducible representations (see Theorem 7.1), with a "list of exceptions" for which we refer the reader to the original paper by Dynkin [9]. This result will finally fill in a gap left by Dynkin [9, p. 247], promising that the computation of the normalizers of the maximal connected subgroups would appear in a separate article, which has apparently never been published.

2. General setup and basic definitions

In order to specify the precise mathematical setup for the problem to be treated in this paper, we must first of all give an exact definition of what we mean by a "maximal subgroup" of a Lie group and, analogously, a "maximal subalgebra" of a Lie algebra. For Lie algebras, this definition is standard.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. A *maximal subalgebra* of $\mathfrak{g}$ is a proper subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that if $\tilde{\mathfrak{m}}$ is any subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \subset \tilde{\mathfrak{m}} \subset \mathfrak{g}$, then $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. The same terminology applies when the expression "subalgebra" is everywhere replaced by the expression "ideal".

In the case of Lie groups, however, the corresponding definition is ambiguous to the extent that it depends on the specific lattice $\mathcal{L}$ of subgroups in which one chooses to search for maximal elements. The general convention on terminology here is that a maximal ... subgroup of a Lie group G is a maximal element in the lattice of ... subgroups of G, where ... is a string of adjectives that defines $\mathcal{L}$ – adjectives such as "closed" or "Lie" or "connected" or "discrete" or "finite", etc.[5] Explicitly, we have, for example:

Definition 2.2. Let G be a Lie group. A *maximal closed subgroup* of G is a proper closed subgroup M of G such that if $\tilde{M}$ is any closed subgroup of G with $M \subset \tilde{M} \subset G$, then $\tilde{M} = M$ or $\tilde{M} = G$. The same terminology applies when the expression "closed subgroup" is everywhere replaced by the expression "closed normal subgroup".

Similarly, we may define the concept of a *maximal Lie subgroup*, as well as those of a *maximal connected closed/Lie subgroup* and of a *maximal discrete closed/Lie subgroup*, of a Lie group G, repeating Definition 2.2 with the word "closed" everywhere replaced by the word "Lie" or by the expressions "connected closed/Lie" and "discrete closed/Lie", respectively.[6]

For connected Lie groups, it is natural to consider only connected (but not necessarily closed) Lie subgroups, since the Lie correspondence [27, p. 47] establishes a bijection between the lattice of connected Lie subgroups of a connected Lie group G_0 and the lattice of subalgebras of its Lie algebra $\mathfrak{g}$: hence it also establishes a bijection between maximal connected Lie subgroups of G_0 and maximal subalgebras of $\mathfrak{g}$. This implies that maximal connected Lie subgroups always exist and that any connected Lie subgroup is contained in a maximal one; in fact, it can always be realized as the last member in a finite descending chain of connected Lie subgroups where each is maximal in the preceding one.[7]

On the other hand, for compact Lie groups, it is natural to consider only closed (but not necessarily connected) subgroups, since these are again compact Lie groups. Unfortunately, the Lie correspondence is almost entirely lost when we exclude Lie subgroups which are not closed but allow for Lie subgroups which are not connected. As a result, we cannot even guarantee that maximal closed subgroups exist at all, nor that a given closed subgroup is contained in a maximal one. In fact, there is an elementary general obstruction:

Proposition 2.3. *A connected abelian Lie group contains no maximal closed subgroups.*

[5] Abstract subgroups of Lie groups which are not Lie subgroups will be disregarded.

[6] Connected Lie subgroups are also called analytic subgroups. Discrete closed/Lie subgroups are simply closed/Lie subgroups with trivial Lie algebra. In the closed case, these are discrete in the topology of G, whereas in the Lie case, they are only discrete in their own topology but not in that of G; quite to the contrary, they may very well be dense in G.

[7] The same favorable situation prevails for finite groups: they always admit maximal subgroups and any subgroup is contained in a maximal one: in fact, it can always be realized as the last member in a finite descending chain of subgroups where each is maximal in the preceding one.

Proof. To begin with, note that it suffices to show that a connected abelian Lie group contains no maximal discrete closed subgroups. (Indeed, if G is a connected abelian Lie group and H is a closed subgroup of G with connected one-component H_0, then G/H_0 is a connected abelian Lie group and H/H_0 is a discrete closed subgroup of G/H_0, so if H were maximal among the closed subgroups of G, H/H_0 would have to be maximal among the discrete closed subgroups of G/H_0.) To prove this, suppose that G is a connected abelian Lie group, so $G = \mathbb{R}^n/\Lambda \cong \mathbb{T}^p \times \mathbb{R}^q$ where Λ is some lattice in $\mathbb{R}^n$, $\mathbb{T}$ denotes the unit circle (one-dimensional torus) and p is the dimension of the subspace of $\mathbb{R}^n$ generated by Λ [38, Theorem 6.20, p. 155]. Now it is well known that the discrete closed subgroups of $\mathbb{R}^n$ are precisely the lattices in $\mathbb{R}^n$ [38, Lemma 6.18, p. 155], so it is clear that discrete closed subgroups of G are of the form L/Λ where L is a lattice in $\mathbb{R}^n$ containing Λ as a sublattice. But for any of these lattices L, there is always a bigger one in which it is contained. ■

Fortunately, the situation is quite different for nonabelian Lie groups – otherwise, the present paper would be pointless. To see this, let us look at a simple example:

Example 2.4. Consider the group $G = SO(3)$ of all rotations in $\mathbb{R}^3$. Then up to conjugacy, the non-discrete proper closed subgroups (and in fact, the non-discrete proper Lie subgroups) of G are

$$\begin{aligned} SO(2) &= \text{the two-dimensional rotation group} , \\ O(2) &= \text{the two-dimensional orthogonal group} , \end{aligned}$$

whereas the discrete closed (and hence finite) subgroups of G are

$$\begin{aligned} \mathbb{Z}_n &= \text{the cyclic group of order } n , \\ \mathrm{D}_n &= \text{the dihedral group of order } 2n , \\ \mathrm{T} &= \text{the tetrahedral group } \cong A_4 , \\ \mathrm{O} &= \text{the cubic/octahedral group } \cong S_4 , \\ \mathrm{I} &= \text{the dodecahedral/icosahedral group } \cong A_5 . \end{aligned}$$

(The first statement follows by noting that if H is a non-discrete proper Lie subgroup of $G = SO(3)$ with connected one-component H_0, then H_0 must be equal to its maximal torus $T = SO(2)$ and H itself must be contained in its normalizer $N_G(T) = O(2)$.[8] But $N_G(T)/T = \mathbb{Z}_2$, so there is no other alternative: H must be either one or the other. For the second statement, see [7, p. 192] and [14, p. 103].) Concerning maximality, note first that $SO(2)$ is a maximal connected Lie subgroup of $SO(3)$, since its Lie algebra $\mathfrak{so}(2)$ is a maximal subalgebra of $\mathfrak{so}(3)$, but fails to be a maximal closed subgroup of $SO(3)$, since $SO(2) \subset O(2)$. However, $O(2)$ is a maximal closed subgroup of $SO(3)$. Geometrically, $SO(2)$ is a circle and $O(2) \cong \mathbb{Z}_2 \ltimes SO(2)$ is the disjoint union of two circles, generated by $SO(2)$ and any orthogonal matrix of determinant -1, such as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

[8] Here, we use the fact that any Lie subgroup must normalize its own connected one-component; this general feature will be used extensively in what follows.

which represents reflection along the main diagonal. Similarly, looking at the list of finite subgroups above, we see that the first three fail to be maximal closed subgroups of $SO(3)$, since $\mathbb{Z}_n \subset D_n \subset O(2)$ and $\mathrm{T} \subset \mathrm{O}$, but the last two are.[9]

Of course, we may wonder what would happen if we were to replace the concept of a maximal closed subgroup by that of a maximal Lie subgroup. To get an idea, let us look again at the subgroups of the rotation group.

Example 2.5. Consider again the group $G = SO(3)$ of all rotations in $\mathbb{R}^3$. Then it follows from the arguments given in Example 2.4 above that the orthogonal subgroup $O(2)$ is a maximal Lie subgroup of $SO(3)$. On the other hand, none of the finite subgroups listed in Example 2.4 above is a maximal Lie subgroup of $SO(3)$. For example, the octahedral group O is contained in a bigger subgroup $SO(3,\mathbb{Q})$, which is clearly a discrete Lie subgroup. The same is true for the icosahedral group I, with $\mathbb{Q}$ replaced by $\mathbb{Q}(\sqrt{5})$. Moroever, none of the discrete Lie subgroups $SO(3,\mathbb{Q}(\alpha_1,\ldots,\alpha_k))$ of $SO(3)$, where $\{\alpha_1,\ldots,\alpha_k\}$ is a finite set of irrational numbers linearly independent over $\mathbb{Q}$, is maximal, since one can always extend any one of them to a bigger one by passing to a bigger field extension of $\mathbb{Q}$, with more generators.

In general, it is clear that maximal Lie subgroups fall into two distinct classes: closed maximal Lie subgroups and dense maximal Lie subgroups. (Indeed, if M is a maximal Lie subgroup of G, then its closure $\bar{M}$ is a Lie subgroup of G containing M and hence $\bar{M} = M$ or $\bar{M} = G$.) Now closed maximal Lie subgroups are maximal closed subgroups, but the two notions do *not* coincide since the former are maximal among all Lie subgroups while the latter are only maximal among all closed subgroups. (Taking into account that the expression "closed Lie subgroup" is a tautology and is always abbreviated to "closed subgroup", we may express this fact by the statement that the adjectives "closed" and "maximal" do not commute.) This discrepancy is not specific to the above example but rather, at least for discrete subgroups, a general and systematic feature, since the same argument as in the example shows that if G is any compact connected matrix Lie group ($G \subset SO(n)$), then a finite subgroup of G can be a maximal closed subgroup of G but cannot be a closed maximal Lie subgroup of G, since it is contained in a dense discrete Lie subgroup of G of the form $G \cap SO(n,\mathbb{Q}(\alpha_1,\ldots,\alpha_k))$, where the α's are chosen such that all matrix elements of all elements of H are rational linear combinations of the α's. Moreover, none of these dense discrete Lie subgroups is a maximal Lie subgroup. In fact, we know of no example of a dense maximal Lie subgroup, and the arguments given here support the conjecture that such subgroups do not exist.

These considerations show that studying maximal Lie subgroups is largely uninteresting: the closed ones can be found among the maximal closed subgroups, and on the dense ones, practically nothing is known, not even regarding the

[9] Although it is true that S_4 can be embedded into A_5 as a subgroup, O cannot be embedded into I within $SO(3)$, even up to conjugacy: the symmetry group of the octahedron is not a subgroup of the symmetry group of the icosahedron.

question whether they exist at all. And even if they do, they are uninteresting at least from the point of view of representation theory, since by continuity, the restriction of an irreducible representation of a Lie group to a dense Lie subgroup remains irreducible: there is no branching. Thus in order to simplify the terminology, we adhere to the general convention already mentioned in the footnote on the title page:

Convention: "maximal subgroup" always means "maximal closed subgroup".

With this convention in mind, let us collect a few initial observations on the problem of determining the maximal subgroups of Lie groups, and in particular, of compact Lie groups. As in the case of maximal connected Lie subgroups, the Lie correspondence (i.e., the one-to one correspondence between connected Lie subgroups of a connected Lie group and subalgebras of its Lie algebra [27, p. 47]) plays an important role, but in a different way, because it is clear that if we allow for Lie subgroups (more specifically, closed subgroups) which are not connected, several different ones may have the same connected one-component and hence correspond to the same Lie subalgebra, so we must face the question which of them, if any, is a maximal one. That this is not completely straightforward can already be seen from Example 2.4 above: it shows that, even when the ambient Lie group G is connected, the classification of its maximal subgroups is by no means equivalent to the classification of the maximal subalgebras of its Lie algebra $\mathfrak{g}$, simply because the connected one-component M_0 of a maximal subgroup M of G and the corresponding subalgebra $\mathfrak{m}$ are not necessarily maximal. Quite to the contrary, M_0 and $\mathfrak{m}$ may even be trivial, which means that M must be discrete, but that does of course not mean that M must be trivial. On the other hand, M_0 may be maximal among the connected Lie subgroups of G and yet fail to be maximal among all the Lie subgroups of G or even the closed subgroups of G. Additional complications arise when the ambient Lie group G is not connected, but these can be handled by introducing a modified concept of maximality, giving rise to the notions of a "maximal invariant" subalgebra or subgroup and of a "(quasi)primitive" subalgebra; these will be specified in a mathematically precise manner in Definitions 3.3, 3.4 and 4.4 below, with the conventions stated in Remarks 3.2 and 3.11.

Before embarking on this program, let us fix some notation that will be used constantly throughout this paper, often without further mention. In all that follows, G will always denote a Lie group and $\mathfrak{g}$ will denote its Lie algebra.[10] The case of main interest to us is when G is compact, but since many of the results in the first part of the paper hold more generally, we have refrained from assuming this right from the start. Similarly, a case of particular interest and importance is when G is connected, but the fact that maximal subgroups of connected Lie groups need not be connected, together with the desire to repeat the procedure of passing to a maximal subgroup in order to construct chains of subgroups where each subgroup is maximal in the previous one, leads us to refrain from assuming this right from the start. Thus in general, G_0 will denote the connected one-

[10] All Lie groups and Lie algebras considered in this paper are assumed to be finite-dimensional, without further mention.

component of G, Γ will denote the component group of G, that is, the discrete quotient group G/G_0, and π will denote the canonical projection from G to Γ, so we have the following short exact sequence:

$$\{1\} \longrightarrow G_0 \longrightarrow G \stackrel{\pi}{\longrightarrow} \Gamma \longrightarrow \{1\}\,. \tag{1}$$

Hence G is an upwards extension of G_0 by Γ (or downwards extension of Γ by G_0):

$$G = G_0\,.\,\Gamma \tag{2}$$

Further, we shall write $\mathrm{Aut}(G_0)$ for the group of automorphisms of G_0, $\mathrm{Inn}(G_0)$ for the normal subgroup of inner automorphisms of G_0 (i.e., automorphisms of G_0 given by conjugation with elements of G_0) and $\mathrm{Out}(G_0)$ for the quotient group

$$\mathrm{Out}(G_0) = \mathrm{Aut}(G_0)/\mathrm{Inn}(G_0)$$

which, by abuse of language, is called the group of outer automorphisms of G_0 even though its elements are not automorphisms but rather equivalence classes of automorphisms. Similarly, we shall write $\mathrm{Aut}(\mathfrak{g})$ for the group of automorphisms of $\mathfrak{g}$, $\mathrm{Inn}(\mathfrak{g})$ for the normal subgroup of inner automorphisms of $\mathfrak{g}$ (i.e., automorphisms of $\mathfrak{g}$ of the form $\mathrm{Ad}(g_0)$ with $g_0 \in G_0$) and $\mathrm{Out}(\mathfrak{g})$ for the quotient group

$$\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$$

which, by abuse of language, is called the group of outer automorphisms of $\mathfrak{g}$ even though its elements are not automorphisms but rather equivalence classes of automorphisms. With this notation, we see that any short exact sequence of the form (1) gives rise to homomorphisms

$$\Gamma \longrightarrow \mathrm{Out}(G_0) \tag{3}$$

and

$$\Gamma \longrightarrow \mathrm{Out}(\mathfrak{g}) \tag{4}$$

defined by taking the left coset gG_0 of an element g of G to the equivalence class of the automorphism of G_0 given by conjugation with g and of the automorphism of $\mathfrak{g}$ given by $\mathrm{Ad}(g)$, respectively; its image is a subgroup of the respective outer automorphism group $\mathrm{Out}(G_0)$ or $\mathrm{Out}(\mathfrak{g})$ which we shall typically (and in either case) denote by Σ. Of course, a natural question to ask is whether these homomorphisms can be lifted to homomorphisms

$$\Gamma \longrightarrow \mathrm{Aut}(G_0) \tag{5}$$

and

$$\Gamma \longrightarrow \mathrm{Aut}(\mathfrak{g}) \tag{6}$$

respectively. A particularly important case where this happens is when the extension (2) is *split*, that is, when there exists a homomorphism of Γ into G which is a right inverse to π: then identifying Γ with its image by this homomorphism allows to consider Γ as a subgroup of G and G as the semidirect product of G_0 and Γ:

$$G = G_0 : \Gamma \tag{7}$$

Unfortunately, this is not always the case, even for compact G [20]. However, there is a slightly weaker statement that does hold for any compact Lie group, known as Lee's supplement theorem,[11] according to which G always contains a finite subgroup Λ, called a *Lee supplement* for G_0 in G, which is such that (a) Λ and G_0 generate G and (b) their intersection $\Lambda_0 = \Lambda \cap G_0$ is a normal subgroup of G contained in the center of G_0 [22, p. 272]: then obviously, there is a canonical isomorphism $\Gamma \cong \Lambda / \Lambda_0$ obtained by restricting the projection $\pi : G \to \Gamma$ to Λ and observing that this restriction is still surjective, due to (a), and has Λ_0 as its kernel. (The split case corresponds to $\Lambda = \Gamma$, $\Lambda_0 = \{1\}$.) As a result, we obtain, as in the split case, actions of $\Gamma \cong \Lambda / \Lambda_0$ on G_0 and on $\mathfrak{g}$ by automorphisms, defined by restricting the respective actions of G on G_0 by conjugation and on $\mathfrak{g}$ by Ad to Λ and observing that Λ_0 acts trivially because it is contained in the center of G_0. Therefore, in the compact case, the homomorphisms (3) and (4) can always be lifted to homomorphisms of the form (5) and (6), respectively, and hence Σ can be realized as a subgroup of $\mathrm{Aut}(G_0)$ or $\mathrm{Aut}(\mathfrak{g})$, rather than just of $\mathrm{Out}(G_0)$ or $\mathrm{Out}(\mathfrak{g})$, respectively.

One of the most important tools in the study of maximal subgroups is the concept of the normalizer of a subgroup. Suppose that H is a Lie subgroup of G with connected one-component H_0 and $\mathfrak{h}$ is the Lie algebra of both H and H_0. We denote by

$$N_G(H_0) \;=\; \{\, g \in G \mid g H_0 g^{-1} \subset H_0 \,\} \tag{8}$$

the normalizer of H_0 in G and similarly by

$$N_G(\mathfrak{h}) \;=\; \{\, g \in G \mid \mathrm{Ad}(g)\mathfrak{h} \subset \mathfrak{h} \,\} \tag{9}$$

the normalizer of $\mathfrak{h}$ in G with respect to the adjoint representation. Obviously, we have

$$N_G(H_0) \;=\; N_G(\mathfrak{h}) \tag{10}$$

and we shall denote this group simply by N whenever there is no danger of confusion. It is easy to see that N is always a closed subgroup of G since using a decomposition of the Lie algebra $\mathfrak{g}$ into the direct sum of the subalgebra $\mathfrak{h}$ and an arbitrarily chosen complementary subspace, N can be written as the inverse image of the "block triangular subgroup" $\{\, T \in \mathrm{GL}(\mathfrak{g}) \mid T(\mathfrak{h}) \subset \mathfrak{h} \,\}$ under the homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. The Lie algebra of N, denoted by $\mathfrak{n}$, is

$$\mathfrak{n} \;=\; \{\, X \in \mathfrak{g} \mid \mathrm{ad}(X)\mathfrak{h} \subset \mathfrak{h} \,\} \tag{11}$$

and is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. Of course, we may also consider the normalizer $N_G(H)$ of H in G which is defined analogously but will be needed only occasionally,[12] and we then have the following sequence of inclusions which, although very simple to prove, is fundamental:

$$H_0 \subset H \subset N_G(H) \subset N_G(H_0)\;. \tag{12}$$

[11] We are grateful to one of the referees for drawing our attention to this theorem.

[12] In passing, we note that the normalizer of a general Lie subgroup H may fail to be closed and hence it cannot even be guaranteed "a priori" that it is a Lie subgroup, but it is an easy exercise to check that this cannot happen when H is closed (or, as argued before, when H is connected).

Indeed, the last inclusion follows by observing that for any $n \in N_G(H)$, we have $nHn^{-1} \subset H$ and hence $nH_0n^{-1} \subset H_0$ since conjugation by n is a homeomorphism of H and thus preserves its connected one-component. The corresponding inclusion at the level of Lie algebras is

$$\mathfrak{h} \subset \mathfrak{n} \,, \tag{13}$$

which is also obvious from equation (11).

Remark 2.6. In general, N may be larger than H_0 or H and $\mathfrak{n}$ may be larger than $\mathfrak{h}$. An extreme case is when N contains G_0, i.e., H_0 is a normal subgroup of G_0, or equivalently, when $\mathfrak{n} = \mathfrak{g}$, i.e., $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. The other extreme case occurs when N has H_0 as its connected one-component, or equivalently, when N/H_0 is a discrete group, which in turn is equivalent to the condition that $\mathfrak{n} = \mathfrak{h}$.

Definition 2.7. Let G be a Lie group and $\mathfrak{g}$ be a Lie algebra. A closed subgroup H of G is called *self-normalizing* if it coincides with its own normalizer $N_G(H)$ in G. Similarly, a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *self-normalizing* if it coincides with its own normalizer $\mathfrak{n}$ in $\mathfrak{g}$.

In order to complete the picture, consider the general case where $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$. Denoting its normalizer in $\mathfrak{g}$ by $\mathfrak{n}_1$, the normalizer of $\mathfrak{n}_1$ in $\mathfrak{g}$ by $\mathfrak{n}_2$ and so on, we obtain an ascending sequence of subalgebras of $\mathfrak{g}$ which, for dimensional reasons, ends at some proper subalgebra $\mathfrak{n}_r$ of $\mathfrak{g}$:

$$\mathfrak{h} \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \ldots \subset \mathfrak{n}_r \subset \mathfrak{g} \,. \tag{14}$$

Analogously, using equation (12), we obtain an ascending sequence of subgroups of G,

$$H \subset N_1 \subset N_2 \subset \ldots \subset N_r \subset G \,, \tag{15}$$

where $N_1 = N_G(\mathfrak{h}), N_2 = N_G(\mathfrak{n}_1), \ldots, N_r = N_G(\mathfrak{n}_{r-1})$. There are then two possibilities: either $\mathfrak{n}_r$ is an ideal of $\mathfrak{g}$ or else $\mathfrak{n}_r$ is self-normalizing.

3. The three types of maximal subgroups

The first main issue in this paper is to show that the problem of classifying the maximal subgroups of an arbitrary Lie group G can be reduced to (a) that of classifying the maximal subgroups of its component group Γ together with (b) that of classifying the maximal Γ-invariant subgroups of its connected one-component G_0, and these split naturally into two types: those whose connected one-component is a normal subgroup and those who are the normalizer of their own connected one-component. The first step in this reduction, which in the context of Lie groups has apparently never been formulated explicitly (although its analogue in the context of finite groups seems to be well known), will be quite elementary once we figure out the precise meaning of the term "maximal Γ-invariant subgroup" of G_0, the main obstacle being that there is no canonical way to make Γ act on G_0. (Here, the emphasis is on the word "canonical" because we have already mentioned that, at least for compact Lie groups, the homomorphism (3) can always be lifted to a homomorphism of the form (5), so the main problem with understanding the meaning of "Γ-invariance" is not lack of existence but rather lack of uniqueness: the lifting is not canonical since it depends on the choice of the splitting, in the split case, or of the Lee supplement, in the general compact case.) However, and this is the key observation, Γ does act naturally on conjugacy classes of elements and also of subgroups of G_0, and this will suffice since the property of maximality of subgroups is invariant under conjugation.

To explain this in more detail, let us introduce some more notation. First, given any element γ of the component group Γ, we shall denote the corresponding connected component of G by G_γ, i.e., $G_\gamma = \pi^{-1}(\{\gamma\})$, so that in particular, $G_1 \equiv G_0$. Similarly, given any subset H of G, we write $H_\gamma = H \cap G_\gamma$ and, in particular, $H_1 = H \cap G_1$. Note that in the case of interest to us here, namely when H is a closed subgroup (or Lie subgroup) of G with connected one-component H_0 and Lie algebra $\mathfrak{h}$, there is no reason for H_1 to be connected, so in general, H_1 will be an intermediate closed subgroup (or Lie subgroup) of G_0 that lies in between H_0 and H: just like H_0, it is necessarily normal in H, and of course, H, H_1 and H_0 all have the same Lie algebra $\mathfrak{h}$. Next, we shall write $[g_0]$ for the conjugacy class of an element g_0 of G_0 in G_0 and, similarly, $[H_0]$ for the conjugacy class of a subset (subgroup / closed subgroup / Lie subgroup) H_0 of G_0 in G_0; explicitly, as sets,

$$[g_0] \;=\; \{\, \tilde{g}_0 g_0 \tilde{g}_0^{-1} \mid \tilde{g}_0 \in G_0 \,\} \quad , \quad [H_0] \;=\; \{\, \tilde{g}_0 g_0 \tilde{g}_0^{-1} \mid \tilde{g}_0 \in G_0 \,,\, g_0 \in H_0 \,\} \,.$$

With this notation, given $g \in G_\gamma$, we define

$$\gamma \cdot [g_0] \;=\; [g g_0 g^{-1}] \quad , \quad \gamma \cdot [H_0] \;=\; [g H_0 g^{-1}] \,. \tag{16}$$

Observe that this provides well defined actions of Γ on the set of conjugacy classes of elements of G_0 and on the set of conjugacy classes of subsets (subgroups / closed subgroups / Lie subgroups) of G_0, since if we replace g by any other representative of γ, say $g\hat{g}_0$, and g_0 by any other representative of the same conjugacy class, say $\tilde{g}_0 g_0 \tilde{g}_0^{-1}$, the resulting element $(g\hat{g}_0)(\tilde{g}_0 g_0 \tilde{g}_0^{-1})(g\hat{g}_0)^{-1}$ equals $\big(g(\hat{g}_0\tilde{g}_0)g^{-1}\big) g g_0 g^{-1} \big(g(\hat{g}_0\tilde{g}_0)g^{-1}\big)^{-1}$ which is conjugate to $g g_0 g^{-1}$. Similarly, we shall write $[X]$ for the $\mathrm{Ad}(G_0)$-orbit of an element X of $\mathfrak{g}$ and, similarly, $[\mathfrak{h}]$ for the $\mathrm{Ad}(G_0)$-orbit of a subset (subspace / subalgebra) $\mathfrak{h}$ of $\mathfrak{g}$; explicitly, as sets,

$$[X] \;=\; \{\, \mathrm{Ad}(\tilde{g}_0)X \mid \tilde{g}_0 \in G_0 \,\} \quad , \quad [\mathfrak{h}] \;=\; \{\, \mathrm{Ad}(\tilde{g}_0)X \mid \tilde{g}_0 \in G_0 \,,\, X \in \mathfrak{h} \,\} \,.$$

With this notation, given $g \in G_\gamma$, we define

$$\gamma \cdot [X] \;=\; [\mathrm{Ad}(g)X] \quad , \quad \gamma \cdot [\mathfrak{h}] \;=\; [\mathrm{Ad}(g)\mathfrak{h}] \,. \tag{17}$$

Observe that this provides well defined actions of Γ on the set of $\mathrm{Ad}(G_0)$-orbits of elements of G_0 and on the set of $\mathrm{Ad}(G_0)$-orbits of subsets (subspaces / subalgebras) of $\mathfrak{g}$, since if we replace g by any other representative of γ, say $g\hat{g}_0$, and X by any other representative of the same $\mathrm{Ad}(G_0)$-orbit, say $\mathrm{Ad}(\tilde{g}_0)X$, the resulting element $\mathrm{Ad}(g\hat{g}_0)\mathrm{Ad}(\tilde{g}_0)X$ equals $\mathrm{Ad}(g(\hat{g}_0\tilde{g}_0)g^{-1})\mathrm{Ad}(g)X$ which is in the same $\mathrm{Ad}(G_0)$-orbit as $\mathrm{Ad}(g)X$. All these actions are canonical since they do not depend on any additional data such as the choice or even the existence of a splitting or, more generally, a Lee supplement.

Concerning the relation between subgroups of G and subgroups of G_0, we note that of course every closed subgroup (or Lie subgroup) H of G gives rise to a closed subgroup (or Lie subgroup) H_1 of G_0 by taking its intersection with G_0, but the converse is not true: there may be closed subgroups (or Lie subgroups) of G_0 that cannot be obtained from closed subgroups (or Lie subgroups) of G in this way. The situation is clarified by the following simple observation.

Proposition 3.1. *Let G be a Lie group with connected one-component G_0 and component group Γ, and let H_1 be a closed subgroup (or Lie subgroup) of G_0. Then H_1 extends to a closed subgroup (or Lie subgroup) H of G such that $H \cap G_0 = H_1$ and $\pi(H) = \Gamma$ if and only if its conjugacy class $[H_1]$ in G_0 is Γ-invariant. In this case, both H and H_1 have the same connected one-component H_0 whose conjugacy class $[H_0]$ in G_0 is also Γ-invariant, and all have the same Lie algebra $\mathfrak{h}$ whose $\mathrm{Ad}(G_0)$-orbit is also Γ-invariant.*

Proof. Assume first that H is a closed subgroup (or Lie subgroup) of G such that $H \cap G_0 = H_1$ and $\pi(H) = \Gamma$. Then for any $\gamma \in \Gamma$, $H_\gamma \neq \emptyset$, and picking any $h_\gamma \in H_\gamma$, we get

$$h_\gamma^{-1} H_\gamma \;=\; H_1 \quad , \quad H_\gamma h_\gamma^{-1} \;=\; H_1$$

since both left and right translation by h_γ are bijections that map G_0 onto G_γ and H onto itself. But this means that conjugation by h_γ leaves H_1 invariant and therefore, being a homeomorphism, also leaves its connected one-component H_0 invariant, while $\mathrm{Ad}(h_\gamma)$ leaves $\mathfrak{h}$ invariant, so $[H_1]$, $[H_0]$ and $[\mathfrak{h}]$ are all Γ-invariant. (Of course, h_γ is not unique but may be freely multiplied by elements of H_1, from either side, and that is precisely the possible ambiguity in the choice of h_γ.) Conversely, the condition that γ leaves $[H_1]$ invariant means that, for any representative g_γ of γ in G_γ, the subgroups $g_\gamma H_1 g_\gamma^{-1}$ and H_1 are conjugate in G_0, i.e., there exists $g_0^{(\gamma)} \in G_0$ such that $g_\gamma H_1 g_\gamma^{-1} = g_0^{(\gamma)} H_1 g_0^{(\gamma)\,-1}$, which implies that conjugation by $h_\gamma = g_\gamma g_0^{(\gamma)\,-1}$ leaves H_1 invariant. Now since $h_\gamma \in G_\gamma$, we can define H by

$$H \;=\; \dot{\bigcup_{\gamma\in\Gamma}} H_\gamma \qquad \text{where} \qquad H_\gamma \;=\; h_\gamma H_1 \;=\; H_1 h_\gamma$$

and check immediately that H is a closed subgroup (or Lie subgroup) of G such that $H \cap G_0 = H_1$ (provided we choose $h_1 = 1$, which is always possible) and $\pi(H) = \Gamma$. (Of course, once again, h_γ is not unique but may be freely multiplied by elements of H_1, from either side: this does not change the definition of H. However, it may even be multiplied by elements of the normalizer of H_1 in G_0, and this can change the definition of H. In such a case, there are several subgroups of G having the same intersection with G_0, but this is no problem since nowhere do we need, nor did we claim, that H is uniquely determined by H_1: this cannot be guaranteed in general.) ■

Remark 3.2. By abuse of language, we shall say that a closed subgroup (or Lie subgroup) H_1 of G_0 is Γ-invariant if its conjugacy class $[H_1]$ in G_0 is Γ-invariant.[13] Similarly, we shall say that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is Γ-invariant if its $\mathrm{Ad}(G_0)$-orbit $[\mathfrak{h}]$ is Γ-invariant. Explicitly, this property can be formulated as the condition that every $\gamma \in \Gamma$ has at least one representative $g_\gamma \in G_\gamma$ such that conjugation by g_γ leaves H_1 invariant and $\mathrm{Ad}(g_\gamma)$ leaves $\mathfrak{h}$ invariant, or in other words, as the condition that the respective normalizers $N_G(H_1)$ of H_1 and $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G meet every connected component of G.

With this language, we can complement Definitions 2.1 and 2.2 as follows.

Definition 3.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and component group Γ. A *maximal* Γ*-invariant subalgebra* of $\mathfrak{g}$ is a proper Γ-invariant subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that if $\tilde{\mathfrak{m}}$ is any Γ-invariant subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \subset \tilde{\mathfrak{m}} \subset \mathfrak{g}$, then $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. The same terminology applies when the expression "subalgebra" is everywhere replaced by the expression "ideal".

Definition 3.4. Let G be a Lie group with connected one-component G_0 and component group Γ. A *maximal* Γ*-invariant closed subgroup* of G_0 is a proper Γ-invariant closed subgroup M_1 of G_0 such that if $\tilde{M}_1$ is any Γ-invariant closed subgroup of G_0 with $M_1 \subset \tilde{M}_1 \subset G_0$, then $\tilde{M}_1 = M_1$ or $\tilde{M}_1 = G_0$. The same terminology applies when the expression "closed subgroup" is everywhere replaced by the expression "closed normal subgroup".

In order for this concept to become useful, we should convince ourselves that non-trivial proper Γ-invariant closed subgroups exist at all. As it turns out, there is only one special situation where this may fail to be true; we give it a name:

Definition 3.5. Let G be a Lie group with connected one-component G_0 and component group Γ. We say that G_0 is Γ*-irrational* if G_0 is abelian and simply connected, i.e., $G_0 \cong \mathbb{R}^n$, and Γ acts on it irreducibly and so that it contains no Γ-invariant lattice.

Proposition 3.6. *Let G be a Lie group with connected one-component G_0 and component group Γ, and suppose G_0 is not Γ-irrational. Then G_0 contains non-trivial proper Γ-invariant closed subgroups.*

Proof. The only information available on the action of Γ on (conjugacy classes of elements and subgroups of) G_0 is that it preserves products; therefore, we shall search for non-trivial proper closed subgroups H of G_0 which are invariant under all automorphisms of G_0, up to conjugacy in G_0.

[13] If one wants to be more precise, one can say that H_1 is Γ-invariant up to conjugacy.

- If G_0 is neither semisimple nor solvable, take H to be its radical, which is then a non-trivial proper connected closed subgroup and invariant under all automorphisms [38, Proposition 10.12, p. 207].

- If G_0 is solvable but not abelian, take H to be one of the non-trivial proper subgroups $(G_0)_k$ of its closed commutator subgroup series, all of which are closed subgroups and invariant under all automorphisms. (Explicitly, let $G_0^{(1)}$ be its algebraic commutator subgroup G_0', whose elements are products of commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ with $g_1, g_2 \in G_0$, define $G_0^{(k)}$ recursively to be the algebraic commutator subgroup $G_0^{(k-1)\,\prime}$ of $G_0^{(k-1)}$, and let $(G_0)_k$ be the closure of $G_0^{(k)}$ in G_0. Then it is easy to see by induction on k that each $G_0^{(k)}$ is an abstract normal subgroup of G_0 and each $(G_0)_k$ is a closed normal subgroup of G_0 such that both $G_0^{(k)}$ and $(G_0)_k$ are invariant under all automorphisms of G_0 and both $G_0^{(k-1)}/G_0^{(k)}$ and $(G_0)_{k-1}/(G_0)_k$ are abelian; moreover, it is also known that G_0 is solvable iff the descending abstract commutator subgroup series $G_0 \supset G_0^{(1)} \supset \ldots \supset G_0^{(k)} \supset \ldots$ terminates after a finite number of steps and also iff the descending closed commutator subgroup series $G_0 \supset (G_0)_1 \supset \ldots \supset (G_0)_k \supset \ldots$ terminates after a finite number of steps [38, Chapter 10.1, pp. 201-204]. Then at least one of the $(G_0)_k$ must be non-trivial and proper. Note that the argument works even if $G_0^{(1)}$ should happen to be dense in G_0, so that $(G_0)_1 = G_0$, but this just means that we have to look at the next step(s) in the series.)

- If G_0 is semisimple and compact but not abelian, take H to be any maximal torus, which is then a non-trivial proper connected closed subgroup and invariant under all automorphisms, up to conjugacy.

- If G_0 is semisimple and non-compact but not abelian, take H to be any maximal compact subgroup, which is then a non-trivial proper connected closed subgroup and invariant under all automorphisms, up to conjugacy.

- If G_0 is abelian but not simply connected, so $G_0 \cong \mathbb{T}^p \times \mathbb{R}^q$ where $\mathbb{T}$ denotes the unit circle (one-dimensional torus) [38, Theorem 6.20, p. 155], with $p > 0$, take H to be $\mathbb{Z}_N^p \times \mathbb{R}^q$, for any value of N, which is then a non-trivial proper closed subgroup and invariant under all automorphisms since $\mathbb{Z}_N^p$ is the subgroup of $\mathbb{T}^p$ formed by the N^{th} roots of unity, and any automorphism of any abelian group preserves the set of N^{th} roots of unity.

- If $G_0 \cong \mathbb{R}^n$ is reducible under the action of Γ, take H to be the non-trivial proper Γ-invariant subspace (and hence connected closed subgroup) of $\mathbb{R}^n$ whose existence is guaranteed by reducibility.

- If $G_0 \cong \mathbb{R}^n$ is irreducible under the action of Γ but contains a Γ-invariant lattice, take H to be this lattice, which is a discrete closed subgroup of $\mathbb{R}^n$ (and all discrete closed subgroups of $\mathbb{R}^n$ are of this form [38, Lemma 6.18, p. 155]).

■

Remark 3.7. From a practical point of view, the exceptional case of G_0 being Γ-irrational is completely irrelevant: it cannot occur if G_0 is compact, which is the case of main interest in this paper, or if G_0 is connected, or if G_0 is not abelian. And even when $G_0 \cong \mathbb{R}^n$, it cannot occur for standard group actions, such as those of reflection groups or crystallographic groups, which always leave some lattice invariant.

As before, we could, in principle, consider the option of replacing the concept of a maximal Γ-invariant closed subgroup by that of a maximal Γ-invariant Lie subgroup, but again this will not lead to anything interesting: closed maximal Γ-invariant Lie subgroups are maximal Γ-invariant closed subgroups, and on dense maximal Γ-invariant Lie subgroups, nothing is known, not even regarding their existence. Thus once again, we shall adhere to the general convention already mentioned in the footnote on the title page:

Convention: "maximal Γ-invariant subgroup" always means "maximal Γ-invariant closed subgroup".

Now we are ready to formulate our first main theorem, which provides a splitting of the maximal subgroups of an arbitrary Lie group G into three distinct classes:

Theorem 3.8. *Let G be a Lie group with connected one-component G_0 and component group Γ, and let M be a maximal subgroup of G with connected one-component M_0. Set $M_1 = M \cap G_0$, so $M_0 \subset M_1 \subset M$. Then one of the following three alternatives holds.*

1. *$M_0 = G_0$, $M_1 = G_0$ and M is an (upwards) extension of G_0 by a maximal subgroup Ξ of Γ:*
$$M \;=\; G_0 \,.\, \Xi \,. \tag{18}$$
*We then say that M is of **trivial type**.*

2. *M_0 is a proper Γ-invariant closed connected normal subgroup of G_0, or equivalently, a proper connected closed normal subgroup of G, M_1 is an (upwards) extension of M_0 by a discrete maximal Γ-invariant subgroup D_0 of the quotient group G_0/M_0 and M is an (upwards) extension of M_1 by Γ:*
$$M_1 \;=\; M_0 \,.\, D_0 \quad , \quad M \;=\; M_1 \,.\, \Gamma \,. \tag{19}$$
*We then say that M and M_1 are of **normal type**. Here, we may have $M_1 = M_0$, i.e., D_0 is trivial, but according to Proposition 3.6, this can only occur when the quotient group G_0/M_0 is Γ-irrational, and hence we shall refer to this exceptional situation[14] as the Γ-irrational type.*

[14] See Remark 3.7.

3. M_0 is a proper Γ-invariant closed connected non-normal subgroup of G_0, M_1 is its normalizer in G_0, M is its normalizer in G and M is an (upwards) extension of M_1 by Γ:

$$M_1 \;=\; N_{G_0}(M_0) \quad , \quad M \;=\; N_G(M_0) \quad , \quad M \;=\; M_1 \,.\, \Gamma \;. \qquad (20)$$

*We then say that M and M_1 are of **normalizer type**.*[15]

In the last two cases, M meets every connected component of G, and M_1 is a maximal Γ-invariant subgroup of G_0 which, except in the case of the Γ-irrational type, is self-normalizing and contains the center of G_0. Conversely, the condition that Ξ should be maximal in Γ, in the first case, and that M_1 should be maximal Γ-invariant in G_0, in the last two cases, guarantees that M will be maximal in G.

Remark 3.9. Note that Theorem 3.8 does not provide a full "if and only if" statement because one aspect is still missing: the restrictions that must be imposed on M_0 in order to guarantee that M is really maximal, or equivalently, that M_1 is really maximal Γ-invariant. This question will be dealt with later; see Theorem 5.3.

Proof. Assume M is a maximal subgroup of G, set $\Xi = \pi(M)$ and let $\tilde{M}$ be its preimage in G (i.e., $\tilde{M} = \pi^{-1}(\pi(M))$), N be the normalizer of M_0 in G and N_1 be the normalizer of M_0 in G_0: then obviously, $N_1 = N \cap G_0$ and, by Proposition 3.1, N_1 is Γ-invariant. Since $\tilde{M}$ is a closed subgroup of G such that $M \subset \tilde{M} \subset G$, maximality of M in G implies that either $\tilde{M} = M$ or $\tilde{M} = G$. In the first case ($\tilde{M} = M$), we conclude that M is the (disjoint) union of all connected components of G that project to elements of Ξ (in particular, $G_0 \subset M$), and M being maximal in G is then equivalent to Ξ being maximal in Γ: this is the first alternative of the theorem. In the second case ($\tilde{M} = G$), we conclude that M meets every connected component of G (in particular, $\Xi = \Gamma$), so $\Gamma = M/M_1$ and M is an extension of M_1 by Γ. It then follows from Proposition 3.1 that M_1 is Γ-invariant and that M being maximal in G is equivalent to M_1 being maximal Γ-invariant in G_0. Note, however, that M_0 is just Γ-invariant and proper but may or may not be maximal Γ-invariant. Indeed, the fact that M is maximal in G only implies that either $N = M$, $N_1 = M_1$ or $N = G$, $N_1 = G_0$, corresponding to the third and to the second alternative of the theorem, respectively. Similarly, letting N_2 be the normalizer of M_1 in G_0 and noting that N_2 is closed and Γ-invariant since M_1 is, the fact that M_1 is maximal Γ-invariant in G_0 implies that either $N_2 = M_1$ or $N_2 = G_0$. In the first case, we conclude that M_1 is self-normalizing and contains the center of G_0 (since the normalizer of any subgroup always contains the center). Therefore, we are left with the task of analyzing in more detail what happens when M_0 and/or M_1 are normal. To handle part of this analysis in one stroke, assume that $\bar{M}$ is any Γ-invariant closed subgroup of G_0 with connected one-component M_0 (in particular, it can be either M_0 or M_1), and suppose that $\bar{M}$ is normal. Then we may consider the quotient groups $G/\bar{M}$

[15] Here, the term "normalizer" may be thought of as an abbreviation for "normalizer of its own connected one-component" or "normalizer of its own Lie algebra".

and $G_0/\bar{M}$, which will fit into a short exact sequence

$$\{1\} \longrightarrow G_0/\bar{M} \longrightarrow G/\bar{M} \xrightarrow{\bar{\pi}} \Gamma \longrightarrow \{1\}\ ,$$

where $\bar{\pi}$ is induced from π by passing to the quotient. Repeating the arguments at the beginning of this section, we see that the concept of Γ-invariant subgroups of $G_0/\bar{M}$ is well-defined, and it is then obvious that the non-trivial proper Γ-invariant closed subgroups of $G_0/\bar{M}$ will, by taking the inverse image under the canonical projection from G_0 to $G_0/\bar{M}$, correspond to the Γ-invariant closed subgroups of G_0 properly containing $\bar{M}$ and properly contained in G_0, with the additional property that the former are discrete if and only if the latter have connected one-component M_0. In the case of the second alternative of the theorem, we can use this construction with $\bar{M} = M_0$ to conclude that since M_1 is a maximal Γ-invariant subgroup of G_0, $D_0 = M_1/M_0$ must be a discrete maximal Γ-invariant subgroup of G_0/M_0. For the final statements of the theorem about M_1, we can use the same construction with $\bar{M} = M_1$, combined with Proposition 3.6 applied to the quotient group G_0/M_1, to conclude that if M_1 is normal, then G_0/M_1 must be Γ-irrational. In particular, $G_0/M_1 \cong \mathbb{R}^n$, which is simply connected, and an elementary application of the exact homotopy sequence to the bundle $G_0 \longrightarrow G_0/M_1$ now shows that M_1 must be connected; hence we are in the case of the normal type, with $M_1 = M_0$. ∎

Remark 3.10. A substantial part of Theorem 3.8, namely that concerning the relation between M and M_1, has been motivated by a corresponding statement for finite groups relating maximal subgroups of finite groups to maximal subgroups of their extensions by groups of outer automorphisms (this is what we are dealing with here if the homomorphism (3) is supposed to have trivial kernel): a brief summary can be found in Section 5 of the Introduction to the ATLAS [6, p. xix]. (Note, however, that the notion of "maximal Γ-invariant" subgroup does not appear there.) In particular, in Ref. [6], a maximal subgroup M of G, viewed as an extension of a subgroup of G_0, is called "trivial" if it is of the form $G_0\,.\,\Xi$ with Ξ a maximal subgroup of Γ, as in the first alternative of Theorem 3.8, and we have decided to follow this terminology, while it is called "ordinary" or "novel" if it is of the form $M_1\,.\,\Gamma$ with M_1 a maximal Γ-invariant subgroup of G_0, as in the last two alternatives of Theorem 3.8, according to whether M_1 is a maximal or non-maximal subgroup of G_0. Note that "novel" maximal subgroups M (for which M_1 is non-maximal) can only appear when M_1 is contained in several maximal subgroups $M_1^{(i)}$ of G_0 that are not conjugate in G_0 but are abstractly isomorphic (as Lie groups) and whose conjugacy classes in G_0 are permuted transitively by Γ: then none of them can be extended to a maximal subgroup of G and M_1 will be a maximal Γ-invariant subgroup of G_0 which is "smaller than expected" since it is properly contained in each of the $M_1^{(i)}$.

Remark 3.11. Note that the condition of Γ-invariance of a closed subgroup (or Lie subgroup) of G_0 or of a subalgebra of $\mathfrak{g}$ that plays a central role in this paper depends only on the "outer automorphism part" of Γ, that is, on the image Σ of Γ under the homomorphism (3) or (4), respectively. The full component group Γ is an extension of Σ by the kernel of the respective homomorphism, but for the purpose of deciding whether a closed subgroup (or Lie subgroup) H_1 of G_0 or a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is Γ-invariant, this kernel is irrelevant, so throughout the discussion in this section, we could equally well have used the term "Σ-invariant", instead of "Γ-invariant", where Σ is the aforementioned subgroup of $\mathrm{Out}(G_0)$ or $\mathrm{Out}(\mathfrak{g})$, respectively.

4. Quasiprimitive and primitive subalgebras

In Theorem 3.8, the maximality condition for a closed subgroup M of a Lie group G which meets every connected component of G is reduced to a maximality condition on its intersection M_1 with the connected one-component G_0 of G, but what is still missing is to translate that into an appropriate maximality condition on the corresponding connected one-component M_0, or equivalently, on the corresponding Lie algebra $\mathfrak{m}$ – which, as we shall see, is weaker than that of being maximal Γ-invariant or maximal Σ-invariant and also somewhat more intricate. This is the problem we shall turn to now.

We begin with a preliminary definition that should be more or less obvious.

Definition 4.1. Given a Lie algebra $\mathfrak{g}$ and a subgroup Σ of its outer automorphism group $\mathrm{Out}(\mathfrak{g})$, we say that $\mathfrak{g}$ is Σ-*simple* if it is not abelian and contains no non-trivial proper Σ-invariant ideal. When Σ is the image under the homomorphism (4) of the component group Γ of a Lie group G with Lie algebra $\mathfrak{g}$, we also use the term "Γ-simple" as a synonym for "Σ-simple".

For a correct understanding of this terminology, it is useful to note that an ideal of $\mathfrak{g}$ is, by definition, invariant under inner automorphisms of $\mathfrak{g}$ and so its invariance under an arbitrary automorphism of $\mathfrak{g}$ depends only on the equivalence class of the latter, as an element of $\mathrm{Out}(\mathfrak{g})$.

Proposition 4.2. *Let $\mathfrak{g}$ be a Lie algebra and Σ be a subgroup of its outer automorphism group $\mathrm{Out}(\mathfrak{g})$. Then $\mathfrak{g}$ is Σ-simple if and only if it is the direct sum of various copies of the same simple Lie algebra which under the action of Σ are permuted transitively among themselves.*

Proof. The "if" part is obvious, whereas the arguments for the "only if" part are very similar to those used in the last part of the proof of Theorem 3.8 above:

- If $\mathfrak{g}$ is neither solvable nor semisimple, then its radical is a non-trivial proper Σ-invariant ideal of $\mathfrak{g}$.
- If $\mathfrak{g}$ is solvable but not abelian, then its commutator subalgebra is a non-trivial proper Σ-invariant ideal of $\mathfrak{g}$.
- If $\mathfrak{g}$ is semisimple, then decomposing it into the direct sum of its simple ideals and assembling these into Σ-orbits shows that if there is more than one such orbit, any one of them will be a non-trivial proper Σ-invariant ideal of $\mathfrak{g}$. ∎

A similar terminology is used for connected Lie groups.

Definition 4.3. Given a connected Lie group G_0 and a subgroup Σ of its outer automorphism group $\mathrm{Out}(G_0)$, we say that G_0 is Σ*-simple* if it is not abelian and contains no non-trivial proper connected Σ-invariant normal Lie subgroup. When Σ is the image under the homomorphism (3) of the component group Γ of a Lie group G with connected one-component G_0, we also use the term "Γ-simple" as a synonym for "Σ-simple".

To describe the maximality condition for subalgebras that we are after, let us assume that $\mathfrak{g}$ is a Lie algebra and Σ is a subgroup of its outer automorphism group $\mathrm{Out}(\mathfrak{g})$, as before, and that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. These data give rise to a subgroup

$$\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h}) \;=\; \mathrm{Aut}_\Sigma(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g},\mathfrak{h}) \tag{21}$$

of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$, where $\mathrm{Aut}_\Sigma(\mathfrak{g})$ denotes the inverse image of Σ under the canonical projection from $\mathrm{Aut}(\mathfrak{g})$ to $\mathrm{Out}(\mathfrak{g})$ (i.e., the group of automorphisms of $\mathfrak{g}$ that project to Σ) and

$$\mathrm{Aut}(\mathfrak{g},\mathfrak{h}) \;=\; \{\, \phi \in \mathrm{Aut}(\mathfrak{g}) \mid \phi(\mathfrak{h}) = \mathfrak{h} \,\} \;. \tag{22}$$

denotes the stability group of $\mathfrak{h}$ in $\mathrm{Aut}(\mathfrak{g})$. When Σ is trivial ($\Sigma = \{1\}$), this becomes

$$\mathrm{Inn}(\mathfrak{g},\mathfrak{h}) \;=\; \{\, \phi \in \mathrm{Inn}(\mathfrak{g}) \mid \phi(\mathfrak{h}) = \mathfrak{h} \,\} \;. \tag{23}$$

In what follows, we shall only be interested in Σ-invariant subalgebras and thus note that, according to Remarks 3.2 and 3.11, the condition that $\mathfrak{h}$ should be Σ-invariant means that every element of Σ should have a representative in $\mathrm{Aut}_\Sigma(\mathfrak{g})$ which leaves $\mathfrak{h}$ invariant, or in other words, that under the canonical projection from $\mathrm{Aut}(\mathfrak{g})$ to $\mathrm{Out}(\mathfrak{g})$, the subgroup $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$ should be mapped *onto* the subgroup Σ: this will ensure that, for any other subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$, $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$-invariance is a stronger condition than Σ-invariance. (Of course, by the very definition of $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$, $\mathfrak{h}$ itself is always $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$-invariant.) Note also that in the special case where $\mathfrak{h}$ is a Σ-invariant ideal, we have $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h}) = \mathrm{Aut}_\Sigma(\mathfrak{g})$ (independently of $\mathfrak{h}$), so that for any other subalgebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$, $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$-invariance of $\tilde{\mathfrak{h}}$ is then equivalent to $\tilde{\mathfrak{h}}$ being a Σ-invariant ideal as well.

Definition 4.4. Let $\mathfrak{g}$ be a Lie algebra and Σ be a subgroup of its outer automorphism group $\mathrm{Out}(\mathfrak{g})$. A Σ*-quasiprimitive subalgebra* of $\mathfrak{g}$ is a proper Σ-invariant subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ which is maximal among all $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebras of $\mathfrak{g}$, that is, such that if $\tilde{\mathfrak{m}}$ is any $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \subset \tilde{\mathfrak{m}} \subset \mathfrak{g}$, then $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. A Σ*-primitive subalgebra* of $\mathfrak{g}$ is a Σ-quasiprimitive subalgebra of $\mathfrak{g}$ which contains no non-trivial proper Σ-invariant ideal of $\mathfrak{g}$. When Σ is the image under the homomorphism (4) of the component group Γ of a Lie group G with Lie algebra $\mathfrak{g}$, we also use the term "Γ-(quasi)primitive" as a synonym for "Σ-(quasi)primitive", and when Σ is trivial ($\Sigma = \{1\}$), we omit the prefix and simply speak of a (quasi)primitive subalgebra.

Remark 4.5. Note that the normalizer $\mathfrak{n}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h})$-invariant, for any choice of Σ, because it is even $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$-invariant. Indeed, for $\phi \in \mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ and $X \in \mathfrak{n}$, we have

$$Y \in \mathfrak{h} \quad \Longrightarrow \quad \mathrm{ad}(\phi(X))(Y) \;=\; [\phi(X), \phi(\phi^{-1}(Y))] \;=\; \phi\big(\mathrm{ad}(X)(\phi^{-1}(Y))\big) \in \mathfrak{h} \;,$$

that is, $\phi(X) \in \mathfrak{n}$. Therefore, a Σ-quasiprimitive subalgebra is either self-normalizing or is an ideal. In the second case, the observation immediately preceding Definition 4.4 shows that the condition of being Σ-quasiprimitive is equivalent to that of being a maximal Σ-invariant ideal. Thus a Σ-primitive subalgebra is always self-normalizing. Moreover, the concepts of Σ-quasiprimitive subalgebra and Σ-primitive subalgebra coincide when the ambient Lie algebra is Σ-simple.

Remark 4.6. The position of a Σ-quasiprimitive subalgebra $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ relative to the center $\mathfrak{z}$ of $\mathfrak{g}$ is strongly restricted, since using the fact that their sum $\mathfrak{m}+\mathfrak{z}$ is obviously an $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$, we immediately conclude that either $\mathfrak{m}$ contains $\mathfrak{z}$ (this is necessarily the case when $\mathfrak{m}$ is self-normalizing, since any self-normalizing subalgebra contains the center) or else $\mathfrak{m}$ is a maximal Σ-invariant ideal of $\mathfrak{g}$ which, together with $\mathfrak{z}$ which is also a Σ-invariant ideal of $\mathfrak{g}$, spans all of $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{m}+\mathfrak{z}$ (note that this sum is not necessarily direct). However, it follows from Theorem 3.8 that the second case (which may occur even when $\mathfrak{m}$ is primitive and in this context corresponds to the so-called affine primitive examples mentioned in [12, Proposition 2.3]) does not arise in the study of maximal subgroups, and hence it will be discarded in what follows.[16]

So far, we have not imposed any "a priori" restrictions on the choice of Σ, but the idea is of course that Σ should be discrete. (For example, if a Lie algebra $\mathfrak{g}$ is Σ-simple, then due to Proposition 4.2, $\mathfrak{g}$ is necessarily semisimple, so $\mathrm{Out}(\mathfrak{g})$ is discrete, and hence so is Σ.) This is also what is required in order to make contact with the situation encountered in the previous section, where G is a Lie group with connected one-component G_0, component group Γ and Lie algebra $\mathfrak{g}$ and where $\Sigma \subset \mathrm{Out}(\mathfrak{g})$ is the image of Γ under the homomorphism (4). Indeed, take the adjoint representation of G, viewed as a Lie group homomorphism from G to $GL(\mathfrak{g})$ which we shall denote by Ad or sometimes by Ad_G in order to avoid ambiguities whenever this seems convenient: then the very definition of the homomorphism (4), together with the fact that, also by definition, the group of inner automorphisms of $\mathfrak{g}$ is

$$\mathrm{Inn}(\mathfrak{g}) \;=\; \mathrm{Ad}(G_0) \;, \tag{24}$$

implies that applying Ad to the short exact sequence (1) gives the corresponding

[16] A nice example is provided by considering $\mathfrak{m} = \mathfrak{su}(n)$ as a primitive subalgebra of $\mathfrak{g} = \mathfrak{u}(n)$: there is no maximal subgroup of $U(n)$ which has $SU(n)$ as its connected one-component, since we may extend the latter to closed subgroups with arbitrarily large finite component groups $\mathbb{Z}_p$ by multiplying matrices in $SU(n)$ with p-th roots of unity.

adjoint short exact sequence in the form

$$\{1\} \longrightarrow \mathrm{Ad}(G_0) \longrightarrow \mathrm{Ad}(G) \longrightarrow \Sigma \longrightarrow \{1\}\ . \tag{25}$$

This shows that there is a Lie group (even a linear Lie group) which has Σ as its component group, namely the adjoint group $\mathrm{Ad}(G)$, and also implies that

$$\mathrm{Aut}_\Sigma(\mathfrak{g}) \;=\; \mathrm{Ad}(G)\ . \tag{26}$$

Suppose further that $\mathfrak{h}$ is a Σ-invariant subalgebra of $\mathfrak{g}$, and let N be its normalizer in G. Then

$$\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h}) \;=\; \mathrm{Ad}_G(N)\ . \tag{27}$$

What seems to be more difficult is to formulate, for an arbitrary Lie algebra $\mathfrak{g}$, reasonably simple conditions that would allow to state precisely which subgroups Σ of the group $\mathrm{Out}(\mathfrak{g})$ can arise from a Lie group G having $\mathfrak{g}$ as its Lie algebra, especially in those cases where $\mathrm{Inn}(\mathfrak{g})$ fails to be closed in $\mathrm{Aut}(\mathfrak{g})$ and hence $\mathrm{Out}(\mathfrak{g})$ is not even a Lie group (see [21, Chap. 2, Exercise D3] for an example). But we have not attempted to do so since we are ultimately interested in compact Lie groups, for which the answer is immediate: Σ must be finite.

The importance of the concept of a quasiprimitive subalgebra in the present context stems from its relation to that of a maximal subgroup. However, a precise formulation of this relation depends on additional hypotheses, one of which is that the Lie algebra $\mathfrak{g}$ be reductive.

5. Reductive Lie algebras and groups

In order to continue our analysis, we shall now impose an important restriction: we consider only reductive Lie algebras and groups. Recall that, according to one of various possible definitions (all of which are equivalent), a Lie algebra $\mathfrak{g}$ is reductive if it can be written as the direct sum of its center $\mathfrak{z}$ and a semisimple ideal; the latter is then necessarily equal to the derived subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$, or what is the same, the commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$. As is well known, any compact Lie algebra (i.e., any Lie algebra that can be obtained as the Lie algebra of some compact Lie group) is reductive, and of course so is any semisimple Lie algebra. For Lie groups, the situation is somewhat more complicated since there is a certain amount of ambiguity in the literature as to what are the precise conditions required of a reductive Lie group, over and above the obvious one that its Lie algebra should be reductive. (Another standard requirement is that its component group should be finite.) We shall return to this subject below and then give a precise definition (see Definition 5.5), but before that, we want to state and prove the basic theorem about the relation between quasiprimitive subalgebras and maximal subgroups in the reductive setting, which does not depend on any of the more sophisticated aspects of that definition.

As a preliminary step, we specify criteria to ensure that certain connected normal subgroups of certain connected Lie groups are automatically closed.

Lemma 5.1. *Let G_0 be a connected Lie group with Lie algebra $\mathfrak{g}$ and let H_0 be a connected Lie subgroup of G_0 with Lie algebra $\mathfrak{h}$. Suppose that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and, in addition, either one of the following two hypotheses is satisfied:*

1. *G_0 is simply connected.*
2. *$\mathfrak{g}$ is reductive and $\mathfrak{h}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$.*

Then H_0 is closed in G_0.

Proof. Following the idea outlined in [5, p. 127], consider the quotient algebra $\mathfrak{l} = \mathfrak{g}/\mathfrak{h}$ and assume L to be any connected Lie group which has Lie algebra $\mathfrak{l}$. Suppose we can lift the canonical projection from $\mathfrak{g}$ to $\mathfrak{l}$, which is a Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow \mathfrak{l}$, to a Lie group homomorphism $F : G_0 \longrightarrow L$. Then it is obvious that H_0 is equal to the connected one-component of the kernel of F (since both are generated by the same subalgebra of $\mathfrak{g}$, namely $\mathfrak{h}$), and hence is closed in G_0. This proves the first statement, since the desired lifting of f to F always exists if G_0 is simply connected. If not, let $\tilde{G}_0$ be the universal covering group of G_0, with covering homomorphism $\pi : \tilde{G}_0 \longrightarrow G_0$ whose kernel will be denoted by $\tilde{N}$, so that $G_0 = \tilde{G}_0/\tilde{N}$, and lift the Lie algebra homomorphism $f : \mathfrak{g} \longrightarrow \mathfrak{l}$ to a Lie group homomorphism $\tilde{F} : \tilde{G}_0 \longrightarrow L$. In order for the previous argument to continue working, we must show that $\tilde{F}$ is trivial on $\tilde{N}$, since this guarantees that it factors to yield a Lie group homomorphism $F : G_0 \longrightarrow L$, as before. Now $\tilde{N}$ is a discrete normal subgroup of $\tilde{G}_0$ contained in its center $Z(\tilde{G}_0)$, and $\tilde{F}$, being surjective since f is surjective and L is connected, maps the center $Z(\tilde{G}_0)$ of $\tilde{G}_0$ to the center $Z(L)$ of L. Thus it suffices to show that L can be chosen to be centerfree. This is guaranteed by taking L to be the adjoint group of $\mathfrak{l}$, provided $\mathfrak{l}$ is centerfree. But if $\mathfrak{g}$ is reductive and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ containing its center $\mathfrak{z}$, then $\mathfrak{l} = \mathfrak{g}/\mathfrak{h} \cong [\mathfrak{g},\mathfrak{g}]/([\mathfrak{g},\mathfrak{g}] \cap \mathfrak{h})$ is semisimple and hence centerfree. ∎

Remark 5.2. It may be worthwhile to point out that the conclusion of the second statement in Lemma 5.1 would become wrong if we were to omit any one of its hypotheses. The source of the counterexamples is always the same, namely, the irrational flow on the torus.

1. Take G_0 to be the two-dimensional torus $\mathbb{T}^2$ itself. Of course, G_0 is abelian and hence any subgroup H_0 of G_0 is normal. Thus taking H_0 to be the irrational flow on this torus, we obtain a connected normal Lie subgroup of G_0 which is dense in G_0, rather than closed, even though $\mathfrak{g}$, being abelian, is reductive. However, $\mathfrak{h}$ does not contain the center $\mathfrak{z}$ of $\mathfrak{g}$, since $\mathfrak{z} = \mathfrak{g}$.
2. Take G_0 to be the group of upper triangular complex (3×3)-matrices of determinant 1 and diagonal entries from the unit circle $\mathbb{T}$: its commutator subgroup G_0' is the group of upper triangular complex (3×3)-matrices with diagonal entries all equal to 1. Here, G_0 is solvable, with discrete center $Z(G_0) \cong \mathbb{Z}_3$, whereas G_0' is nilpotent, and by the very definition of the commutator subgroup, any subgroup H_0 of G_0 containing G_0' is normal.

In fact, the qoutient group G_0/G_0' is the two-dimensional torus $\mathbb{T}^2$, so taking H_0 to be the inverse image of the irrational flow on this torus under the canonical projection from G_0 to G_0/G_0', we obtain a connected normal Lie subgroup of G_0 which is dense in G_0, rather than closed, even though $\mathfrak{h}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$, since $\mathfrak{z} = 0$. However, $\mathfrak{g}$ is not reductive, since it is solvable without being abelian.

Now we are ready to formulate the first main theorem of this section.

Theorem 5.3. *Let G be a Lie group with connected one-component G_0, component group Γ and Lie algebra $\mathfrak{g}$, and let M be a closed subgroup of G with connected one-component M_0 and Lie algebra $\mathfrak{m}$ such that M meets every connected component of G. Set $M_1 = M \cap G_0$, so $M_0 \subset M_1 \subset M$. Moreover, suppose that the Lie algebra $\mathfrak{g}$ is reductive and the quotient group M_1/M_0 is finite. Then M is a maximal subgroup of G, or equivalently, M_1 is a maximal Γ-invariant subgroup of G_0, if and only if $\mathfrak{m}$ is a Γ-quasiprimitive subalgebra of $\mathfrak{g}$ containing its center $\mathfrak{z}$ and, more specifically, one of the following two alternatives holds:*

- ***Normal type:*** *$\mathfrak{m}$ is a maximal Γ-invariant ideal of $\mathfrak{g}$, or equivalently, M_0 is a maximal Γ-invariant connected normal subgroup of G_0, and M_1/M_0 is a finite maximal Γ-invariant subgroup of G_0/M_0: from these data, M_1 is recovered by taking the inverse image under the canonical projection from G_0 to G_0/M_0 while M is recovered using Proposition 3.1.*

- ***Normalizer type:*** *$\mathfrak{m}$ is a self-normalizing Γ-quasiprimitive subalgebra of $\mathfrak{g}$: then M_1 is its normalizer in G_0 while M is its normalizer in G.*

Proof. This is naturally divided into two parts.

The first part of the proof consists in showing that if M is maximal, or equivalently, M_1 is maximal Γ-invariant, then $\mathfrak{m}$ must be Γ-quasiprimitive. First of all, it is clear from Theorem 3.8 that $\mathfrak{m}$ is Γ-invariant and must either be an ideal of $\mathfrak{g}$ or else be self-normalizing, depending on whether M_0 is normal or not. Now suppose that $\tilde{\mathfrak{m}}$ is an $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$, where Σ is the image of Γ under the homomorphism (4), as usual: we want to show that either $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. This will be done in three steps. The first step consists in noting that in the case of the Γ-irrational type, where G_0/M_0 is $\mathbb{R}^n$ (as an abelian Lie group) and hence $\mathfrak{g}/\mathfrak{m}$ is $\mathbb{R}^n$ (as an abelian Lie algebra), this is trivial, since an intermediate Γ-invariant subalgebra between $\mathfrak{m}$ and $\mathfrak{g}$ would correspond to a non-trivial proper Γ-invariant subspace of $\mathfrak{g}/\mathfrak{m}$, and there are no such subspaces because Γ acts irreducibly on G_0/M_0 and hence also on $\mathfrak{g}/\mathfrak{m}$. Therefore, according to Theorem 3.8, we may assume, without loss of generality, that M_1 is self-normalizing and contains the center of G_0, implying that $\mathfrak{m}$ contains the center $\mathfrak{z}$ of $\mathfrak{g}$. The second step consists in showing that we may, without loss of generality, assume $\tilde{\mathfrak{m}}$ to be a Γ-invariant ideal of $\mathfrak{g}$. Indeed, if $\mathfrak{m}$ is a Γ-invariant ideal of $\mathfrak{g}$, $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$ is equal to $\mathrm{Ad}(G)$ and contains $\mathrm{Ad}(G_0)$, so the conclusion is immediate. If on the other hand $\mathfrak{m}$ is a Γ-invariant self-normalizing subalgebra of $\mathfrak{g}$, $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$ is equal to $\mathrm{Ad}_G(M)$ and contains $\mathrm{Ad}_G(M_1)$ since in this case, according to Theorem 3.8, M is the normalizer of $\mathfrak{m}$ in G and M_1 is the normalizer of $\mathfrak{m}$ in G_0, so $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariance of $\tilde{\mathfrak{m}}$ is equivalent to the statement that its normalizer $\tilde{N}$ in G contains M and also that its normalizer $\tilde{N}_1$ in G_0 is Γ-invariant and contains M_1. Maximality of M now implies that either $\tilde{N} = M$, $\tilde{N}_1 = M_1$ or $\tilde{N} = G$, $\tilde{N}_1 = G_0$. Thus if $\tilde{\mathfrak{n}}$ denotes the normalizer of $\tilde{\mathfrak{m}}$ in $\mathfrak{g}$, which is just the Lie algebra of $\tilde{N}$ and of $\tilde{N}_1$, we conclude that in the first case, $\tilde{\mathfrak{m}} \subset \tilde{\mathfrak{n}} = \mathfrak{m}$, which is one of the two possible conclusions that we ultimately want to arrive at, while in the second case, $\tilde{\mathfrak{n}} = \mathfrak{g}$, i.e., $\tilde{\mathfrak{m}}$ is an ideal of $\mathfrak{g}$. The third step consists in showing that, independently of whether $\mathfrak{m}$ is an ideal or is self-normalizing, $\tilde{\mathfrak{m}}$ being an ideal containing $\mathfrak{m}$, which in turn contains $\mathfrak{z}$, leads to the conclusion that either $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. To this end, let $\tilde{M}_0$ be the connected Lie subgroup of G_0 corresponding to $\tilde{\mathfrak{m}}$ and let $\tilde{M}_1$ be the abstract subgroup of G_0 generated by M_1 and $\tilde{M}_0$; explicitly, $\tilde{M}_1 = M_1\tilde{M}_0$ since $\tilde{\mathfrak{m}}$ is an ideal and hence $\tilde{M}_0$ is normal. Now since $\mathfrak{g}$ is assumed to be reductive, we can apply Lemma 5.1 to conclude that $\tilde{M}_0$ is a closed subgroup of G_0. Moreover, it follows that $\tilde{M}_1$ is also a closed subgroup of G_0 and that it has connected one-component $\tilde{M}_0$. [Indeed, since as we have just seen, $\tilde{M}_0$ is a closed normal subgroup of G_0, the quotient $G_0/\tilde{M}_0$ is a well-defined connected Lie group and the image of M_1 under the canonical projection from G_0 to $G_0/\tilde{M}_0$ is the quotient group $M_1/(M_1 \cap \tilde{M}_0)$; let us denote it by Q, for the sake of brevity. Now according to the first and second isomorphism theorems of group theory, we have $\tilde{M}_1/\tilde{M}_0 \cong Q \cong (M_1/M_0)\,/\,((M_1 \cap \tilde{M}_0)/M_0)$: the second isomorphism shows that Q is a quotient group of M_1/M_0, which is assumed to be finite, and hence Q itself must be finite, whereas the first isomorphism shows that $\tilde{M}_1$ is precisely the inverse image of Q under the canonical projection from G_0 to $G_0/\tilde{M}_0$. Therefore, $\tilde{M}_1$ is a closed subgroup of G_0 with connected one-component $\tilde{M}_0$ and component group Q.] Maximality of M now implies that either $\tilde{M}_1 = M_1$ or $\tilde{M}_1 = G_0$ and hence either $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$, thus completing the proof that $\mathfrak{m}$ is Γ-quasi-primitive. In particular, note that if M and M_1 are of normal type, this means that $\mathfrak{m}$ is a maximal Γ-invariant ideal of $\mathfrak{g}$ and hence M_0 is a maximal Γ-invariant connected normal subgroup of G_0.

The second part of the proof is devoted to showing that the conditions stated in Theorem 5.3 are not only necessary but also sufficient to guarantee that M is a maximal subgroup of G, or equivalently, M_1 is a maximal Γ-invariant subgroup of G_0. This will be done separately for each of the two types.

- Normal type: First of all, it follows from Lemma 5.1 that the connected normal Lie subgroup $\tilde{M}_0$ of G_0 corresponding to an ideal $\tilde{\mathfrak{m}}$ of $\mathfrak{g}$ containing its center $\mathfrak{z}$ is automatically closed: this implies that as soon as $\mathfrak{z} \subset \mathfrak{m}$, $\mathfrak{m}$ is maximal Γ-invariant if and only if if M_0 is maximal Γ invariant. Assuming this to be the case, and assuming D_0 to be a finite maximal Γ-invariant subgroup of G_0/M_0, let M_1 be the inverse image of D_0 under the canonical projection from G_0 to G_0/M_0 and M be the closed subgroup of G obtained from M_1 according to Proposition 3.1. Now if $\tilde{M}_1$ is a Γ-invariant closed subgroup of G_0 containing M_1, it is clear that $\tilde{M}_1/M_0$ is a Γ-invariant closed subgroup of G_0/M_0 containing $D_0 = M_1/M_0$. (Here, we just use the definition of the quotient topology, which states that a subset A (U) of G_0/M_0 is closed (open) in G_0/M_0 if and only if its inverse image $\pi^{-1}(A)$ $(\pi^{-1}(U))$ under the canonical projection π from G_0 to G_0/M_0 is closed (open) in G_0. But $\tilde{M}_1$ is a subgroup containing M_1 and hence M_0, so that $\pi^{-1}(\tilde{M}_1/M_0) = \tilde{M}_1$.) Maximality of D_0 now implies that either $\tilde{M}_1/M_0 = M_1/M_0$ or $\tilde{M}_1/M_0 = G_0/M_0$, so either $\tilde{M}_1 = M_1$ or $\tilde{M}_1 = G_0$.

- Normalizer type: Suppose that $\mathfrak{m}$ is a self-normalizing Γ-quasiprimitive subalgebra of $\mathfrak{g}$, M_1 is its normalizer in G_0 and M is its normalizer in G. Now if $\tilde{M}$ is a closed subgroup of G containing M and $\tilde{\mathfrak{m}}$ is its Lie algebra, it is obvious that $\tilde{\mathfrak{m}}$ is an $\mathrm{Ad}_G(M)$-invariant subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$. Thus in view of equation (27), Γ-quasiprimitivity of $\mathfrak{m}$ implies that either $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$ and hence either $\tilde{M} \subset M$, according to equations (10) and (12), so in fact $\tilde{M} = M$, or else $\tilde{M} = G$.

■

Remark 5.4. We emphasize here that the conclusions stated in Theorem 5.3 for the normal type are not independent, in the sense that the requirement on $\mathfrak{m}$ – namely the condition that $\mathfrak{m}$ should be a maximal Γ-invariant ideal of $\mathfrak{g}$, which is equivalent to the condition that $\mathfrak{g}/\mathfrak{m}$ should not admit any non-trivial proper Γ-invariant ideals – is a necessary prerequisite for the existence of *finite* maximal Γ-invariant subgroups of G_0/M_0. Indeed, if $\mathfrak{m}$ is not maximal, we can choose a non-trivial proper Γ-invariant ideal of $\mathfrak{g}/\mathfrak{m}$, say $\mathfrak{h}$, and consider the connected normal Lie subgroup H_0 of G_0/M_0 corresponding to $\mathfrak{h}$, noting that according to Lemma 5.1, H_0 is closed because $\mathfrak{g}$ is reductive and $\mathfrak{m}$ contains its center $\mathfrak{z}$, so $\mathfrak{g}/\mathfrak{m}$ is semisimple, i.e., reductive with trivial center. Therefore, if D_0 is any given finite Γ-invariant subgroup of G_0/M_0, the abstract subgroup H of G_0/M_0 generated by D_0 and H_0, explicitly given by $H = D_0H_0$ since $\mathfrak{h}$ is an ideal and hence H_0 is normal, is a closed subgroup of G_0/M_0 with connected one-component H_0 properly containing D_0 and properly contained in G_0/M_0, which shows that D_0 cannot be maximal. (The fact that H is also a closed subgroup of G_0/M_0 and that it has connected one-component H_0 is proved in exactly the same way as the corresponding statement in the first part of the proof of Theorem 5.3, replacing G_0 by G_0/M_0, M_0 by $\{1\}$, M_1 by D_0 and $\tilde{M}_0$ by H_0.)

Returning to the discussion of reductive Lie algebras and groups initiated at the very beginning of this section, we proceed to formulate the precise definition of the concept of a reductive Lie group announced there:

Definition 5.5. Let G be a Lie group with connected one-component G_0, component group Γ and Lie algebra $\mathfrak{g}$. Assuming that $\mathfrak{g}$ is a reductive Lie algebra, we say that G is a *reductive Lie group* if it comes equipped with the following additional structure: a compact subgroup K of G, an involutive automorphism θ of $\mathfrak{g}$ and an $\mathrm{Ad}(G)$-invariant as well as θ-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$ such that if we write

$$\mathfrak{g} \;=\; \mathfrak{k} \oplus \mathfrak{p} \,, \tag{28}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ denote the eigenspaces of θ for eigenvalue $+1$ and -1, respectively – which implies the commutation relations

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \quad , \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \quad , \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \, , \tag{29}$$

expressing the requirement that θ should be an automorphism, and forces $\mathfrak{k}$ and $\mathfrak{p}$ to be orthogonal under B – we have

(i) $\mathfrak{k}$ is precisely the subalgebra of $\mathfrak{g}$ corresponding to the closed subgroup K of G,

(ii) B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$,

(iii) the map

$$\begin{array}{ccc} K \times \mathfrak{p} & \longrightarrow & G \\ (k,X) & \longmapsto & k \, \exp(X) \end{array} \tag{30}$$

is a global diffeomorphism,

(iv) the derived subgroup G_0' of G_0 (defined as the connected Lie subgroup of G_0 corresponding to the derived subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$) has finite center.

As in the semisimple case, K is called the *maximal compact subgroup* of G (its maximality follows easily from condition (iii) above), θ is called the *Cartan involution*, the direct decomposition (28) is called the *Cartan decomposition* and B is called the *invariant bilinear form.*

Note that this definition is slightly weaker than that of a "reductive Lie group in the Harish-Chandra class" given in Chapter 7, Section 2 of Ref. [27], in that among the conditions (i)-(vi) listed there, we impose conditions (i)-(iv) and (vi) but discard condition (v). There are various advantages that speak in favor of this omission. The basic one is that any compact Lie group and any semisimple Lie group with finite component group [17] and finite center is reductive in our sense: in particular, this avoids an annoying flaw of the definition given in Ref. [27], according to which compact Lie groups G whose component group Γ acts on their Lie algebra $\mathfrak{g}$ by outer automorphisms, such as the full orthogonal groups in even dimensions, are not reductive. More precisely, it excludes all compact Lie groups for which the image Σ of Γ under the homomorphism (4), which according to equation (25) is also the component group of $\mathrm{Ad}(G)$, is non-trivial, i.e., $\Sigma \neq \{1\}$: such groups are of great interest for our work and we cannot see any convincing reason for excluding them from the category of reductive Lie groups. Fortunately, many of the important structural results on reductive Lie groups stated in Ref. [27], especially Propositions 7.19, 7.20 and 7.21, remain valid in our slightly broader context. In particular, G_0' is a closed subgroup of G_0. More generally, we have the following analogue of Lemma 5.1, whose proof is however much less elementary.

[17] We do not adhere to the convention adopted in Ref. [27] according to which semisimple Lie groups are automatically supposed to be connected.

Proposition 5.6. *Let G be a reductive Lie group with connected one-component G_0, component group Γ and Lie algebra $\mathfrak{g}$. Then for any semisimple ideal of $\mathfrak{g}$ and, more generally, for any θ-invariant semisimple subalgebra of $\mathfrak{g}$, the corresponding connected Lie subgroup of G_0 is closed and has finite center.*

Proof. For the second statement, see Proposition 7.20 (b) of Ref. [27]. Using this fact, the proof of the first statement for the general case is identical with that for the special case of the derived subgroup G_0' of G_0, which is the content of Proposition 7.20 (a) of Ref. [27]. ■

In particular, this proposition can be applied to the simple ideals of $\mathfrak{g}$ as well as to the Γ-simple ideals of $\mathfrak{g}$ (each of which is, according to Proposition 4.2, the direct sum of a certain number of copies of the same simple ideal, transitively permuted among themselves under the action of Γ) to deduce that the connected Lie subgroups of G_0 corresponding to these ideals are closed subgroups of G_0 with finite center: they will be called the *simple factors* and the Γ-*simple factors* of G_0, respectively. Note that the latter, being Γ-invariant, extend to closed subgroups of G, each of which meets every connected component of G.

In what follows, our main goal will be to show how, very roughly speaking, the study of the maximal subgroups of a reductive Lie group can be reduced to that of the maximal subgroups of its simple factors. We start at the Lie algebra level, where this reduction can be achieved through the following two theorems.

Theorem 5.7. *Let $\mathfrak{g}$ be a Lie algebra and Σ be a subgroup of its outer automorphism group* $\mathrm{Out}(\mathfrak{g})$. *Assuming that $\mathfrak{g}$ is reductive, with canonical decomposition*

$$\mathfrak{g} \;=\; \mathfrak{z} \oplus \mathfrak{g}' \quad , \quad \mathfrak{g}' \;=\; \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma \tag{31}$$

into its center $\mathfrak{z}$, its derived algebra $\mathfrak{g}'$ and its Σ-simple ideals $\mathfrak{g}_1^\Sigma, \ldots, \mathfrak{g}_r^\Sigma$, let $\mathfrak{m}$ be a Σ-quasiprimitive subalgebra of $\mathfrak{g}$ containing $\mathfrak{z}$. Then denoting the intersection of $\mathfrak{m}$ with $\mathfrak{g}'$ by $\mathfrak{m}'$, so that $\mathfrak{m} = \mathfrak{z} \oplus \mathfrak{m}'$, one of the following two alternatives holds.

1. *$\mathfrak{m}'$ is the direct sum of all Σ-simple ideals of $\mathfrak{g}$ except one, say $\mathfrak{g}_i^\Sigma$, plus a Σ-primitive subalgebra $\mathfrak{m}_i^\Sigma$ of $\mathfrak{g}_i^\Sigma$:*

$$\mathfrak{m}' \;=\; \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{g}_{i-1}^\Sigma \oplus \mathfrak{m}_i^\Sigma \oplus \mathfrak{g}_{i+1}^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma \;. \tag{32}$$

   *We then say that $\mathfrak{m}$ is of **Σ-simple type**.*

2. *$\mathfrak{m}'$ is the direct sum of all Σ-simple ideals of $\mathfrak{g}$ except two isomorphic ones, say $\mathfrak{g}_i^\Sigma$ and $\mathfrak{g}_j^\Sigma$, with the "diagonal" subalgebra $\mathfrak{g}_{ij}^\Sigma$ of $\mathfrak{g}_i^\Sigma \oplus \mathfrak{g}_j^\Sigma$,*

$$\mathfrak{m}' \;=\; \bigoplus_{\substack{k=1 \\ k\neq i,j}}^{r} \mathfrak{g}_k^\Sigma \oplus \mathfrak{g}_{ij}^\Sigma \;, \tag{33}$$

where "diagonal" means that under suitable Σ-equivariant isomorphisms $\mathfrak{g}_i^\Sigma \cong \mathfrak{g}_s^\Sigma$ and $\mathfrak{g}_j^\Sigma \cong \mathfrak{g}_s^\Sigma$, the subalgebra $\mathfrak{g}_{ij}^\Sigma$ of $\mathfrak{g}_i^\Sigma \oplus \mathfrak{g}_j^\Sigma$ corresponds to the subalgebra

$$\mathrm{diag}\ \mathfrak{g}_s^\Sigma \;=\; \{(X,X) \mid X \in \mathfrak{g}_s^\Sigma\}$$

*of $\mathfrak{g}_s^\Sigma \oplus \mathfrak{g}_s^\Sigma$. We then say that $\mathfrak{m}$ is of Σ-**diagonal type**. In this case, $\mathfrak{m}$ is a maximal Σ-invariant subalgebra of $\mathfrak{g}$.*

Moreover, $\mathfrak{m}$ will be a maximal Σ-invariant ideal of $\mathfrak{g}$ if and only if $\mathfrak{m}$ is of Σ-simple type and $\mathfrak{m}_i^\Sigma = \{0\}$; in all other cases, $\mathfrak{m}$ is self-normalizing.

Proof. Letting π_i^Σ denote the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i^\Sigma$, which is an $\mathrm{Aut}_\Sigma(\mathfrak{g})$-equivariant Lie algebra homomorphism, and $\mathfrak{m}_i^\Sigma$ denote the image of $\mathfrak{m}$ under this projection, we note first of all that there can be at most one index, say i, for which $\mathfrak{m}_i^\Sigma$ is properly contained in $\mathfrak{g}_i^\Sigma$. Indeed, if there were two such indices, say i and j, then

$$\mathfrak{z} \oplus \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{m}_i^\Sigma \oplus \ldots \oplus \mathfrak{g}_j^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma$$

and

$$\mathfrak{z} \oplus \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{g}_i^\Sigma \oplus \ldots \oplus \mathfrak{m}_j^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma$$

would be proper $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebras of $\mathfrak{g}$ and

$$\mathfrak{z} \oplus \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{m}_i^\Sigma \oplus \ldots \oplus \mathfrak{m}_j^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma$$

would be another $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$ properly contained in any of the two preceding ones and containing $\mathfrak{m}$ (though perhaps not properly), which contradicts maximality of $\mathfrak{m}$. Similarly, if there is one such index, say i, so that $\mathfrak{m}'$ is of the form given in equation (32), then $\mathfrak{m}_i^\Sigma$ must be a Σ-primitive subalgebra of $\mathfrak{g}_i^\Sigma$ since otherwise, we could find a proper $\mathrm{Aut}_\Sigma(\mathfrak{g}_i^\Sigma,\mathfrak{m}_i^\Sigma)$-invariant subalgebra $\tilde{\mathfrak{m}}_i^\Sigma$ of $\mathfrak{g}_i^\Sigma$ properly containing $\mathfrak{m}_i^\Sigma$, and then

$$\mathfrak{z} \oplus \mathfrak{g}_1^\Sigma \oplus \ldots \oplus \mathfrak{g}_{i-1}^\Sigma \oplus \tilde{\mathfrak{m}}_i^\Sigma \oplus \mathfrak{g}_{i+1}^\Sigma \oplus \ldots \oplus \mathfrak{g}_r^\Sigma$$

would be a proper $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{m}$, which contradicts maximality of $\mathfrak{m}$. (Note that there are then two possibilities: either $\mathfrak{m}_i^\Sigma = \{0\}$, which means that $\mathfrak{m}$ is an ideal, or $\mathfrak{m}_i^\Sigma \neq \{0\}$, which implies that both $\mathfrak{m}$, as a subalgebra of $\mathfrak{g}$, and $\mathfrak{m}_i^\Sigma$, as a subalgebra of $\mathfrak{g}_i^\Sigma$, must be self-normalizing.) Finally, we must deal with the case that there is no such index, i.e., that $\mathfrak{m}_i^\Sigma = \mathfrak{g}_i^\Sigma$ for all i. (This of course implies that $\mathfrak{m}$ cannot be an ideal and hence must be self-normalizing. Note also that this case cannot occur when $r = 1$, so we may assume without loss of generality that $r \geqslant 2$.) Then for any fixed value of i, $\mathfrak{m} + \mathfrak{g}_i^\Sigma$ is an $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$, which due to maximality of $\mathfrak{m}$ implies that either $\mathfrak{m} + \mathfrak{g}_i^\Sigma = \mathfrak{m}$, i.e., $\mathfrak{g}_i^\Sigma \subset \mathfrak{m}$, or $\mathfrak{m} + \mathfrak{g}_i^\Sigma = \mathfrak{g}$, and the second alternative must occur for at least two different values of i since if it did not occur at all, $\mathfrak{m}$ would have to be all of $\mathfrak{g}$, and if it occurred for only one value of i, $\mathfrak{m}$ would contain the kernel of π_i^Σ and hence, in order to satisfy the condition $\mathfrak{m}_i^\Sigma = \mathfrak{g}_i^\Sigma$, would once again have to be all of $\mathfrak{g}$. (As we shall see soon, the alternative in question must in fact occur for exactly two different values of i.) This in turn implies that for any fixed value of i, $\mathfrak{m} \cap \mathfrak{g}_i^\Sigma$

is a Σ-invariant ideal of $\mathfrak{g}$, since we may choose $j \neq i$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}_j^\Sigma$ and then use the obvious inclusions $[\mathfrak{m} \cap \mathfrak{g}_i^\Sigma, \mathfrak{m}] \subset \mathfrak{m} \cap \mathfrak{g}_i^\Sigma$ and $[\mathfrak{m} \cap \mathfrak{g}_i^\Sigma, \mathfrak{g}_j^\Sigma] = \{0\}$ to derive that $[\mathfrak{m} \cap \mathfrak{g}_i^\Sigma, \mathfrak{g}] \subset \mathfrak{m} \cap \mathfrak{g}_i^\Sigma$. But $\mathfrak{g}_i^\Sigma$ is Σ-simple, so we must have either $\mathfrak{m} \cap \mathfrak{g}_i^\Sigma = \mathfrak{g}_i^\Sigma$, i.e., $\mathfrak{g}_i^\Sigma \subset \mathfrak{m}$, or $\mathfrak{m} \cap \mathfrak{g}_i^\Sigma = \{0\}$, i.e., $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_i^\Sigma$. Now let $\mathfrak{g}''$ denote the Σ-invariant semisimple subalgebra of $\mathfrak{g}$ obtained by taking the direct sum of all those Σ-simple ideals $\mathfrak{g}_i^\Sigma$ of $\mathfrak{g}$ that intersect $\mathfrak{m}$ trivially, and let $\mathfrak{m}''$ denote the intersection of $\mathfrak{m}$ and $\mathfrak{g}''$. Moreover, let $\mathfrak{g}'' = \mathfrak{g}_1'' \oplus \mathfrak{g}_2''$ be any non-trivial direct decomposition of $\mathfrak{g}''$ into two complementary Σ-invariant ideals of $\mathfrak{g}$ (each of which must therefore be the direct sum of some of the Σ-simple ideals $\mathfrak{g}_i^\Sigma$ of $\mathfrak{g}$ which intersect $\mathfrak{m}$ trivially), and let $\pi_1'' : \mathfrak{g}'' \to \mathfrak{g}_1''$ and $\pi_2'' : \mathfrak{g}'' \to \mathfrak{g}_2''$ be the corresponding projections. Then we claim that, for $k = 1$ as well as for $k = 2$, the restriction of π_k'' to $\mathfrak{m}''$ establishes a Σ-equivariant Lie algebra isomorphism between $\mathfrak{m}''$ and $\mathfrak{g}_k''$.

> To prove this for $k=1$, say, note that for any Σ-simple ideal $\mathfrak{g}_j^\Sigma$ of $\mathfrak{g}$ contained in $\mathfrak{g}_2''$, we have $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_j^\Sigma$ and hence $\mathfrak{g}'' = \mathfrak{m}'' \oplus \mathfrak{g}_j^\Sigma$. Therefore, we may decompose any element X'' of $\mathfrak{g}''$ in the form $X'' = X_{\mathfrak{m}}'' + X_j''$ with $X_{\mathfrak{m}}'' \in \mathfrak{m}''$ and $X_j'' \in \mathfrak{g}_j^\Sigma$ to get $\pi_1''(X'') = \pi_1''(X_{\mathfrak{m}}'')$, which shows that π_1'' maps $\mathfrak{m}''$ onto $\mathfrak{g}_1''$. On the other hand, we can repeat the argument employed above to show that $\mathfrak{m}'' \cap \mathfrak{g}_2'' = \mathfrak{m} \cap \mathfrak{g}_2''$ is a Σ-invariant ideal of $\mathfrak{g}$ (choose any Σ-simple ideal $\mathfrak{g}_i^\Sigma$ of $\mathfrak{g}$ contained in $\mathfrak{g}_1''$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_i^\Sigma$ and use the obvious inclusions $[\mathfrak{m} \cap \mathfrak{g}_2'', \mathfrak{m}] \subset \mathfrak{m} \cap \mathfrak{g}_2''$ and $[\mathfrak{m} \cap \mathfrak{g}_2'', \mathfrak{g}_i^\Sigma] = \{0\}$ to derive that $[\mathfrak{m} \cap \mathfrak{g}_2'', \mathfrak{g}] \subset \mathfrak{m} \cap \mathfrak{g}_2''$), so it must be the direct sum of some of the Σ-simple ideals $\mathfrak{g}_j^\Sigma$ of $\mathfrak{g}$ contained in $\mathfrak{g}_2''$, which is impossible since by the definition of $\mathfrak{g}''$, any such $\mathfrak{g}_j^\Sigma$ intersects $\mathfrak{m}$ trivially. This shows that $\mathfrak{m}'' \cap \mathfrak{g}_2'' = \{0\}$, and since $\mathfrak{g}_2''$ is the kernel of π_1'', that the restriction of π_1'' to $\mathfrak{m}''$ is one-to-one.

But this can only happen if both $\mathfrak{g}_1''$ and $\mathfrak{g}_2''$ are Σ-simple and isomorphic to $\mathfrak{m}''$, which completes the proof of the main statement of the theorem.

What remains to be shown is that $\operatorname{diag} \mathfrak{g}_s^\Sigma$ is really a maximal Σ-invariant subalgebra of $\mathfrak{g}_s^\Sigma \oplus \mathfrak{g}_s^\Sigma$. To this end, assume that $\mathfrak{h}$ is a Σ-invariant subalgebra of $\mathfrak{g}_s^\Sigma \oplus \mathfrak{g}_s^\Sigma$ containing $\operatorname{diag} \mathfrak{g}_s^\Sigma$ and define $\mathfrak{h}_1 = \mathrm{pr}_1\big(\mathfrak{h} \cap (\mathfrak{g}_s^\Sigma \oplus \{0\})\big)$ and $\mathfrak{h}_2 = \mathrm{pr}_2\big(\mathfrak{h} \cap (\{0\} \oplus \mathfrak{g}_s^\Sigma)\big)$ where pr_1 and pr_2 denote the first and second projection from $\mathfrak{g}_s^\Sigma \oplus \mathfrak{g}_s^\Sigma$ to $\mathfrak{g}_s^\Sigma$, respectively. Obviously, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Σ-invariant subalgebras of $\mathfrak{g}_s^\Sigma$, but they are even ideals: this follows by observing that since $\mathfrak{h}$ contains $\operatorname{diag} \mathfrak{g}_s^\Sigma$, the projections pr_1 and pr_2 both map $\mathfrak{h}$ onto $\mathfrak{g}_s^\Sigma$, and hence

$$
\begin{aligned}
X_1 \in \mathfrak{g}_s^\Sigma\ ,\ Y_1 \in \mathfrak{h}_1 \quad &\Longrightarrow \quad \exists\, X_2 \in \mathfrak{g}_s^\Sigma \text{ such that } (X_1, X_2) \in \mathfrak{h}\ ,\ (Y_1, 0) \in \mathfrak{h} \\
&\Longrightarrow \quad ([X_1, Y_1], 0) = [(X_1, X_2), (Y_1, 0)] \in \mathfrak{h} \\
&\Longrightarrow \quad [X_1, Y_1] \in \mathfrak{h}_1
\end{aligned}
$$

$$
\begin{aligned}
X_2 \in \mathfrak{g}_s^\Sigma\ ,\ Y_2 \in \mathfrak{h}_2 \quad &\Longrightarrow \quad \exists\, X_1 \in \mathfrak{g}_s^\Sigma \text{ such that } (X_1, X_2) \in \mathfrak{h}\ ,\ (0, Y_2) \in \mathfrak{h} \\
&\Longrightarrow \quad (0, [X_2, Y_2]) = [(X_1, X_2), (0, Y_2)] \in \mathfrak{h} \\
&\Longrightarrow \quad [X_2, Y_2] \in \mathfrak{h}_2
\end{aligned}
$$

But $\mathfrak{g}_s^\Sigma$ is Σ-simple, so it follows that either $\mathfrak{h}_1 = \mathfrak{g}_s^\Sigma$ or $\mathfrak{h}_2 = \mathfrak{g}_s^\Sigma$ or $\mathfrak{h}_1 = \{0\}$ and $\mathfrak{h}_2 = \{0\}$. In the first two cases, we can use the hypothesis that $\mathfrak{h}$ contains diag $\mathfrak{g}_s^\Sigma$ to conclude that $\mathfrak{h} = \mathfrak{g}_s^\Sigma \oplus \mathfrak{g}_s^\Sigma$, whereas in the third case, it follows that $\mathfrak{h} = \mathrm{diag}\ \mathfrak{g}_s^\Sigma$. ■

Theorem 5.8. *Let $\mathfrak{g}$ be a Lie algebra and Σ be a subgroup of its outer automorphism group* $\mathrm{Out}(\mathfrak{g})$*. Assuming that $\mathfrak{g}$ is Σ-simple, with canonical decomposition*

$$\mathfrak{g} \;=\; \mathfrak{g}_s \oplus \ldots \oplus \mathfrak{g}_s \quad (n \text{ summands}) \tag{34}$$

into the direct sum of n copies of the same simple Lie algebra $\mathfrak{g}_s$, let $\mathfrak{m}$ be a Σ-primitive subalgebra of $\mathfrak{g}$. Then one of the following two alternatives holds.

1. *Up to "twisting" with an appropriate automorphism of $\mathfrak{g}$, $\mathfrak{m}$ is the direct sum of n copies of the same Σ_s-primitive subalgebra $\mathfrak{m}_s$ of $\mathfrak{g}_s$,*

$$\mathfrak{m} \;=\; \mathfrak{m}_s \oplus \ldots \oplus \mathfrak{m}_s \quad (n \text{ summands})\,, \tag{35}$$

   where the subgroup Σ_s of $\mathrm{Out}(\mathfrak{g}_s)$ *is obtained from the subgroup Σ of* $\mathrm{Out}(\mathfrak{g})$ *by projection. We then say that $\mathfrak{m}$ is of **simple type**.*

2. *Up to "twisting" with an appropriate automorphism of $\mathfrak{g}$, $\mathfrak{m}$ is the direct sum of a certain number (say q) of copies of the diagonal subalgebra of the direct sum of a certain number (say p) of copies of $\mathfrak{g}_s$,*

$$\begin{array}{rcl} \mathfrak{m} & = & \mathrm{diag}_p\,\mathfrak{g}_s \oplus \ldots \oplus \mathrm{diag}_p\,\mathfrak{g}_s \quad (q \text{ summands}) \\ \mathrm{diag}_p\,\mathfrak{g}_s & = & \{(X_s, \ldots, X_s) \mid X_s \in \mathfrak{g}_s\} \quad (p \text{ summands}) \end{array}\,, \tag{36}$$

   where p and q are divisors of n, with $p > 1$ and $q < n$, chosen such that p is the minimum and q the maximum possible value for which the resulting partition

$$\{1, \ldots, n\} \;=\; \big\{\{1, \ldots, p\}, \ldots, \{n-p+1, \ldots, n\}\big\} \tag{37}$$

   *is Σ-invariant. We then say that $\mathfrak{m}$ is of **diagonal type**.*

Before turning to the proof, let us comment on the structure of the automorphism group of a semisimple Lie algebra $\mathfrak{g}$ which, as in the above theorem, is the direct sum of a certain number of copies (say n) of one and the same simple Lie algebra $\mathfrak{g}_s$: this information will be needed for properly understanding some of the statements of the theorem. In this situation, the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ is the *wreath product* of the automorphism group $\mathrm{Aut}(\mathfrak{g}_s)$ of $\mathfrak{g}_s$ with the permutation group S_n, that is, the semidirect product $\mathrm{Aut}(\mathfrak{g}_s)^n : S_n$, and since inner automorphisms preserve the summands in the direct sum (34), so $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}_s)^n$, we conclude that similarly, the outer automorphism group $\mathrm{Out}(\mathfrak{g})$ of $\mathfrak{g}$ is the *wreath product* of the outer automorphism group $\mathrm{Out}(\mathfrak{g}_s)$ of $\mathfrak{g}_s$ with the permutation group S_n, that is, the semidirect product $\mathrm{Out}(\mathfrak{g}_s)^n : S_n$. Note that the projection from this semidirect product to S_n is a homomorphism while that to $\mathrm{Out}(\mathfrak{g}_s)^n$ is not, nor is its composition with any one of the projections to $\mathrm{Out}(\mathfrak{g}_s)$. Remarkably, however, its composition with the i-th projection to $\mathrm{Out}(\mathfrak{g}_s)$ does become a homomorphism when restricted to the semidirect product $\mathrm{Out}(\mathfrak{g}_s)^n : (S_n)_i$, where $(S_n)_i$ is the stability group of i in S_n:

$$(S_n)_i \ = \ \{ \pi \in S_n \mid \pi(i) = i \} \ . \tag{38}$$

Next, let Σ be a given subgroup of $\mathrm{Out}(\mathfrak{g})$, and denote by S_n^Σ the subgroup of S_n obtained as the image of Σ under the projection from $\mathrm{Out}(\mathfrak{g})$ to S_n. Thus elements σ of Σ are certain $(n+1)$-tuples $(\sigma_1, \ldots, \sigma_n; \pi)$, where $\sigma_1, \ldots, \sigma_n \in \mathrm{Out}(\mathfrak{g}_s)$ and $\pi \in S_n$: they can be represented by $(n+1)$-tuples $(\varphi_1, \ldots, \varphi_n; \pi)$, where $\varphi_1, \ldots, \varphi_n \in \mathrm{Aut}(\mathfrak{g}_s)$ and $\pi \in S_n$ (each φ_i being defined up to multiplication by inner automorphisms of $\mathfrak{g}_s$), acting on n-tuples $(X_1, \ldots, X_n)$ of elements $X_1, \ldots, X_n$ of $\mathfrak{g}_s$ according to

$$(\varphi_1, \ldots, \varphi_n; \pi) \cdot (X_1, \ldots, X_n) \ = \ (\varphi_1(X_{\pi(1)}), \ldots, \varphi_n(X_{\pi(n)})) \ . \tag{39}$$

As noticed above, the projection taking σ to π is a homomorphism whereas the ones taking σ to $(\sigma_1, \ldots, \sigma_n)$ or to σ_i are not, but the latter does become a homomorphism when restricted to the intersection of Σ with the semidirect product $\mathrm{Out}(\mathfrak{g}_s)^n : (S_n)_i$. Its image is a subgroup of $\mathrm{Out}(\mathfrak{g}_s)$ that we shall denote by $\Sigma_{s,i}$ and whose definition can be described explicitly as follows: an element σ_s of $\mathrm{Out}(\mathfrak{g}_s)$ belongs to $\Sigma_{s,i}$ if and only if there exist elements $\sigma_1, \ldots, \sigma_n$ of $\mathrm{Out}(\mathfrak{g}_s)$ such that $\sigma_i = \sigma_s$ and a permutation $\pi \in S_n$ satisfying $\pi(i) = i$ such that, when each σ_k is interpreted as representing an (equivalence class of) isomorphism(s) from $\mathfrak{g}_{s,\pi(k)}$ to $\mathfrak{g}_{s,k}$, $(\sigma_1, \ldots, \sigma_n; \pi)$ belongs to Σ. (Here and in what follows, we shall, for $1 \leqslant i \leqslant n$, refer to the ideal

$$\mathfrak{g}_{s,i} = \{0\} \oplus \ldots \oplus \mathfrak{g}_s \oplus \ldots \oplus \{0\}$$

of $\mathfrak{g}$, with $\mathfrak{g}_s$ in the i-th position, as its i-th simple summand.) Substantial further simplifications can be achieved by using the freedom of modifying the identification (34) above, "twisting" it with an appropriate automorphism of $\mathfrak{g}$, which amounts to replacing the subgroup Σ of $\mathrm{Out}(\mathfrak{g})$ with a conjugate subgroup. For example, we can perform such a "twist" with an automorphism preserving each of the simple summands $\mathfrak{g}_{s,i}$ of $\mathfrak{g}$ (i.e., belonging to the subgroup $\mathrm{Aut}(\mathfrak{g}_s)^n$ of $\mathrm{Aut}(\mathfrak{g})$), in order to guarantee that, for $1 \leqslant i, j \leqslant n$, there exists an element $\sigma_{i,j} \in \Sigma$ which can be represented by an automorphism $\phi_{i,j} \in \mathrm{Aut}_\Sigma(\mathfrak{g})$ taking the j-th simple summand $\mathfrak{g}_{s,j}$ to the i-th simple summand $\mathfrak{g}_{s,i}$ and acting as the identity on $\mathfrak{g}_s$, i.e.,[18]

$$\phi_{i,j} \cdot (\underbrace{0, \ldots, 0}_{j-1}, X, \underbrace{0, \ldots, 0}_{n-j}) \ = \ (\underbrace{0, \ldots, 0}_{i-1}, X, \underbrace{0, \ldots, 0}_{n-i}) \ .$$

Note that once this is achieved, the fact that $\Sigma_{s,i}$ and $\Sigma_{s,j}$ are conjugate under $\phi_{i,j}$ implies that the subgroups $\Sigma_{s,1}, \ldots, \Sigma_{s,n}$ of $\mathrm{Out}(\mathfrak{g}_s)$ will all be equal, so we may drop the index i and simply denote them by Σ_s. Similarly, given any Σ-invariant subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, its image under the i-th projection from $\mathfrak{g}$ to $\mathfrak{g}_s$ will

[18]Unfortunately, there is no information about how $\phi_{i,j}$ acts on the remaining simple summands, which is why we use quotation marks when referring to these automorphisms as "shift operators".

not depend on i and will be a Σ_s-invariant subalgebra of $\mathfrak{g}_s$ which we may simply denote by $\mathfrak{h}_s$. Moreover, we obtain the inclusion

$$\mathrm{Aut}_\Sigma(\mathfrak{g}) \ \subset \ \mathrm{Aut}_{\Sigma_s}(\mathfrak{g}_s)^n : S_n^\Sigma \tag{40}$$

which implies the inclusion

$$\Sigma \ \subset \ \Sigma_s^n : S_n^\Sigma \tag{41}$$

and, for any Σ-invariant subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the inclusion

$$\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{h}) \ \subset \ \mathrm{Aut}_{\Sigma_s}(\mathfrak{g}_s,\mathfrak{h}_s)^n : S_n^\Sigma \tag{42}$$

Indeed, to see that given an element $(\varphi_1,\ldots,\varphi_n;\pi)$ of $\mathrm{Aut}_\Sigma(\mathfrak{g})$, its j-th component φ_j really belongs to $\mathrm{Aut}_{\Sigma_s}(\mathfrak{g}_s)$, we use that it is also the j-th component of the product $(\varphi_1,\ldots,\varphi_n;\pi)\,\phi_{\pi(j),j}$, which by definition maps to $(S_n)_j$ under the last projection and still belongs to $\mathrm{Aut}_\Sigma(\mathfrak{g})$, and it is then clear that if $\mathfrak{h}$ is Σ-invariant, $(\varphi_1,\ldots,\varphi_n;\pi)$ and $\phi_{\pi(j),j}$ both preserve $\mathfrak{h}$ and hence φ_j preserves $\mathfrak{h}_s$.

> The construction of such a "twisting automorphism" is elementary, using the hypothesis that $\mathfrak{g}$ is Σ-simple, i.e., that Σ permutes the simple summands of $\mathfrak{g}$ transitively: it implies that for $1 \leqslant j \leqslant n$, there exists an element $\sigma_j \in \Sigma$ which can be represented by an automorphism $\varphi_j \in \mathrm{Aut}_\Sigma(\mathfrak{g})$ taking $\mathfrak{g}_{s,j}$ to $\mathfrak{g}_{s,1}$, say, so it will act on $\mathfrak{g}_{s,j}$ by shifting elements of $\mathfrak{g}_s$ from the j-th to the first position and then applying some automorphism $(\varphi_j)_1$ of $\mathfrak{g}_s$. Of course, we have $\sigma_1 = 1$ and hence $(\varphi_1)_1$ can be taken as the identity of $\mathfrak{g}_s$. Then the subgroup $\Sigma' = \sigma\Sigma\sigma^{-1}$ of $\mathrm{Out}(\mathfrak{g})$, where $\sigma \in \mathrm{Out}(\mathfrak{g})$ is the class of $\varphi = (\varphi_1)_1 \times \ldots \times (\varphi_n)_1 \in \mathrm{Aut}(\mathfrak{g})$, has the required property, with $\sigma'_{1,j} = \sigma\sigma_j\sigma^{-1}$ and $\sigma'_{i,j} = \sigma\sigma_i^{-1}\sigma_j\sigma^{-1}$.

Similarly, given any Σ-invariant equivalence relation in the set $\{1,\ldots,n\}$, we can perform a further "twisting" with an automorphism which is a "pure permutation" of $\mathfrak{g}$ (i.e., belongs to the subgroup S_n of $\mathrm{Aut}(\mathfrak{g})$) such that the corresponding partition into equivalence classes takes the "block" form given by equation (37) with $1 \leqslant p,q \leqslant n$ satisfying $pq = n$; then S_n^Σ becomes a subgroup of the wreath product of the permutation group S_p with the permutation group S_q, that is, of the semidirect product $S_p^q : S_q$.

> Again, the construction of such a "twisting automorphism" is elementary, using the hypothesis that $\mathfrak{g}$ is Σ-simple, i.e., that Σ permutes the simple summands of $\mathfrak{g}$ transitively: it implies that all equivalence classes must have the same cardinality, say p, which must therefore divide n.

With these preliminaries out of the way, we can proceed to the proof of Theorem 5.8.

Proof. To begin with, we observe that, once the aforementioned simplifications by means of appropriate "twistings" have been performed, it follows from the inclusion (42) that if $\mathfrak{h}$ is any Σ-invariant subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_s$ is the Σ_s-invariant subalgebra of $\mathfrak{g}_s$ obtained by any one of the projections from $\mathfrak{g}$ to $\mathfrak{g}_s$, then the direct sum $\mathfrak{h}_s \oplus \ldots \oplus \mathfrak{h}_s$ will be a subalgebra of $\mathfrak{g}$ which is not only again Σ-invariant but is also $\mathrm{Aut}_\Sigma(\mathfrak{g}, \mathfrak{h})$-invariant. (The first part of this statement can also be inferred directly by noting that, essentially, invariance under Σ is equivalent to invariance under Σ_s together with invariance under the "shift operators" introduced above, and for the direct sum $\mathfrak{h}_s \oplus \ldots \oplus \mathfrak{h}_s$, the latter is manifest.) Thus if $\mathfrak{m}$ is a Σ-primitive subalgebra of $\mathfrak{g}$, there are precisely two distinct possibilities: either $\mathfrak{m}_s$ is a proper subalgebra of $\mathfrak{g}_s$, $\mathfrak{m} = \mathfrak{m}_s \oplus \ldots \oplus \mathfrak{m}_s$ and, as is then easy to see, $\mathfrak{m}_s$ is a Σ_s-primitive subalgebra of $\mathfrak{g}_s$, or else $\mathfrak{m}_s = \mathfrak{g}_s$. To handle the second case, consider, for $1 \leqslant i, j \leqslant n$, the image $\mathrm{pr}_j\big(\mathfrak{m} \cap \ker\, \mathrm{pr}_i\big)$ of the intersection of $\mathfrak{m}$ with the kernel of the i-th projection pr_i from $\mathfrak{g}$ to $\mathfrak{g}_s$ under the j-th projection pr_j from $\mathfrak{g}$ to $\mathfrak{g}_s$ and note that this is an ideal of $\mathfrak{g}_s$.

> Indeed, suppose that $X \in \mathrm{pr}_j\big(\mathfrak{m} \cap \ker\, \mathrm{pr}_i\big)$ and $X' \in \mathfrak{g}_s$. This means that there exists $(X_1, \ldots, X_n) \in \mathfrak{m}$ such that $X_j = X$ and $X_i = 0$, and since pr_j maps $\mathfrak{m}$ onto $\mathfrak{g}_s$, there also exists $(X_1', \ldots, X_n') \in \mathfrak{m}$ such that $X_j' = X'$. Since $\mathfrak{m}$ is a subalgebra and $\ker\, \mathrm{pr}_i$ is an ideal of $\mathfrak{g}$, it follows that
>
> $$([X_1', X_1], \ldots, [X_n', X_n]) \;=\; [(X_1', \ldots, X_n'), (X_1, \ldots, X_n)]$$
>
> belongs to $\mathfrak{m} \cap \ker\, \mathrm{pr}_i$ and therefore, $[X', X] \in \mathrm{pr}_j\big(\mathfrak{m} \cap \ker\, \mathrm{pr}_i\big)$.

Since $\mathfrak{g}_s$ is simple, there are only two possibilites: either $\mathrm{pr}_j\big(\mathfrak{m} \cap \ker\, \mathrm{pr}_i\big) = \{0\}$, which means that $\mathfrak{m} \cap \ker\, \mathrm{pr}_i \subset \mathfrak{m} \cap \ker\, \mathrm{pr}_j$, or else $\mathrm{pr}_j\big(\mathfrak{m} \cap \ker\, \mathrm{pr}_i\big) = \mathfrak{g}_s$. Using that not only pr_j but also pr_i maps $\mathfrak{m}$ onto $\mathfrak{g}_s$ and that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$, we can sharpen the second alternative to the statement that for any two elements of $\mathfrak{g}_s$, there exists $(X_1, \ldots, X_n) \in \mathfrak{m}$ such that X_i equals the first and X_j equals the second. As a result, it becomes evident that both alternatives are symmetric under the exchange of i and j and therefore, the first alternative provides an equivalence relation $\sim$ in $\{1, \ldots, n\}$, defined by

$$i \sim j \quad \Longleftrightarrow \quad \mathfrak{m} \cap \ker\, \mathrm{pr}_i \;=\; \mathfrak{m} \cap \ker\, \mathrm{pr}_j \;.$$

When i and j are in the same equivalence class, one projection factors over the kernel of the other to provide mutually inverse automorphisms $\varphi_{i,j}$ and $\varphi_{j,i}$ of $\mathfrak{g}_s$ such that $\varphi_{ji} \circ \mathrm{pr}_i|_{\mathfrak{m}} = \mathrm{pr}_j|_{\mathfrak{m}}$ and $\varphi_{ij} \circ \mathrm{pr}_j|_{\mathfrak{m}} = \mathrm{pr}_i|_{\mathfrak{m}}$. Together with an appropriate permutation to bring the resulting partition of $\{1, \ldots, n\}$ into the form (49), these can be used to define the "twist" automorphism that maps $\mathfrak{m}$ into the subalgebra given by equation (48). The requirement that $\mathfrak{m}$ should be Σ-primitive (which incudes the condition that it should not be equal to all of $\mathfrak{g}$) is then guaranteed by the condition that p should be chosen as small as possible and q as large as possible, with the restriction that $p > 1$, $q < n$. ∎

In order to perform the same reduction at the Lie group level, we proceed in several steps, assuming as before that G is a Lie group with connected one-component G_0, component group Γ and Lie algebra $\mathfrak{g}$ (all steps refer to properties of G_0):

- *Step 1:* from general reductive to centerfree semisimple Lie groups,
- *Step 2:* from centerfree semisimple to centerfree Γ-simple Lie groups,
- *Step 3:* from centerfree Γ-simple to centerfree simple Lie groups.

The first step is elementary: it simply consists in dividing out the center $Z(G_0)$ of the connected one-component G_0 of G, which is a Γ-invariant closed normal subgroup of G_0 and which, according to Theorem 3.8, is contained in any maximal Γ-invariant subgroup of G_0. Thus considering the quotient Lie group $\hat{G} = G/Z(G_0)$, with connected one-component $\hat{G}_0 = G_0/Z(G_0)$, component group $\hat{\Gamma} = \Gamma$ and Lie algebra $\mathfrak{g}/\mathfrak{z}$, it becomes obvious that every maximal subgroup M of G is the inverse image of a maximal subgroup $\hat{M}$ of $\hat{G}$ (namely $\hat{M} = M/Z(G_0)$) under the projection from G to $\hat{G}$ and similarly every maximal Γ-invariant subgroup M_1 of G_0 is the inverse image of a maximal Γ-invariant subgroup $\hat{M}_1$ of $\hat{G}_0$ (namely $\hat{M}_1 = M_1/Z(G_0)$) under the projection from G_0 to $\hat{G}_0$: in fact, this prescription establishes a one-to-one correspondence between maximal subgroups M of G and maximal subgroups $\hat{M}$ of $\hat{G}$ and similarly between maximal Γ-invariant subgroups M_1 of G_0 and maximal Γ-invariant subgroups $\hat{M}_1$ of $\hat{G}_0$. Note that this procedure is completely general (it works for any Lie group), so we must only convince ourselves that if G is reductive, $\hat{G}$ will be semisimple and will have trivial center. To this end, assume that $\mathfrak{g}$ is reductive and consider the derived subgroup G_0' of G_0, which is semisimple and whose center $Z(G_0')$ is equal to its intersection $Z(G_0) \cap G_0'$ with the center $Z(G_0)$ of G_0. (The only non-trivial statement in this equality is the inclusion $Z(G_0') \subset Z(G_0)$, which follows by observing that $g_0 \in Z(G_0)$ is equivalent to $\mathrm{Ad}(g_0) = \mathrm{id}_{\mathfrak{g}}$ while $g_0 \in Z(G_0')$ is equivalent to $\mathrm{Ad}(g_0)|_{\mathfrak{g}'} = \mathrm{id}_{\mathfrak{g}'}$, since G_0 and G_0' are connected, but $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ and obviously $\mathrm{Ad}(g_0)|_{\mathfrak{z}} = \mathrm{id}_{\mathfrak{z}}$ for any $g_0 \in G_0$, since G_0 is connected.) Therefore, $\hat{G}_0$ can be identified with the quotient group $G_0'/Z(G_0')$, implying that $\hat{G}_0$ has trivial center (see Proposition 6.30 of Ref. [27]): it is the adjoint group of G_0'.

The second and third step will be summarized in the form of two theorems which result from transferring Theorems 5.7 and 5.8 from the Lie algebra to the Lie group context: this can be done, e.g., by invoking Theorems 3.8 and 5.3. Direct proofs can also be given but will be left to the reader since they are completely analogous to the proofs of Theorems 5.7 and 5.8 given above.

Theorem 5.9. *Let G be a Lie group with connected one-component G_0 and component group Γ, and let $\Sigma \subset \mathrm{Out}(G_0)$ be the image of Γ under the homomorphism (4). Assuming that G_0 is semisimple with trivial center and that*

$$G_0 \;=\; G_{1,0}^{\Sigma} \times \ldots \times G_{r,0}^{\Sigma} \tag{43}$$

is its canonical decomposition into its Σ-simple factors $G_{1,0}^{\Sigma}, \ldots, G_{r,0}^{\Sigma}$, let M be a maximal subgroup of G with connected one-component M_0 such that M meets every connected component of G. Set $M_1 = M \cap G_0$, so $M_0 \subset M_1 \subset M$. Then one of the following two alternatives holds:

- *M_1 is the direct product of all Σ-simple factors of G_0 except one, say $G^{\Sigma}_{i,0}$, with a maximal Σ-invariant subgroup $M^{\Sigma}_{i,1}$ of $G^{\Sigma}_{i,0}$:*

$$M_1 \;=\; G^{\Sigma}_{1,0} \times \ldots \times G^{\Sigma}_{i-1,0} \times M^{\Sigma}_{i,1} \times G^{\Sigma}_{i+1,0} \times \ldots \times G^{\Sigma}_{r,0} \tag{44}$$

*We then say that M and M_1 are of Σ**-simple type**.*

- *M_1 is the direct product of all Σ-simple factors of G_0 except two isomorphic ones, say $G^{\Sigma}_{i,0}$ and $G^{\Sigma}_{j,0}$, with the "diagonal" subgroup $G^{\Sigma}_{ij,0}$ of $G^{\Sigma}_{i,0} \times G^{\Sigma}_{j,0}$,*

$$M_1 \;=\; \prod_{\substack{k=1 \\ k\neq i,j}}^{r} G^{\Sigma}_{k,0} \,\times\, G^{\Sigma}_{ij,0} \;, \tag{45}$$

where "diagonal" means that under suitable Σ-equivariant isomorphisms $G^{\Sigma}_{i,0} \cong G^{\Sigma}_{s,0}$ and $G^{\Sigma}_{j,0} \cong G^{\Sigma}_{s,0}$, the subgroup $G^{\Sigma}_{ij,0}$ of $G^{\Sigma}_{i,0} \times G^{\Sigma}_{j,0}$ corresponds to the subgroup

$$\mathrm{diag}\, G^{\Sigma}_{s,0} \;=\; \{(g_0, g_0) \mid g_0 \in G^{\Sigma}_{s,0}\}$$

*of $G^{\Sigma}_{s,0} \times G^{\Sigma}_{s,0}$. We then say that M and M_1 are of Σ**-diagonal type**.*

Moreover, M and M_1 will be of normal type if and only if they are of Σ-simple type and $M^{\Sigma}_{i,1}$ is discrete; in all other cases, M and M_1 will be of normalizer type.

Theorem 5.10. *Let G be a Lie group with connected one-component G_0 and component group Γ, and let $\Sigma \subset \mathrm{Out}(G_0)$ be the image of Γ under the homomorphism (4). Assuming that G_0 is Σ-simple with trivial center and that*

$$G_0 \;=\; G_s \times \ldots \times G_s \quad (n \text{ factors}) \tag{46}$$

is its canonical decomposition into the direct product of n copies of the same simple Lie group G_s, let M be a maximal subgroup of G with connected one-component M_0 such that M meets every connected component of G. Set $M_1 = M \cap G_0$, so $M_0 \subset M_1 \subset M$. Then one of the following two alternatives holds:

1. *Up to "twisting" with an appropriate automorphism of G_0, M_1 is the direct product of n copies of the same maximal Σ_s-invariant subgroup $M_{s,1}$ of G_s,*

$$M_1 \;=\; M_{s,1} \,\times \ldots \times\, M_{s,1} \quad (n \text{ factors}) \;, \tag{47}$$

*where the subgroup Σ_s of $\mathrm{Out}(G_s)$ is obtained from the subgroup Σ of $\mathrm{Out}(G_0)$ by projection. We then say that M and M_1 are of **simple type**.*

2. *Up to "twisting" with an appropriate automorphism of G_0, M_1 is the direct product of a certain number (say q) of copies of the diagonal subgroup of the direct product of a certain number (say p) of copies of G_s,*

$$\begin{aligned} M_1 \;&=\; \mathrm{diag}_p\, G_s \times \ldots \times \mathrm{diag}_p\, G_s \quad (q \text{ factors}) \\ \mathrm{diag}_p\, G_s \;&=\; \{(g_s, \ldots, g_s) \mid g_s \in G_s\} \quad (p \text{ factors}) \end{aligned} \;, \tag{48}$$

where p and q are divisors of n, with $p > 1$ and $q < n$, chosen such that p is the minimum and q the maximum possible value for which the resulting partition

$$\{1,\ldots,n\} \;=\; \big\{\{1,\ldots,p\},\ldots,\{n-p+1,\ldots,n\}\big\} \tag{49}$$

is Σ-invariant. We then say that M and M_1 are of ***diagonal type****.*

Moreover, M and M_1 will be of normal type if and only if they are of simple type and $M_{s,1}$ is discrete; in all other cases, M and M_1 will be of normalizer type.

Regarding the proof of Theorem 5.10, there is only one case which is not fully covered by Theorem 5.8, namely the one where M and M_1 and $M_{s,1}$ are discrete. However, it is a simple exercise to perform the necessary adaptations, observing that when G_0 is a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, then if G_0 is simply connected or – as in the case of interest here – if G_0 is centerfree, every automorphism of $\mathfrak{g}$ can be lifted to an automorphism of G_0 and hence the groups $\mathrm{Aut}(G_0)$ and $\mathrm{Aut}(\mathfrak{g})$ and similarly the groups $\mathrm{Out}(G_0)$ and $\mathrm{Out}(\mathfrak{g})$ are canonically isomorphic.

It is instructive to illustrate the phenomena that appear in Theorems 5.9 and 5.10 by examples. The first of them will also show that the maximal subgroups of a non-connected compact Lie group may differ considerably from those of its connected one-component.

Example 5.11. Consider the group $O(4)$ of all orthogonal transformations in $\mathbb{R}^4$, whose connected one-component is the special orthogonal group $SO(4)$. The latter has a non-trivial center Z and a non-trivial outer automorphism group Σ, both isomorphic to $\mathbb{Z}_2$: the non-trivial element of Z is the matrix -1_4, whereas the non-trivial element of Σ can be represented by conjugation with the reflection matrix $\mathrm{diag}(1,-1,-1,-1)$, say: thus $SO(4)$ is not simple but is Σ-simple. As explained before Theorem 5.9, the first step consists in dividing by Z, descending to the adjoint group $SO(3) \times SO(3)$, on which the non-trivial element of Σ acts by switching the factors. (In passing, we note that $SO(4)$ lies "in between" its universal covering group $SU(2) \times SU(2)$ and its adjoint group $SO(3) \times SO(3)$; more precisely, $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$ with $\mathbb{Z}_2 = \{\,(1_2,1_2),(-1_2,-1_2)\}$ and $SO(4)/Z \cong SO(3) \times SO(3)$.)[19] Therefore, using the classification of the maximal subgroups of $SO(3)$ given in Example 2.4 – according to which there are two of normal type (which, as we recall, must be finite, since $SO(3)$ is simple), namely the cubic/octahedral group O and the dodecahedral/icosahedral group I, while there is only one of normalizer type, namely $O(2)$ – we see that Theorems 5.9 and 5.10 provide the following list of maximal subgroups and of maximal Σ-invariant subgroups of $SO(4)/Z$ (which give the corresponding ones of $SO(4)$ by taking the inverse image under the quotient homomorphism):

[19]Note that in this example, we encounter no less than three different $\mathbb{Z}_2$ groups, which must be clearly distinguished since they play very different roles.

Type	Maximal subgroups	Maximal Σ-invariant subgroups
simple/normal	$\mathrm{O} \times SO(3)$, $SO(3) \times \mathrm{O}$ $\mathrm{I} \times SO(3)$, $SO(3) \times \mathrm{I}$	$\mathrm{O} \times \mathrm{O}$, $\mathrm{I} \times \mathrm{I}$
simple/normalizer	$O(2) \times SO(3)$, $SO(3) \times O(2)$	$O(2) \times O(2)$
diagonal	$\mathrm{diag}_2\, SO(3)$	$\mathrm{diag}_2\, SO(3)$

Table 1: Maximal subgroups and maximal Σ-invariant subgroups of $SO(4)/Z$ ($SO(4)/Z \cong SO(3) \times SO(3)$) ($Z \cong \mathbb{Z}_2$ = center of $SO(4)$, $\Sigma \cong \mathbb{Z}_2$ = outer automorphism group of $SO(4)$)

Example 5.12. Assume G_0 to be a centerfree connected semisimple Lie group which is the direct product of four copies of the same centerfree connected simple Lie group G_s,

$$G_0 \;=\; G_s \times G_s \times G_s \times G_s \;,$$

and suppose that $\Gamma = \Sigma$ is the cyclic group $\mathbb{Z}_4$. Then G_0 is $\mathbb{Z}_4$-simple since $\mathbb{Z}_4$ acts transitively on the set $\{1, 2, 3, 4\}$, and it is easily checked that the maximal $\mathbb{Z}_4$-invariant subgroup of G_0 of diagonal type is

$$\{\, (g_1, g_2, g_1, g_2) \mid g_1, g_2 \in G_s \,\}$$

which is isomorphic to $G_s \times G_s$ and conjugate to the subgroup $\mathrm{diag}_2\, G_s$ of equation (48) under the transposition $2 \leftrightarrow 3$.

6. Real and complex primitive subalgebras

As pointed out in the introduction, we shall for the rest of this paper concentrate on maximal subgroups of compact Lie groups which are of normalizer type, that is, maximal subgroups which are the normalizer of their own Lie subalgebra. From the theorems proved in the previous section, it follows that the problem of determining all such subgroups can be reduced to that of classifying the Σ-primitive subalgebras of compact simple Lie algebras $\mathfrak{g}$, where Σ is a subgroup of the (finite) outer automorphism group $\mathrm{Out}(\mathfrak{g})$ of $\mathfrak{g}$. The best way to do so is by transferring the corresponding classification for complex simple Lie algebras, which (at least for the case of trivial Σ) is known, to their compact real forms. This requires studying the behavior of primitive subalgebras of a compact simple Lie algebra under complexification.

It is well known that the complexification of a real Lie algebra defines a functor from the category of compact Lie algebras to the category of complex reductive Lie algebras, both with homomorphisms of Lie algebras as morphisms.

Moreover, this functor establishes a bijective correspondence

$$\text{compact Lie algebras} \longleftrightarrow \text{complex reductive Lie algebras} \tag{50}$$

at the level of isomorphism classes, since according to the Weyl existence and conjugacy theorem for compact real forms [27], every complex reductive Lie algebra admits compact real forms and any two of these are conjugate.

If we fix a compact real form $\mathfrak{g}$ of a complex reductive Lie algebra, denoted by $\mathfrak{g}^{\mathbb{C}}$ to indicate that it is the complexification of $\mathfrak{g}$, then the above functor induces a bijective correspondence between the lattice of conjugacy classes of subalgebras of $\mathfrak{g}$ and the lattice of conjugacy classes of reductive subalgebras of $\mathfrak{g}^{\mathbb{C}}$:

$$\begin{array}{c} \text{lattice of conjugacy classes of} \\ \text{subalgebras of the} \\ \text{compact Lie algebra } \mathfrak{g} \end{array} \longleftrightarrow \begin{array}{c} \text{lattice of conjugacy classes of} \\ \text{reductive subalgebras of the} \\ \text{complex reductive Lie algebra } \mathfrak{g}^{\mathbb{C}} \end{array} \tag{51}$$

Now note that complex reductive Lie algebras contain non-reductive subalgebras, such as the parabolic subalgebras. Taking into account that every compact Lie algebra is reductive and so is its complexification, these non-reductive subalgebras cannot be obtained as the complexification of any subalgebra of any compact real form. Thus in order to obtain the primitive subalgebras of a compact simple Lie algebra from the primitive subalgebras of its complexification, we must restrict ourselves to reductive subalgebras and therefore we should relativize the notions of maximal and (quasi)primitive to the lattice of reductive subalgebras. This leads to the following modification of Definitions 3.3 and 4.4.

Definition 6.1. Let $\mathfrak{g}$ be a reductive Lie algebra and Σ be a subgroup of its outer automorphism group $\mathrm{Out}(\mathfrak{g})$. A *maximal Σ-invariant reductive subalgebra* of $\mathfrak{g}$ is a proper Σ-invariant reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that if $\tilde{\mathfrak{m}}$ is any Σ-invariant reductive subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \subset \tilde{\mathfrak{m}} \subset \mathfrak{g}$, then $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. A *$\Sigma$-quasiprimitive reductive subalgebra* of $\mathfrak{g}$ is a proper Σ-invariant reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ which is maximal among all $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant reductive subalgebras of $\mathfrak{g}$, that is, such that if $\tilde{\mathfrak{m}}$ is any $\mathrm{Aut}_\Sigma(\mathfrak{g},\mathfrak{m})$-invariant reductive subalgebra of $\mathfrak{g}$ with $\mathfrak{m} \subset \tilde{\mathfrak{m}} \subset \mathfrak{g}$, then $\tilde{\mathfrak{m}} = \mathfrak{m}$ or $\tilde{\mathfrak{m}} = \mathfrak{g}$. A *$\Sigma$-primitive reductive subalgebra* of $\mathfrak{g}$ is a Σ-quasiprimitive reductive subalgebra of $\mathfrak{g}$ which contains no non-trivial proper Σ-invariant ideal of $\mathfrak{g}$. When Σ is the image under the homomorphism (4) of the component group Γ of a Lie group G with Lie algebra $\mathfrak{g}$, we also use the term "maximal Γ-invariant" as a synonym for "maximal Σ-invariant" and the term "Γ-(quasi)primitive reductive" as a synonym for "Σ-(quasi)primitive reductive", and when Σ is trivial ($\Sigma = \{1\}$), we omit the reference to this group and simply speak of a maximal reductive subalgebra and of a (quasi)primitive reductive subalgebra, respectively.

With this terminology, we have the following

Proposition 6.2. *Let $\mathfrak{g}$ be a compact Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ be its complexification, and let Σ be a subgroup of their outer automorphism group $\mathrm{Out}(\mathfrak{g}) \cong \mathrm{Out}(\mathfrak{g})^{\mathbb{C}}$. Then there is a bijective correspondence between the Σ-(quasi)primitive subalgebras of $\mathfrak{g}$ and the Σ-(quasi)primitive reductive subalgebras of $\mathfrak{g}^{\mathbb{C}}$ which preserves* *conjugacy classes of subalgebras. Moreover, this correspondence takes maximal Σ-invariant subalgebras of $\mathfrak{g}$ to maximal Σ-invariant reductive subalgebras of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. The proof is straightforward and is left to the reader. ∎

7. Primitive subalgebras of classical Lie algebras

In this section, we summarize the classification of the primitive subalgebras and, more generally, the Σ-primitive subalgebras of compact classical Lie algebras. Our presentation is based on a combination of results obtained by various authors [4,8,9,12,30]. As has already been mentioned in the introduction, all these papers, with the exception of parts of Ref. [30], refer to the complex case. Since we are ultimately interested in compact Lie groups, we shall restate them by using Proposition 6.2 to translate to the respective compact real forms. Correspondingly, the term "classical Lie algebra" will in what follows mean one of the compact classical Lie algebras in the standard representation as given by equation (52) below.

When working with classical Lie algebras the adequate method for analyzing the inclusion of subalgebras is to use their standard realization as matrix Lie algebras – more precisely, as Lie algebras of complex $(n\times n)$-matrices. Concretely,

$$\begin{array}{cccl} A\text{-series} & \mathfrak{su}(n) & = & \{\, X \in \mathfrak{u}(n) \mid \mathrm{tr}(X) = 0 \,\} \\ (A_r,\ r \geqslant 1) & & & \qquad (n = r+1) \\ B\text{-series} & \mathfrak{so}(n) & = & \{\, X \in \mathfrak{u}(n) \mid X^T + X = 0 \,\} \\ (B_r,\ r \geqslant 2) & & & \qquad (n = 2r+1 \text{ odd}) \\ C\text{-series} & \mathfrak{sp}(n) & = & \{\, X \in \mathfrak{u}(n) \mid X^T J + JX = 0 \,\} \\ (C_r,\ r \geqslant 3) & & & \qquad (n = 2r \text{ even}) \\ D\text{-series} & \mathfrak{so}(n) & = & \{\, X \in \mathfrak{u}(n) \mid X^T + X = 0 \,\} \\ (D_r,\ r \geqslant 4) & & & \qquad (n = 2r \text{ even}) \end{array} \tag{52}$$

where

$$\mathfrak{u}(n) \;=\; \{\, X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^\dagger + X = 0 \,\} \tag{53}$$

is the Lie algebra of antihermitean complex $(n\times n)$-matrices (as usual, the symbols $.^T$ and $.^\dagger$ denote transpose and hermitean adjoint, respectively),

$$J \;=\; \begin{pmatrix} 0 & -1_r \\ 1_r & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix} \tag{54}$$

is the standard symplectic $(2r \times 2r)$-matrix (the position of the $-$ sign being a matter of taste), and the constraints on the values of r, or n, in equation (52) are imposed in order to avoid repetitions due to the well-known canonical isomorphisms between classical Lie algebras of low rank ($A_1 = B_1 = C_1$, D_1 is abelian, $B_2 = C_2$, $D_2 = A_1 \oplus A_1$, $A_3 = D_3$). We also note that the outer automorphism groups of the simple Lie algebras (including the five exceptional ones) are

$$\mathrm{Out}(\mathfrak{g}) \;=\; \left\{ \begin{array}{ll} \{1\} & \text{for } A_1,\, B_r,\, C_r,\, E_7,\, E_8,\, F_4,\, G_2 \\ \mathbb{Z}_2 & \text{for } A_r\ (r \geqslant 2),\, D_r\ (r \geqslant 5),\, E_6 \\ S_3 & \qquad\qquad \text{for } D_4 \end{array} \right\}. \tag{55}$$

This implies that, apart from the exceptional case of the algebra D_4 ($\mathfrak{so}(8)$), Σ-primitive subalgebras of compact simple Lie algebras come in just two types: primitive and $\mathbb{Z}_2$-primitive. Clearly, the latter exist only within the algebras A_r ($r \geqslant 2$), D_r ($r \geqslant 4$) and E_6, and except for D_4, they coincide with the almost primitive subalgebras of Ref. [30]. In the exceptional case of the algebra D_4 ($\mathfrak{so}(8)$), there are other possibilities, since Σ can be chosen to be one of the other two $\mathbb{Z}_2$-subgroups of S_3, the cyclic subgroup $\mathbb{Z}_3$ or all of S_3; here, it is the S_3-primitive subalgebras that coincide with the almost primitive subalgebras of Ref. [30]. In what follows, we shall not deal with these other types of Σ-primitive subalgebras of D_4, nor with the $\mathbb{Z}_2$-primitive subalgebras of E_6: they require a separate analysis. However, we shall present results for the remaining cases, which are the great majority, corresponding to the algebras A_r ($r \geqslant 2$) and D_r ($r \geqslant 5$): they will be handled by noting that any automorphism $\phi \in \mathrm{Aut}(\mathfrak{g})$ representing the non-trivial element of $\mathrm{Out}(\mathfrak{g}) = \mathbb{Z}_2$ can be realized as conjugation with an antiunitary transformation on $\mathbb{C}^n$, in the case of $\mathfrak{su}(n)$ ($n \geqslant 3$), and as conjugation with an orthogonal transformation on $\mathbb{R}^n$ of determinant -1, in the case of $\mathfrak{so}(n)$ (n even, $n \geqslant 10$). This procedure also works for $\mathfrak{so}(8)$, except that it will in this case not generate all of $\mathrm{Out}(\mathfrak{g}) = S_3$ but only one of its three $\mathbb{Z}_2$-subgroups.

With these preliminaries out of the way, we shall divide the set of subalgebras $\mathfrak{s}$ of a given classical Lie algebra $\mathfrak{g}$ into several types, according to two natural criteria. The first criterion refers to the intrinsic nature of $\mathfrak{s}$: it can be

- abelian,
- simple,
- truly semisimple, i.e., not simple,
- truly reductive, i.e., with non-trivial center and non-trivial derived subalgebra.

The second criterion refers to the nature of the inclusion of $\mathfrak{s}$ in $\mathfrak{g}$. Since $\mathfrak{g}$ is a matrix algebra acting on an n-dimensional (complex) vector space $V \cong \mathbb{C}^n$, we can also think of this inclusion as a (faithful) representation

$$\pi : \mathfrak{s} \longrightarrow \mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{C}) \tag{56}$$

of $\mathfrak{s}$ on V, which must be of one of the following three types:

- Type 0: such representations are said not to be self-conjugate, or to be truly complex, and correspond to inclusions into $\mathfrak{g} = \mathfrak{su}(n)$.
- Type $+1$: such representations are said to be real, or orthogonal, and correspond to inclusions into $\mathfrak{g} = \mathfrak{so}(n)$; abstractly, elements of $\mathfrak{so}(n)$ are characterized by the property of commuting with a given antilinear transformation τ on V of square $+1$ (complex conjugation).
- Type -1: such representations are said to be pseudo-real, or quaternionic, or symplectic, and correspond to inclusions into $\mathfrak{g} = \mathfrak{sp}(n)$; abstractly, elements of $\mathfrak{sp}(n)$ are characterized by the property of commuting with a given antilinear transformation τ on V of square -1 (complex conjugation combined with application of J).

There are then two distinct possibilities:

- π is irreducible,
- π is reducible.

As was first observed by Dynkin in the case of maximal subalgebras, the procedure for classifying the simple ones is rather different from that for classifying the remaining ones. As it turns out, the same goes for primitive and $\mathbb{Z}_2$-primitive subalgebras.

To explain Dynkin's strategy, note first of all that when $\mathfrak{s}$ is a simple maximal subalgebra of a classical Lie algebra $\mathfrak{g}$, then with only one exception, $\mathfrak{s}$ is always irreducible, and the same statement holds for primitive and $\mathbb{Z}_2$-primitive subalgebras. (The proof follows as a corollary from the general classification of reducible primitive and $\mathbb{Z}_2$-primitive subalgebras of classical Lie algebras carried out later in this section; see the discussion following Lemma 7.11 below.) What Dynkin realized was that conversely, almost every irreducible representation of a simple Lie algebra $\mathfrak{s}$ provides an inclusion of $\mathfrak{s}$ as a maximal subalgebra of the pertinent classical Lie algebra $\mathfrak{g}$, where "almost all" means that there is only a handful of exceptions, which can be listed explicitly:

Theorem 7.1. (Dynkin [8,9], Chekalov [4], Komrakov [30]) *Let $\mathfrak{g}$ be a compact classical Lie algebra. Then every simple maximal subalgebra, every simple primitive subalgebra and every simple $\mathbb{Z}_2$-primitive subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is irreducible, with the only exception of the inclusions*

$$\mathfrak{so}(n-1) \subset \mathfrak{so}(n) \quad (n \geqslant 6)\ . \tag{57}$$

Conversely, every simple irreducible subalgebra $\mathfrak{s}$ is maximal, which implies that every simple irreducible subalgebra $\mathfrak{s}$ is primitive and every $\mathbb{Z}_2$-invariant simple irreducible subalgebra $\mathfrak{s}$ is $\mathbb{Z}_2$-primitive, unless the inclusion $\mathfrak{s} \subset \mathfrak{g}$ is one of the 18 exceptions listed in Table 1 of [8, p. 364] or Table 7 of [37, p. 236]. Among these exceptions, there is only one inclusion $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{s}$ is primitive, given by [4, p. 279], [30, p. 200]

$$\mathfrak{so}(12) \subset \mathfrak{so}(495)\ , \tag{58}$$

and none such that $\mathfrak{s}$ is $\mathbb{Z}_2$-primitive.

Remark 7.2. Although the theorem does not provide an explicit list of simple maximal or primitive or $\mathbb{Z}_2$-primitive subalgebras of $\mathfrak{g}$, it is the cornerstone for finding all such subalgebras. Namely, fixing the Lie algebra $\mathfrak{g}$ with its natural representation on $\mathbb{C}^n$, we apply representation theory – more specifically, the Weyl dimension formula – to first find all irreducible representations of all simple Lie algebras $\mathfrak{s}$ (classical and exceptional) of dimension n. Next, we determine which among these provide inclusions of $\mathfrak{s}$ in $\mathfrak{g}$ by deciding whether the irreducible representation of $\mathfrak{s}$ under consideration is self-conjugate, and if so, whether it is real (orthogonal) or pseudo-real/quaternionic (symplectic). Finally, we apply the theorem to eliminate the exceptions.

To handle the general case of subalgebras which are not simple, we must distinguish between two different constructions of the pertinent representation, depending on whether the subalgebra in question is reducible or irreducible:

Definition 7.3. Let $\pi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_2)$ be representations of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on vector spaces V_1 and V_2, respectively. The *external direct sum* of π_1 and π_2 is the representation $\pi_1 \boxplus \pi_2$ of the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on the direct sum $V_1 \oplus V_2$ of the vector spaces V_1 and V_2 given by

$$\begin{array}{rccc} \pi_1 \boxplus \pi_2 : & \mathfrak{g}_1 \oplus \mathfrak{g}_2 & \longrightarrow & \mathfrak{gl}(V_1 \oplus V_2) \\ & X_1 + X_2 & \longmapsto & \pi_1(X_1) \oplus \pi_2(X_2) \end{array} . \tag{59}$$

The *external tensor product* of π_1 and π_2 is the representation $\pi_1 \boxtimes \pi_2$ of the Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$ on the tensor product $V_1 \otimes V_2$ of the vector spaces V_1 and V_2 given by

$$\begin{array}{rccc} \pi_1 \boxtimes \pi_2 : & \mathfrak{g}_1 \times \mathfrak{g}_2 & \longrightarrow & \mathfrak{gl}(V_1 \otimes V_2) \\ & (X_1, X_2) & \longmapsto & \pi_1(X_1) \otimes 1 \, + \, 1 \otimes \pi_2(X_2) \end{array} . \tag{60}$$

Remark 7.4. Observe that the resulting Lie algebra is the same in both cases, namely the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Still, in the case of the tensor product, we prefer to use the symbol $\times$, rather than $\oplus$, since experience shows that this helps to avoid confusion. (On the other hand, we avoid the symbol $\otimes$ in this context, since the concept of tensor product of Lie algebras does not exist.) In particular, using both notations in parallel is helpful to characterize inclusions of subalgebras by representations: the symbol $\oplus$ indicates that the inclusion into $\mathfrak{g}$ is by the direct sum construction, whereas the symbol $\times$ indicates that the inclusion into $\mathfrak{g}$ is by the tensor product construction. This will greatly simplify the tables.

The main difference between the two constructions lies in the fact that, even when both π_1 and π_2 are irreducible, the external direct sum $\pi_1 \boxplus \pi_2$ is always a reducible representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, whereas the external tensor product $\pi_1 \boxtimes \pi_2$ is always an irreducible representation of $\mathfrak{g}_1 \times \mathfrak{g}_2$; moreover, it can be shown that every irreducible representation of $\mathfrak{g}_1 \times \mathfrak{g}_2$ is given by this construction. (See [15, Proposition 3.1.8, p. 123] for the proof of an entirely analogous statement for groups, which can be adapted to Lie algebras replacing the group algebra by the universal enveloping algebra.) Finally, by iteration of both constructions, it is clear that we can define the direct sum and the tensor product of an arbitrary finite number of representations.

When working with external direct sums and tensor products, it is useful to consider the following generalization of the concept of equivalence between representations, adapted from [26, p. 55]:

Definition 7.5. Two representations $\pi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_2)$ of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on vector spaces V_1 and V_2, respectively, are said to be *quasiequivalent* if there is a pair (ϕ_{21}, g_{21}) consisting of a Lie algebra isomorphism $\phi_{21} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ and a linear isomorphism $g_{21} : V_1 \longrightarrow V_2$ such that

$$\pi_2(\phi_{21}(X_1)) \;=\; g_{21}\,\pi_1(X_1)\,g_{21}^{-1} \quad \text{for } \; X_1 \in \mathfrak{g}_1 \; . \tag{61}$$

In this case, we may of course assume, without loss of generality, that $\mathfrak{g}_1 = \mathfrak{g} = \mathfrak{g}_2$ and $\phi \in \mathrm{Aut}(\mathfrak{g})$. Obviously, the usual notion of equivalence is recovered when $\phi \in \mathrm{Inn}(\mathfrak{g})$, which motivates the terminology.

The concept of quasiequivalence is needed to formulate an important case distinction in the calculation of normalizers for direct sums and tensor products of irreducible representations.[20]

Lemma 7.6. *Let $\pi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_2)$ be irreducible representations of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on finite-dimensional complex vector spaces V_1 and V_2, respectively, and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be their external direct sum: $\pi = \pi_1 \boxplus \pi_2$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $V = V_1 \oplus V_2$. For simplicity, assume π_1 and π_2 to be faithful (if not, replace $\mathfrak{g}_1$ by $\mathfrak{g}_1/\ker \pi_1$ and $\mathfrak{g}_2$ by $\mathfrak{g}_2/\ker \pi_2$). Then considering $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ as matrix Lie algebras (more precisely, $\mathfrak{g}_i \subset \mathfrak{g} \subset \mathfrak{gl}(V)$ via π and also $\mathfrak{g}_i \subset \mathfrak{gl}(V_i)$ via π_i), their centralizers and normalizers in the respective general linear groups are related as follows.*

1. *For the centralizers,*

$$\begin{aligned} Z_{GL(V)}(\mathfrak{g}_1) &= \mathbb{C}^\times \, \mathrm{id}_{V_1} \oplus GL(V_2) \,, \\ Z_{GL(V)}(\mathfrak{g}_2) &= GL(V_1) \oplus \mathbb{C}^\times \, \mathrm{id}_{V_2} \,, \end{aligned} \tag{62}$$

   and taking the intersection,

$$Z_{GL(V)}(\mathfrak{g}) = \mathbb{C}^\times \, \mathrm{id}_{V_1} \oplus \mathbb{C}^\times \, \mathrm{id}_{V_2} \,. \tag{63}$$

2. *For the normalizers,*

$$\begin{aligned} N_{GL(V)}(\mathfrak{g}_1) &= N_{GL(V_1)}(\mathfrak{g}_1) \oplus GL(V_2) \,, \\ N_{GL(V)}(\mathfrak{g}_2) &= GL(V_1) \oplus N_{GL(V_2)}(\mathfrak{g}_2) \,, \end{aligned} \tag{64}$$

   whereas

$$N_{GL(V)}(\mathfrak{g}) = N_{GL(V_1)}(\mathfrak{g}_1) \oplus N_{GL(V_2)}(\mathfrak{g}_2) \,, \tag{65}$$

   except when π_1 and π_2 are quasiequivalent, in which case

$$N_{GL(V)}(\mathfrak{g}) = \big(N_{GL(V_1)}(\mathfrak{g}_1) \oplus N_{GL(V_2)}(\mathfrak{g}_2) \big) : S_2 \,. \tag{66}$$

The generalization to more than two direct summands is the obvious one.

Here, for the sake of brevity, we use the following notation: given any two subgroups H_1 of $GL(V_1)$ and H_2 of $GL(V_2)$, $H_1 \oplus H_2$ will denote the subgroup of $GL(V_1 \oplus V_2)$ given by

$$H_1 \oplus H_2 = \{ h_1 \oplus h_2 \mid h_1 \in H_1 \,, \, h_2 \in H_2 \} \,.$$

[20] In what follows, $\mathbb{C}^\times$ denotes the multiplicative group of non-zero complex numbers.

Proof.[21] The proof of the inclusions "$\supset$" in equations (62)-(66) is straightforward provided we properly specify the action of S_2 involved in the definition of the semi-direct product in equation (66) which, roughly speaking, consists in switching the direct summands. More precisely, choosing the pair (ϕ_{21}, g_{21}) as in Definition 7.5 and setting $\phi_{12} = \phi_{21}^{-1}$, $g_{12} = g_{21}^{-1}$, we define an involutive automorphism ϕ on $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and a linear involution g on $V = V_1 \oplus V_2$ by setting

$$\begin{array}{rrcc} \phi: & \mathfrak{g}_1 \oplus \mathfrak{g}_2 & \longrightarrow & \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\ & (X_1, X_2) & \longmapsto & (\phi_{12}(X_2), \phi_{21}(X_1)) \\ & & & \\ g: & V_1 \oplus V_2 & \longrightarrow & V_1 \oplus V_2 \\ & (v_1, v_2) & \longmapsto & (g_{12}v_2, g_{21}v_1) \end{array} \tag{67}$$

and see that equation (61) becomes equivalent to the condition that g normalizes $\mathfrak{g}$, since

$$\begin{aligned} (\phi(X)\, g)(v_1, v_2) &= \big((\phi_{12}(X_2)\, g_{12})(v_2)\, , (\phi_{21}(X_1)\, g_{21})(v_1)\big)\ , \\ (g\, X)(v_1, v_2) &= \big((g_{12}\, X_2)(v_2)\, , (g_{21}\, X_1)(v_1)\big)\ . \end{aligned}$$

To prove the converse inclusions "$\subset$", suppose first that $g \in GL(V)$ normalizes $\mathfrak{g}$ and write ϕ for the automorphism of $\mathfrak{g}$ induced by conjugation with g:

$$gX \;=\; \phi(X)\, g \quad \text{for} \;\; X \in \mathfrak{g}\ .$$

Explicitly, this means that if we write ϕ in the form

$$\phi\big(X_1, X_2\big) \;=\; \big(\phi_{11}(X_1) + \phi_{12}(X_2)\, , \, \phi_{21}(X_1) + \phi_{22}(X_2)\big) \tag{68}$$

with $\phi_{11} \in \mathrm{End}(\mathfrak{g}_1)$, $\phi_{12} \in L(\mathfrak{g}_2, \mathfrak{g}_1)$, $\phi_{21} \in L(\mathfrak{g}_1, \mathfrak{g}_2)$, $\phi_{22} \in \mathrm{End}(\mathfrak{g}_2)$, and similarly g in the form

$$g\big(v_1, v_2\big) \;=\; \big(g_{11}v_1 + g_{12}v_2\, , \, g_{21}v_1 + g_{22}v_2\big) \tag{69}$$

with $g_{11} \in \mathrm{End}(V_1)$, $g_{12} \in L(V_2, V_1)$, $g_{21} \in L(V_1, V_2)$, $g_{22} \in \mathrm{End}(V_2)$, then for all $X = (X_1, X_2)$ in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $(v_1, v_2) \in V_1 \oplus V_2$, the expression

$$\big(gX\big)\big(v_1, v_2\big) \;=\; g\big(X_1v_1, X_2v_2\big) \;=\; \big(g_{11}X_1v_1 + g_{12}X_2v_2\, , \, g_{21}X_1v_1 + g_{22}X_2v_2\big)$$

must be equal to the expression

$$\begin{aligned} \big(\phi(X)\, g\big)\big(v_1, v_2\big) &= \phi(X)\big(g_{11}v_1 + g_{12}v_2\, , \, g_{21}v_1 + g_{22}v_2\big) \\ &= \Big(\big(\phi_{11}(X_1) + \phi_{12}(X_2)\big)\big(g_{11}v_1 + g_{12}v_2\big)\, , \\ &\qquad \big(\phi_{21}(X_1) + \phi_{22}(X_2)\big)\big(g_{21}v_1 + g_{22}v_2\big)\Big) \end{aligned}$$

leading us to the following system of equations, in which the ones in the first two lines are obtained by leaving v_1 arbitrary and setting $v_2 = 0$, with X_1 arbitrary and $X_2 = 0$ (left column) and with $X_1 = 0$ and X_2 arbitrary (right column),

[21] The explicit proof given here serves as a kind of "warm-up exercise" for the proof of the next lemma.

while the ones in the last two lines are obtained by setting $v_1 = 0$ and leaving v_2 arbitrary, with $X_1 = 0$ and X_2 arbitrary (left column) and with X_1 arbitrary and $X_2 = 0$ (right column):

$$
\begin{array}{rclcrcl}
g_{11}X_1 & = & \phi_{11}(X_1)\, g_{11} & , & \phi_{12}(X_2)\, g_{11} & = & 0 \,, \\
g_{21}X_1 & = & \phi_{21}(X_1)\, g_{21} & , & \phi_{22}(X_2)\, g_{21} & = & 0 \,, \\
g_{12}X_2 & = & \phi_{12}(X_2)\, g_{12} & , & \phi_{11}(X_1)\, g_{12} & = & 0 \,, \\
g_{22}X_2 & = & \phi_{22}(X_2)\, g_{22} & , & \phi_{21}(X_1)\, g_{22} & = & 0 \,.
\end{array}
\tag{70}
$$

Now assume that $g \in GL(V)$ normalizes $\mathfrak{g}_1$: then these calculations can be applied if we take $X_2 = 0$ with $\phi_{11} \in \mathrm{Aut}(\mathfrak{g}_1)$, $\phi_{21} \equiv 0$ and ϕ_{12}, ϕ_{22} undetermined. Obviously, in this situation, the first equation in the first line states that $g_{11} \in GL(V_1)$ normalizes $\mathfrak{g}_1$, the first equation in the second line implies $g_{21} = 0$, the second equation in the third line implies $g_{12} = 0$ since π_1 is irreducible, and all other equations are trivial: this proves the inclusion "$\subset$" in the first line of equation (64) and, as an immediate corollary, that in the first line of equation (62) (the second one being completely analogous).

Finally, if $g \in GL(V)$ normalizes $\mathfrak{g}$, as supposed above, we can use the above system of equations to argue as follows. If g_{11} (g_{22}) is invertible, then according to the second equation in the first (fourth) line, $\phi_{12} \equiv 0$ ($\phi_{21} \equiv 0$), and according to the first equation in the third (second) line, $g_{12} = 0$ ($g_{21} = 0$), since π_2 (π_1) is irreducible. But g being invertible, this forces g_{22} (g_{11}) to be invertible as well, so repeating the other part of the argument, we conclude that g and hence ϕ are block diagonal. If on the other hand both g_{11} and g_{22} are not invertible, then $\ker g_{11}$ is a non-trivial $\mathfrak{g}_1$-invariant subspace of V_1 and $\ker g_{22}$ is a non-trivial $\mathfrak{g}_2$-invariant subspace of V_2, implying $g_{11} = 0$ and $g_{22} = 0$, since π_1 and π_2 are irreducible. But g being invertible, this forces g_{12} and g_{21} to be invertible and, up to a constant multiple, to be each other's inverse, due to Schur's lemma, so we conclude that g and hence ϕ are block off-diagonal and π_1 and π_2 are quasiequivalent. ■

Similarly, we have

Lemma 7.7. *Let $\pi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_2)$ be irreducible representations of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on finite-dimensional complex vector spaces V_1 and V_2, respectively, and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be their external tensor product: $\pi = \pi_1 \boxtimes \pi_2$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, $V = V_1 \otimes V_2$. For simplicity, assume π_1 and π_2 to be faithful (if not, replace $\mathfrak{g}_1$ by $\mathfrak{g}_1/\ker \pi_1$ and $\mathfrak{g}_2$ by $\mathfrak{g}_2/\ker \pi_2$) and traceless (this is automatic if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple or, more generally, perfect). Then considering $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ as Lie algebras of traceless matrices (more precisely, $\mathfrak{g}_i \subset \mathfrak{g} \subset \mathfrak{sl}(V)$ via π and also $\mathfrak{g}_i \subset \mathfrak{sl}(V_i)$ via π_i), their centralizers and normalizers in the respective general linear groups are related as follows.*

1. *For the centralizers,*

$$
\begin{array}{rcl}
Z_{GL(V)}(\mathfrak{g}_1) & = & \{\mathrm{id}_{V_1}\} \otimes GL(V_2) \,, \\
Z_{GL(V)}(\mathfrak{g}_2) & = & GL(V_1) \otimes \{\mathrm{id}_{V_2}\} \,,
\end{array}
\tag{71}
$$

and taking the intersection,

$$Z_{GL(V)}(\mathfrak{g}) \;=\; \mathbb{C}^\times \, \mathrm{id}_V \; . \tag{72}$$

2. For the normalizers,

$$\begin{aligned} N_{GL(V)}(\mathfrak{g}_1) &= N_{GL(V_1)}(\mathfrak{g}_1) \otimes GL(V_2) \; , \\ N_{GL(V)}(\mathfrak{g}_2) &= GL(V_1) \otimes N_{GL(V_2)}(\mathfrak{g}_2) \; , \end{aligned} \tag{73}$$

and when both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simple,

$$N_{GL(V)}(\mathfrak{g}) \;=\; N_{GL(V_1)}(\mathfrak{g}_1) \otimes N_{GL(V_2)}(\mathfrak{g}_2) \; , \tag{74}$$

except when π_1 and π_2 are quasiequivalent, in which case

$$N_{GL(V)}(\mathfrak{g}) \;=\; \big(N_{GL(V_1)}(\mathfrak{g}_1) \otimes N_{GL(V_2)}(\mathfrak{g}_2) \big) : S_2 \; . \tag{75}$$

The generalization to more than two tensor factors is the obvious one.

Here, for the sake of brevity, we use the following notation: given any two subgroups H_1 of $GL(V_1)$ and H_2 of $GL(V_2)$, $H_1 \otimes H_2$ will denote the subgroup of $GL(V_1 \otimes V_2)$ given by

$$H_1 \otimes H_2 \;=\; \{ \, h_1 \otimes h_2 \mid h_1 \in H_1 \, , \, h_2 \in H_2 \, \} \; .$$

Remark 7.8. In the formulation of Lemma 7.7, we have tacitly incorporated the statement that if π_1 and π_2 are faithful and traceless, then so is π. To show this, we make use of the notion of partial traces of endomorphisms in tensor products. Given two finite-dimensional complex vector spaces V_1 and V_2, we can use the canonical isomorphism

$$\mathrm{End}(V_1 \otimes V_2) \;\cong\; \mathrm{End}(V_1) \otimes \mathrm{End}(V_2)$$

to argue that the bilinear maps

$$\begin{array}{ccc} \mathrm{End}(V_1) \times \mathrm{End}(V_2) & \longrightarrow & \mathrm{End}(V_1) \\ (A_1, A_2) & \longmapsto & \mathrm{tr}(A_2) \, A_1 \end{array}$$

and

$$\begin{array}{ccc} \mathrm{End}(V_1) \times \mathrm{End}(V_2) & \longrightarrow & \mathrm{End}(V_2) \\ (A_1, A_2) & \longmapsto & \mathrm{tr}(A_1) \, A_2 \end{array}$$

extend uniquely to linear maps

$$\mathrm{tr}_2 : \mathrm{End}(V_1 \otimes V_2) \;\longrightarrow\; \mathrm{End}(V_1)$$

and

$$\mathrm{tr}_1 : \mathrm{End}(V_1 \otimes V_2) \;\longrightarrow\; \mathrm{End}(V_2)$$

known as *partial traces* and characterized by the property that

$$\mathrm{tr}_2(A_1 \otimes A_2) \;=\; \mathrm{tr}_{V_2}(A_2) \, A_1 \; , \tag{76}$$

and

$$\mathrm{tr}_1(A_1 \otimes A_2) \;=\; \mathrm{tr}_{V_1}(A_1)\, A_2 \;, \tag{77}$$

respectively. Obviously, the total trace is obtained by composing any one of these partial traces with the ordinary trace on the remaining tensor factor:

$$\mathrm{tr}_{V_1}(\mathrm{tr}_2(A)) \;=\; \mathrm{tr}_{V_1 \otimes V_2}(A) \;=\; \mathrm{tr}_{V_2}(\mathrm{tr}_1(A)) \;. \tag{78}$$

In particular,

$$\mathrm{tr}_{V_1 \otimes V_2}(A_1 \otimes A_2) \;=\; \mathrm{tr}_{V_1}(A_1)\; \mathrm{tr}_{V_2}(A_2) \;. \tag{79}$$

As a result, we see that A_1 and A_2 can be recovered from $A_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes A_2$, provided both A_1 and A_2 are traceless, namely by the formula

$$\begin{aligned} A_1 \;&=\; \frac{1}{\dim V_2}\; \mathrm{tr}_2\big(A_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes A_2\big) \qquad \text{if} \quad \mathrm{tr}_{V_2}(A_2) = 0 \;, \\ A_2 \;&=\; \frac{1}{\dim V_1}\; \mathrm{tr}_1\big(A_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes A_2\big) \qquad \text{if} \quad \mathrm{tr}_{V_1}(A_1) = 0 \;, \end{aligned} \tag{80}$$

which implies that if π_1 and π_2 are faithful and traceless, then so is π. For later use, we also note the following formulas, valid for any $B \in \mathrm{End}(V_1 \otimes V_2)$ (they are obvious when B is tensor decomposable, $B = B_1 \otimes B_2$, and hence hold in general since both sides of each equation are linear in B):

$$\begin{aligned} \mathrm{tr}_2\big((A_1 \otimes \mathrm{id}_{V_2})\, B\big) \;&=\; A_1\; \mathrm{tr}_2(B) \quad , \quad \mathrm{tr}_2\big(B\, (A_1 \otimes \mathrm{id}_{V_2})\big) \;=\; \mathrm{tr}_2(B)\, A_1 \;. \\ \mathrm{tr}_1\big((\, \mathrm{id}_{V_1} \otimes A_2)\, B\big) \;&=\; A_2\; \mathrm{tr}_1(B) \quad , \quad \mathrm{tr}_1\big(B\, (\, \mathrm{id}_{V_1} \otimes A_2)\big) \;=\; \mathrm{tr}_1(B)\, A_2 \;. \end{aligned} \tag{81}$$

Remark 7.9. Another useful tool that plays an important role in the proof of Lemma 7.7 are the injections and projections that relate the tensor product space $V = V_1 \otimes V_2$ to its factors V_1 and V_2. Namely, fixed vectors v_2 in V_2 and v_1 in V_1 induce inclusions

$$\begin{array}{rccc} \mathrm{i}_{v_2} : & V_1 & \longrightarrow & V_1 \otimes V_2 \\ & v_1 & \longmapsto & v_1 \otimes v_2 \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \mathrm{i}_{v_1} : & V_2 & \longrightarrow & V_1 \otimes V_2 \\ & v_2 & \longmapsto & v_1 \otimes v_2 \end{array}$$

whereas fixed linear forms $v_2^* \in V_2^*$ on V_2 and $v_1^* \in V_1^*$ on V_1 induce projections

$$\begin{array}{rccc} \mathrm{pr}_{v_2^*} : & V_1 \otimes V_2 & \longrightarrow & V_1 \\ & v_1 \otimes v_2 & \longmapsto & \langle v_2^*, v_2 \rangle\, v_1 \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \mathrm{pr}_{v_1^*} : & V_1 \otimes V_2 & \longrightarrow & V_2 \\ & v_1 \otimes v_2 & \longmapsto & \langle v_1^*, v_1 \rangle\, v_2 \end{array}$$

which are equivariant under the respective actions of $\mathfrak{g}_1$ on V_1 and V and of $\mathfrak{g}_2$ on V_2 and V. This implies[22] that the $\mathfrak{g}_1$-irreducible subspaces of V are precisely the subspaces of the form $\mathrm{im}\, \mathrm{i}_{v_2} = V_1 \otimes v_2$ with $v_2 \in V_2 \setminus \{0\}$, where

$$\begin{aligned} V_1 \otimes v_2 \;&=\; V_1 \otimes v_2' && \text{if } v_2 \text{ and } v_2' \text{ are linearly dependent} \\ (V_1 \otimes v_2) \cap (V_1 \otimes v_2') \;&=\; \{0\} && \text{if } v_2 \text{ and } v_2' \text{ are linearly independent} \end{aligned}$$

[22]Note that the space $V = V_1 \otimes V_2$ is irreducible under $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ but of course not under $\mathfrak{g}_1$ or $\mathfrak{g}_2$; rather, it is the direct sum of $\dim V_2$ copies of irreducible representations of $\mathfrak{g}_1$, all equivalent to π_1, and also the direct sum of $\dim V_1$ copies of irreducible representations of $\mathfrak{g}_2$, all equivalent to π_2, each of the corresponding decompositions into irreducible subspaces being highly non-unique.

and similarly that the $\mathfrak{g}_2$-irreducible subspaces of V are precisely the subspaces of the form $\operatorname{im} \mathrm{i}_{v_1} = v_1 \otimes V_2$ with $v_1 \in V_1 \setminus \{0\}$, where

$$
\begin{aligned}
v_1 \otimes V_2 &= v_1' \otimes V_2 && \text{if } v_1 \text{ and } v_1' \text{ are linearly dependent} \\
(v_1 \otimes V_2) \cap (v_1' \otimes V_2) &= \{0\} && \text{if } v_1 \text{ and } v_1' \text{ are linearly independent}
\end{aligned}
$$

Of course, the only non-trivial statement here is that *all* irreducible subspaces are of this form; here is a proof for the first case (the second one being completely analogous): Let W be a $\mathfrak{g}_1$-irreducible subspace of V, $W \neq \{0\}$. For any $v_2^* \in V_2^*$, the projection $\mathrm{pr}_{v_2^*}$, being $\mathfrak{g}_1$-equivariant, maps the $\mathfrak{g}_1$-invariant subspace W of V to a $\mathfrak{g}_1$-invariant subspace of V_1, and V_1 being $\mathfrak{g}_1$-irreducible, this subspace can only be $\{0\}$ or all of V_1. Similarly, the kernel of $\mathrm{pr}_{v_2^*}$ intersects W in a $\mathfrak{g}_1$-invariant subspace of W, and W being $\mathfrak{g}_1$-irreducible, this subspace can only be $\{0\}$ or all of W. This means that the restriction $\mathrm{pr}_{v_2^*}|_W$ of $\mathrm{pr}_{v_2^*}$ to W is either 0 or an isomorphism. It cannot be zero for all $v_2^* \in V_2^*$ because there is *no* non-zero vector in V that is annihilated by all the projections $\mathrm{pr}_{v_2^*}$, $v_2^* \in V_2^*$ (this is easy to see by expanding in a basis). Thus we have shown existence of a $\mathfrak{g}_1$-equivariant isomorphism from W to V_1; let us, for the time being, call it ϕ. But then, for any $v_2^* \in V_2^*$, $\mathrm{pr}_{v_2^*} \circ \phi^{-1} \in \mathrm{End}(V_1)$ centralizes $\mathfrak{g}_1$ and, V_1 being $\mathfrak{g}_1$-irreducible, must, according to Schur's lemma, be a multiple of the identity: $\mathrm{pr}_{v_2^*} \circ \phi^{-1} = \lambda(v_2^*)\, \mathrm{id}_{V_1}$. Clearly, $\lambda(v_2^*)$ is linear in v_2^* because so is $\mathrm{pr}_{v_2^*} \circ \phi^{-1}$, so there is a unique vector $v_2 \in V_2$ such that $\lambda(v_2^*) = \langle v_2^*, v_2 \rangle$ and hence $\mathrm{pr}_{v_2^*} \circ \phi^{-1} = \langle v_2^*, v_2 \rangle\, \mathrm{id}_{V_1} = \mathrm{pr}_{v_2^*} \circ \mathrm{i}_{v_2}$. Since this equation holds for all $v_2^* \in V_2^*$, it follows that $\phi^{-1} = \mathrm{i}_{v_2}$, once again because there is *no* non-zero vector in V that is annihilated by all the projections $\mathrm{pr}_{v_2^*}$, $v_2^* \in V_2^*$, q.e.d..

Proof. [23] As in the previous lemma, the proof of the inclusions "$\supset$" in equations (71)-(75) is straightforward provided we properly specify the action of S_2 involved in the definition of the semidirect product in equation (75) which, roughly speaking, consists in switching the tensor factors. More precisely, choosing the pair (ϕ_{21}, g_{21}) as in Definition 7.5 and setting $\phi_{12} = \phi_{21}^{-1}$, $g_{12} = g_{21}^{-1}$, we define an involutive automorphism ϕ on $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ and a linear involution g on $V = V_1 \otimes V_2$ by setting

$$
\begin{array}{rccc}
\phi: & \mathfrak{g}_1 \times \mathfrak{g}_2 & \longrightarrow & \mathfrak{g}_1 \times \mathfrak{g}_2 \\
& (X_1, X_2) & \longmapsto & (\phi_{12}(X_2), \phi_{21}(X_1)) \\
g: & V_1 \otimes V_2 & \longrightarrow & V_1 \otimes V_2 \\
& v_1 \otimes v_2 & \longmapsto & g_{12} v_2 \otimes g_{21} v_1
\end{array}
\tag{82}
$$

and see that equation (61) becomes equivalent to the condition that g normalizes $\mathfrak{g}$, since

$$
\begin{aligned}
(\phi(X)\, g)(v_1 \otimes v_2) &= (\phi_{12}(X_2)\, g_{12})(v_2) \otimes g_{21}(v_1) + g_{12}(v_2) \otimes (\phi_{21}(X_1)\, g_{21})(v_1) , \\
(g\, X)(v_1 \otimes v_2) &= g_{12}(v_2) \otimes (g_{21}\, X_1)(v_1) + (g_{12}\, X_2)(v_2) \otimes g_{21}(v_1) .
\end{aligned}
$$

[23] A rigorous proof of this lemma does not seem to be readily available in the literature and will therefore be given here.

To prove the converse inclusions "$\subset$", suppose first that $g \in GL(V)$ normalizes $\mathfrak{g}$ and write ϕ for the automorphism of $\mathfrak{g}$ induced by conjugation with g:

$$g X g^{-1} ~=~ \phi(X) \quad \text{for} \quad X \in \mathfrak{g} ~.$$

Explicitly, this means that if we write ϕ in the form

$$\phi\big(X_1, X_2\big) ~=~ \big(\phi_{11}(X_1) + \phi_{12}(X_2) \,,\, \phi_{21}(X_1) + \phi_{22}(X_2)\big) \tag{83}$$

with $\phi_{11} \in \mathrm{End}(\mathfrak{g}_1)$, $\phi_{12} \in L(\mathfrak{g}_2, \mathfrak{g}_1)$, $\phi_{21} \in L(\mathfrak{g}_1, \mathfrak{g}_2)$, $\phi_{22} \in \mathrm{End}(\mathfrak{g}_2)$, as before, then for all $X = (X_1, X_2)$ in $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, we have

$$\begin{aligned} & g \left(X_1 \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes X_2\right) g^{-1} \\ & \quad = ~ \big(\phi_{11}(X_1) + \phi_{12}(X_2)\big) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \big(\phi_{21}(X_1) + \phi_{22}(X_2)\big) ~, \end{aligned}$$

leading us, according to equation (80), to the following system of equations:

$$\begin{aligned} \phi_{11}(X_1) ~&=~ \frac{1}{\dim V_2} ~ \mathrm{tr}_2\big(\, g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1}\big) ~, \\ \phi_{21}(X_1) ~&=~ \frac{1}{\dim V_1} ~ \mathrm{tr}_1\big(\, g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1}\big) ~, \\ \phi_{12}(X_2) ~&=~ \frac{1}{\dim V_2} ~ \mathrm{tr}_2\big(\, g(\, \mathrm{id}_{V_1} \otimes X_2)\, g^{-1}\big) ~, \\ \phi_{22}(X_2) ~&=~ \frac{1}{\dim V_1} ~ \mathrm{tr}_1\big(\, g(\, \mathrm{id}_{V_1} \otimes X_2)\, g^{-1}\big) ~. \end{aligned} \tag{84}$$

Moreover, it is a standard fact (which we have already used extensively) that automorphisms of semisimple Lie algebras permute their simple ideals, without mixing them, so under the additional hypotheses of equations (74) and (75) ($\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ simple), there are precisely two distinct possibilities: either (a) ϕ preserves both $\mathfrak{g}_1$ and $\mathfrak{g}_2$, i.e., $\phi_{11} \in \mathrm{Aut}(\mathfrak{g}_1)$ and $\phi_{22} \in \mathrm{Aut}(\mathfrak{g}_2)$, whereas $\phi_{21} \equiv 0$ and $\phi_{12} \equiv 0$, or (b) ϕ switches $\mathfrak{g}_1$ and $\mathfrak{g}_2$, i.e., $\phi_{11} \equiv 0$ and $\phi_{22} \equiv 0$, whereas $\phi_{21} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ and $\phi_{12} : \mathfrak{g}_2 \longrightarrow \mathfrak{g}_1$ are Lie algebra isomorphisms.

Now assume that $g \in GL(V)$ normalizes $\mathfrak{g}_1$ (but without imposing the additional hypothesis that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ should be simple): then the above calculations can be applied if we take $X_2 = 0$ with $\phi_{11} \in \mathrm{Aut}(\mathfrak{g}_1)$, $\phi_{21} \equiv 0$ and ϕ_{12}, ϕ_{22} undetermined. Here, the main issue is to show that g must be tensor decomposable. To this end, observe first of all that for each $X_1 \in \mathfrak{g}_1$, there exists $X_1' \in \mathfrak{g}_1$ such that $g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1} = X_1' \otimes \mathrm{id}_{V_2}$; then applying tr_2 to this equation and using equation (84), we get $X_1' = \phi_{11}(X_1)$, i.e.,

$$g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1} ~=~ \phi_{11}(X_1) \otimes \mathrm{id}_{V_2} ~.$$

Next, for arbitrary vectors $v_2 \in V_2$ and covectors (linear forms) $v_2^* \in V_2^*$, we use the inclusions and projections introduced in Remark 7.9 to define

$$\mathrm{pr}_{(v_2^*, v_2)}(g) ~=~ \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_2} ~\in~ \mathrm{End}(V_1) ~.$$

Then

$$\begin{aligned}
\mathrm{pr}_{(v_2^*,v_2)}(g)\,X_1 \;&=\; \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_2} \circ X_1 \;=\; \mathrm{pr}_{v_2^*} \circ g(X_1 \otimes \mathrm{id}_{V_2}) \circ \mathrm{i}_{v_2}~, \\
&=\; \mathrm{pr}_{v_2^*} \circ (\phi_{11}(X_1) \otimes \mathrm{id}_{V_2})\, g \circ \mathrm{i}_{v_2} \;=\; \phi_{11}(X_1) \circ \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_2} \\
&=\; \phi_{11}(X_1)\; \mathrm{pr}_{(v_2^*,v_2)}(g)~.
\end{aligned}$$

This implies that both the kernel and the image of each $\mathrm{pr}_{(v_2^*,v_2)}(g)$ are $\mathfrak{g}_1$-invariant subspaces of V_1, so irreducibility forces all of these subspaces to be either $\{0\}$ or the entire space V_1, which means that each $\mathrm{pr}_{(v_2^*,v_2)}(g)$ is either 0 or invertible. They cannot all be zero because there is *no* non-zero linear operator in V that is annihilated by all these projections (this is easy to see by expanding in a basis). Thus we have shown existence of an invertible linear transformation on V_1, which we shall denote by g_{11}, which is equivariant in the sense that

$$g_{11} X_1\, g_{11}^{-1} \;=\; \phi_{11}(X_1) \quad \text{for } X_1 \in \mathfrak{g}_1~,$$

i.e., $g_{11} \in GL(V_1)$ normalizes $\mathfrak{g}_1$ so that conjugation with g_{11} induces ϕ_{11}. But then, for any $(v_2^*, v_2) \in V_2^* \times V_2$, $g_{11}^{-1} \circ \mathrm{pr}_{(v_2^*,v_2)}(g) \in \mathrm{End}(V_1)$ centralizes $\mathfrak{g}_1$ and, V_1 being $\mathfrak{g}_1$-irreducible, must, according to Schur's lemma, be a multiple of the identity:

$$g_{11}^{-1} \circ \mathrm{pr}_{(v_2^*,v_2)}(g) \;=\; \lambda(v_2^*, v_2)\, \mathrm{id}_{V_1}~.$$

Clearly, λ is bilinear because $g_{11}^{-1} \circ \mathrm{pr}_{(v_2^*,v_2)}(g)$ is bilinear in (v_2^*, v_2), so there is a unique linear transformation $g_{22} \in \mathrm{End}(V_2)$ such that $\lambda(v_2^*, v_2) = \langle v_2^*, g_{22} v_2 \rangle$, implying

$$\mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_2} \;=\; \mathrm{pr}_{(v_2^*,v_2)}(g) \;=\; \langle v_2^*, g_{22} v_2 \rangle\, g_{11} \;=\; \mathrm{pr}_{v_2^*} \circ (g_{11} \otimes g_{22}) \circ \mathrm{i}_{v_2}~.$$

Since v_2^* and v_2 are arbitrary, this implies $g = g_{11} \otimes g_{22}$, with $g_{11} \in N_{GL(V_1)}(\mathfrak{g}_1)$ and $g_{22} \in GL(V_2)$ arbitrary: this proves the inclusion "$\subset$" in the first line of equation (73) and, as an immediate corollary, that in the first line of equation (71) (the second one being completely analogous).

Finally, we return to the case where $g \in GL(V)$ is supposed to normalize $\mathfrak{g}$ and both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are supposed to be simple. If (a) ϕ preserves both $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we can immediately apply the previous part of the proof to conclude that

$$g \;\in\; N_{GL(V)}(\mathfrak{g}_1) \cap N_{GL(V)}(\mathfrak{g}_2) \;=\; N_{GL(V_1)}(\mathfrak{g}_1) \otimes N_{GL(V_2)}(\mathfrak{g}_2)~.$$

Thus assume that (b) ϕ switches $\mathfrak{g}_1$ and $\mathfrak{g}_2$, i.e., $\phi_{11} \equiv 0$ and $\phi_{22} \equiv 0$, whereas $\phi_{21} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ and $\phi_{12} : \mathfrak{g}_2 \longrightarrow \mathfrak{g}_1$ are Lie algebra isomorphisms. Then we can argue in a similar manner as before, observing first of all that for each $X_1 \in \mathfrak{g}_1$ and each $X_2 \in \mathfrak{g}_2$, there exist $X_1' \in \mathfrak{g}_2$ and $X_2' \in \mathfrak{g}_1$ such that $g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1} = \mathrm{id}_{V_1} \otimes X_1'$ and $g(\mathrm{id}_{V_1} \otimes X_2)\, g^{-1} = X_2' \otimes \mathrm{id}_{V_2}$; then applying tr_1 to the first equation and tr_2 to the second equation and using equation (84), we get $X_1' = \phi_{21}(X_1)$ and $X_2' = \phi_{12}(X_2)$, i.e.,

$$\begin{aligned}
g(X_1 \otimes \mathrm{id}_{V_2})\, g^{-1} \;&=\; \mathrm{id}_{V_1} \otimes \phi_{21}(X_1)~, \\
g(\mathrm{id}_{V_1} \otimes X_2)\, g^{-1} \;&=\; \phi_{12}(X_2) \otimes \mathrm{id}_{V_2}~.
\end{aligned}$$

Next, for arbitrary vectors $v_1 \in V_1$, $v_2 \in V_2$ and covectors (linear forms) $v_1^* \in V_1^*$, $v_2^* \in V_2^*$, we use the inclusions and projections introduced in Remark 7.9 to define

$$\begin{aligned} \mathrm{pr}_{(v_1^*,v_2)}(g) &= \mathrm{pr}_{v_1^*} \circ g \circ \mathrm{i}_{v_2} \in L(V_1,V_2) \;, \\ \mathrm{pr}_{(v_2^*,v_1)}(g) &= \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_1} \in L(V_2,V_1) \;. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{pr}_{(v_1^*,v_2)}(g) \circ X_1 &= \mathrm{pr}_{v_1^*} \circ g \circ \mathrm{i}_{v_2} \circ X_1 \;=\; \mathrm{pr}_{v_1^*} \circ g(X_1 \otimes \mathrm{id}_{V_2}) \circ \mathrm{i}_{v_2} \;, \\ &= \mathrm{pr}_{v_1^*} \circ (\,\mathrm{id}_{V_1} \otimes \phi_{21}(X_1))\, g \circ \mathrm{i}_{v_2} \;=\; \phi_{21}(X_1) \circ \mathrm{pr}_{v_1^*} \circ g \circ \mathrm{i}_{v_2} \\ &= \phi_{21}(X_1) \circ \mathrm{pr}_{(v_1^*,v_2)}(g) \;, \end{aligned}$$

$$\begin{aligned} \mathrm{pr}_{(v_2^*,v_1)}(g) \circ X_2 &= \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_1} \circ X_2 \;=\; \mathrm{pr}_{v_2^*} \circ g(\,\mathrm{id}_{V_1} \otimes X_2) \circ \mathrm{i}_{v_1} \;, \\ &= \mathrm{pr}_{v_2^*} \circ (\phi_{12}(X_2) \otimes \mathrm{id}_{V_2})\, g \circ \mathrm{i}_{v_1} \;=\; \phi_{12}(X_2) \circ \mathrm{pr}_{v_2^*} \circ g \circ \mathrm{i}_{v_1} \\ &= \phi_{12}(X_2) \circ \mathrm{pr}_{(v_2^*,v_1)}(g) \;. \end{aligned}$$

This implies that the kernel of each $\mathrm{pr}_{(v_1^*,v_2)}(g)$ and the image of each $\mathrm{pr}_{(v_2^*,v_1)}(g)$ is a $\mathfrak{g}_1$-invariant subspace of V_1 and similarly the kernel of each $\mathrm{pr}_{(v_2^*,v_1)}(g)$ and the image of each $\mathrm{pr}_{(v_1^*,v_2)}(g)$ is a $\mathfrak{g}_2$-invariant subspace of V_2, so irreducibility forces all of these subspaces to be either $\{0\}$ or the entire space, which means that each $\mathrm{pr}_{(v_1^*,v_2)}(g)$ and each $\mathrm{pr}_{(v_2^*,v_1)}(g)$ is either 0 or a linear isomorphism. They cannot all be zero because there is *no* non-zero linear operator in V that is annihilated by all these projections (this is easy to see by expanding in a basis). Thus we have shown existence of linear isomorphisms from V_1 to V_2 and from V_2 to V_1 which we shall denote by g_{21} and by g_{12}, respectively, and which are equivariant in the sense that

$$\begin{aligned} \phi_{21}(X_1) &= g_{21} \circ X_1 \circ g_{21}^{-1} \;, \\ \phi_{12}(X_2) &= g_{12} \circ X_2 \circ g_{12}^{-1} \;. \end{aligned}$$

Moreover, using Schur's lemma, we see that the product of g_{21} and g_{12} must be a multiple of the identity, so multiplying one of them with an appropriate scalar factor, we may assume without loss of generality that they are each other's inverse. Thus we conclude not only that the representations π_1 of $\mathfrak{g}_1$ and π_2 of $\mathfrak{g}_2$ are quasiequivalent but also that any invertible linear transformation g on V that normalizes $\mathfrak{g}$ and switches $\mathfrak{g}_1$ and $\mathfrak{g}_2$ provides a realization of this quasiequivalence, in the sense of Definition 7.5. Finally, the product of any such invertible linear transformation on V with the inverse of any other one is of course an invertible linear transformation on V that normalizes both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and hence, by what has been proved before, belongs to $N_{GL(V_1)}(\mathfrak{g}_1) \otimes N_{GL(V_2)}(\mathfrak{g}_2)$. ■

Remark 7.10. To see that equation (74) may fail to be true if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not simple (even when they are semisimple), consider the following situation: take $\mathfrak{g}_1 = \mathfrak{g}_0 \times \tilde{\mathfrak{g}}_1$, $\mathfrak{g}_2 = \mathfrak{g}_0 \times \tilde{\mathfrak{g}}_2$ and $\pi_1 = \pi_0 \boxtimes \tilde{\pi}_1$ on $V_1 = V_0 \otimes \tilde{V}_1$, $\pi_2 = \pi_0 \boxtimes \tilde{\pi}_2$ on $V_2 = V_0 \otimes \tilde{V}_2$, where $\mathfrak{g}_0$, $\tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{g}}_2$ are mutually non-isomorphic simple Lie algebras and π_0, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are irreducible representations of $\mathfrak{g}_0$, $\tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{g}}_2$ on the vector spaces V_0, $\tilde{V}_1$ and $\tilde{V}_2$, respectively. Then obviously, $N_{GL(V)}(\mathfrak{g})$ contains a linear involution g which switches the two copies of V_0 but preserves $\tilde{V}_1$ and $\tilde{V}_2$, so it neither preserves nor switches V_1 and V_2.

The same methods allow us to prove the following fact about invariant bilinear or sesquilinear forms in tensor products, which will be used later on:

Lemma 7.11. *Let $\pi_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V_2)$ be irreducible representations of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on finite-dimensional complex vector spaces V_1 and V_2, respectively, and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be their external tensor product: $\pi = \pi_1 \boxtimes \pi_2$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, $V = V_1 \otimes V_2$. Then every invariant bilinear or sesquilinear form $(.\,,.)$ on V is the tensor product of invariant bilinear or sesquilinear forms $(.\,,.)_1$ on V_1 and $(.\,,.)_2$ on V_2:*

$$(v_1 \otimes v_2 \,, v_1' \otimes v_2') \;=\; (v_1, v_1')_1 \, (v_2, v_2')_2 \;. \tag{85}$$

The generalization to more than two tensor factors is the obvious one.

Proof. Given an invariant bilinear or sesquilinear form $(.\,,.)$ on V, let us define, for any two vectors $v_2, v_2' \in V_2$, an invariant bilinear or sesquilinear form $(.\,,.)_{(v_2,v_2')}$ on V_1 by

$$(v_1, v_1')_{(v_2,v_2')} \;=\; (v_1 \otimes v_2 \,, v_1' \otimes v_2') \;.$$

That this form is really invariant is easy to check, since for $X_1 \in \mathfrak{g}_1$,

$$\begin{aligned} &\big(X_1 v_1, v_1'\big)_{(v_2,v_2')} + \big(v_1, X_1 v_1'\big)_{(v_2,v_2')} \\ &\quad = \; \big((X_1 \otimes \mathrm{id}_{V_2})(v_1 \otimes v_2) \,, v_1' \otimes v_2'\big) + \big(v_1 \otimes v_2 \,, (X_1 \otimes \mathrm{id}_{V_2})(v_1' \otimes v_2')\big) \;. \end{aligned}$$

Therefore, its kernel (formed by the vectors orthogonal to all vectors in V_1) is a $\mathfrak{g}_1$-invariant subspace of V_1, so irreducibility forces it to be either $\{0\}$ or the entire space V_1, which means that each of the forms $(.\,,.)_{(v_2,v_2')}$ is either 0 or non-degenerate. They cannot all be zero, except when the original form $(.\,,.)$ on V is zero, in which case the claim is trivial. Otherwise, we obtain existence of a non-degenerate invariant bilinear or sesquilinear form $(.\,,.)_1$ on V_1. However, due to irreducibility, any two such forms are proportional, so we conclude that, for any $v_2, v_2' \in V_2$, $(.\,,.)_{(v_2,v_2')} = \lambda(v_2, v_2')(.\,,.)_1$. Clearly, λ is bilinear or sequilinear because $(.\,,.)_{(v_2,v_2')}$ is bilinear or sesquilinear in (v_2, v_2'), so we are done. ∎

With these concepts at our disposal, assume now that $\mathfrak{s}$ is a reducible subalgebra of one of the classical Lie algebras $\mathfrak{g}$; note that we do not exclude the case that $\mathfrak{s}$ is simple. Viewing the inclusion of $\mathfrak{s}$ into $\mathfrak{g}$ as a (faithful) representation π of $\mathfrak{s}$ on $V \cong \mathbb{C}^n$ and using the fact that representations of reductive Lie algebras are completely reducible, we may decompose π into its irreducible constituents, which are certain irreducible representations π_i of $\mathfrak{s}$ on $V_i \cong \mathbb{C}^{n_i}$ $(1 \leqslant i \leqslant r)$, as follows:

$$V \;=\; \bigoplus_{i=1}^{r} V_i \otimes W_i \qquad , \qquad \pi(X) \;=\; \bigoplus_{i=1}^{r} \pi_i(X) \otimes \mathrm{id}_{W_i} \quad \text{for } X \in \mathfrak{s} \;. \tag{86}$$

Here, the $W_i \cong \mathbb{C}^{m_i}$ $(1 \leqslant i \leqslant r)$ are auxiliary spaces on which $\mathfrak{s}$ acts trivially: for each i, m_i is the *multiplicity* with which π_i appears in π, while the subspace

$V_i \otimes W_i$ of V is known as the corresponding *isotypic component*. The main advantage of this decomposition into isotypic components is that it is unique, whereas the finer one into irreducible subspaces is not. Another useful fact is that every irreducible subspace of V is contained in precisely one isotypic component. Finally, the representation π is said to be *multiplicity free* if all m_i are equal to 1: in this case, the isotypic components become identical with the irreducible subspaces, and these become unique.

Lemma 7.12. *Let $\mathfrak{s}$ be a Σ-primitive reducible subalgebra of one of the classical Lie algebras $\mathfrak{g}$, where Σ is any subgroup of* $\mathrm{Out}(\mathfrak{g})$. *Then the repesentation of $\mathfrak{s}$ defined by its inclusion in $\mathfrak{g}$ is multiplicity free.*

Proof. This is a simple consequence of the fact that, in general, the isotypic decomposition (86) implies that linear transformations $A \in \mathrm{End}(V)$ of the form

$$A \;=\; \bigoplus_{i=1}^{r} \mathrm{id}_{V_i} \otimes A_i$$

with $A_i \in \mathrm{End}(W_i)$ commute with all $\pi(X)$, $X \in \mathfrak{s}$, and hence if $m_i > 1$ for some i, the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$ contains elements that do no belong to $\mathfrak{s}$, so $\mathfrak{s}$ cannot be self-normalizing. ■

As a result, it becomes clear that for complex representations, that is, when $\mathfrak{g} = \mathfrak{su}(n)$, and with an appropriate choice of basis, $\mathfrak{s}$ is contained in the *complex flag algebra*

$$\mathfrak{f} \;=\; \mathfrak{s}\big(\mathfrak{u}(n_1) \oplus \ldots \oplus \mathfrak{u}(n_r)\big) \quad (n = n_1 + \ldots + n_r)\;. \tag{87}$$

For self-conjugate representations, that is, when $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$, we can say somewhat more: since $\mathfrak{g}$ is pointwise fixed under conjugation with τ and hence so is $\mathfrak{s}$, τ permutes the aforementioned irreducible subspaces and therefore these fall into two classes: single subspaces that are τ-invariant and pairs of subspaces that are switched under the action of τ. As a result, again with an appropriate choice of basis, $\mathfrak{s}$ is contained in the *generalized real flag algebra*

$$\begin{array}{rl} \mathfrak{f} \;=\; & \mathfrak{so}(n_1) \oplus \ldots \oplus \mathfrak{so}(n_r) \oplus \mathfrak{u}(n_{r+1}) \oplus \ldots \oplus \mathfrak{u}(n_{r+s}) \\ & (n = n_1 + \ldots + n_r + 2n_{r+1} + \ldots + 2n_{r+s}) \end{array} \;, \tag{88}$$

when $\mathfrak{g} = \mathfrak{so}(n)$ (using the inclusion $\mathfrak{u}(n_i) \subset \mathfrak{so}(2n_i)$), but is contained in the *generalized quaternionic flag algebra*

$$\begin{array}{rl} \mathfrak{f} \;=\; & \mathfrak{sp}(n_1) \oplus \ldots \oplus \mathfrak{sp}(n_r) \oplus \mathfrak{u}(n_{r+1}) \oplus \ldots \oplus \mathfrak{u}(n_{r+s}) \\ & (n = n_1 + \ldots + n_r + 2n_{r+1} + \ldots + 2n_{r+s}) \end{array} \;, \tag{89}$$

when $\mathfrak{g} = \mathfrak{sp}(n)$ (using the inclusion $\mathfrak{u}(n_i) \subset \mathfrak{sp}(2n_i)$). Moreover, all these (generalized) flag algebras $\mathfrak{f}$ are $\mathrm{Inn}(\mathfrak{g},\mathfrak{s})$-invariant, and when $\mathfrak{s}$ is $\mathbb{Z}_2$-invariant, they are also $\mathrm{Aut}_{\mathbb{Z}_2}(\mathfrak{g},\mathfrak{s})$-invariant.[24] [Indeed, given $\phi \in \mathrm{Inn}(\mathfrak{g},\mathfrak{s})$, write it in

[24] From now on, whenever we speak about $\mathbb{Z}_2$-invariant or $\mathrm{Aut}_{\mathbb{Z}_2}(\mathfrak{g},\mathfrak{h})$-invariant or $\mathbb{Z}_2$-primitive subalgebras of one of the classical Lie algebras $\mathfrak{g}$, we always assume that $\mathfrak{g} = \mathfrak{su}(n)$ or $\mathfrak{g} = \mathfrak{so}(n)$ with n even, even when this is not stated explicitly.

the form $\phi = \mathrm{Ad}(g)$ with $g \in G_0$. Then since ϕ preserves $\mathfrak{s}$, g permutes the aforementioned irreducible subspaces; moreover, when $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$, g commutes with τ and hence transforms irreducible subspaces in the same pair to irreducible subspaces in the same pair. But then $\phi = \mathrm{Ad}(g)$ preserves $\mathfrak{f}$, since $\mathfrak{f}$ consists precisely of *all* elements of $\mathfrak{g}$ that map each of the aforementioned irreducible subspaces to itself. The same argument prevails when $\mathfrak{s}$ is $\mathbb{Z}_2$-invariant and $\phi \in \mathrm{Aut}_{\mathbb{Z}_2}(\mathfrak{g},\mathfrak{s}) \setminus \mathrm{Inn}(\mathfrak{g},\mathfrak{s})$, since we can still write $\phi = \mathrm{Ad}(g)$ where g is now an antiunitary transformation on $\mathbb{C}^n$ in the case of $\mathfrak{su}(n)$ and an orthogonal transformation on $\mathbb{R}^n$ of determinant -1 in the case of $\mathfrak{so}(n)$: the hypothesis that $\mathfrak{s}$ should be $\mathbb{Z}_2$-invariant merely states that the set of such automorphisms ϕ is not empty.] But this implies that if $\mathfrak{s}$ is assumed to be primitive, it must in fact be equal to $\mathfrak{f}$, and the same conclusion holds if $\mathfrak{s}$ is assumed to be $\mathbb{Z}_2$-primitive, provided $\mathfrak{f}$ is $\mathbb{Z}_2$-invariant. Note also that for $\mathfrak{g} = \mathfrak{su}(n)$, this additional condition of $\mathbb{Z}_2$-invariance is automatically satisfied by the complex flag algebras of equation (87), but that for $\mathfrak{g} = \mathfrak{so}(n)$ with n even, it imposes restrictions on the generalized real flag algebras of equation (88) that are allowed; these will be derived below.

Having shown that the reducible primitive or $\mathbb{Z}_2$-primitive subalgebras of classical Lie algebras must be sought among the (generalized) flag algebras or $\mathbb{Z}_2$-invariant (generalized) flag algebras $\mathfrak{f}$ introduced above, our next step will be to figure out the additional constraints on the numbers n_i imposed by primitivity or $\mathbb{Z}_2$-primitivity. Beginning with the primitive case, we first note that any automorphism of $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$ which preserves a (generalized) flag algebra as defined in equations (87), (88) or (89), respectively, can only permute blocks of equal type and equal size. As a result, there are various situations in which a (generalized) flag algebra $\mathfrak{f}$ cannot be primitive, simply because one easily finds a strictly larger proper subalgebra $\tilde{\mathfrak{f}}$ of $\mathfrak{g}$ which is $\mathrm{Inn}(\mathfrak{g},\mathfrak{f})$-invariant, namely:

- If, in the case of $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$, $\mathfrak{f}$ contains "orthogonal blocks" or "symplectic blocks", respectively, together with "unitary blocks", take $\tilde{\mathfrak{f}}$ to be $\mathfrak{so}(p) \oplus \mathfrak{so}(2q)$ or $\mathfrak{sp}(p) \oplus \mathfrak{sp}(2q)$, with $p = n_1 + \ldots + n_r$ and $q = n_{r+1} + \ldots + n_{r+s}$, respectively, to conclude that $\mathfrak{f}$ cannot be primitive.

- If, in the case of $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$, $\mathfrak{f}$ contains no "orthogonal blocks" or "symplectic blocks", respectively, so then $r = 0$, but contains more than one "unitary block", take $\tilde{\mathfrak{f}}$ to be $\mathfrak{so}(2n_1) \oplus \ldots \oplus \mathfrak{so}(2n_s)$ or $\mathfrak{sp}(2n_1) \oplus \ldots \oplus \mathfrak{sp}(2n_s)$, respectively, to conclude that $\mathfrak{f}$ cannot be primitive.

- If, in the case of $\mathfrak{su}(n)$ or in the case of $\mathfrak{so}(n)$ with "orthogonal blocks" only or of $\mathfrak{sp}(n)$ with "symplectic blocks" only, $\mathfrak{f}$ contains more than two blocks whose sizes are not all equal, and arranging the blocks according to their size, say in decreasing order, with i denoting the first index for which $n_i > n_{i+1}$, take $\tilde{\mathfrak{f}}$ to be $\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$ or $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ or $\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$, respectively, with $p = n_1 + \ldots + n_i$ and $q = n_{i+1} + \ldots + n_r$, to conclude that $\mathfrak{f}$ cannot be primitive.

Thus we are left with the following candidates for reducible primitive subalgebras:

$$\mathfrak{g} \;=\; \mathfrak{su}(n): \qquad \begin{cases} \mathfrak{s}\big(\mathfrak{u}(p) \oplus \mathfrak{u}(q)\big) & (n = p+q) \\ \mathfrak{s}\big(\mathfrak{u}(p) \oplus \ldots \oplus \mathfrak{u}(p)\big) & (l \text{ summands, } n = pl\,,\; l \geqslant 3) \end{cases}$$

$$\mathfrak{g} \;=\; \mathfrak{so}(n): \qquad \begin{cases} \mathfrak{so}(p) \oplus \mathfrak{so}(q) & (n = p+q) \\ \mathfrak{so}(p) \oplus \ldots \oplus \mathfrak{so}(p) & (l \text{ summands, } n = pl\,,\; l \geqslant 3) \\ \mathfrak{u}(p) & (n = 2p) \end{cases}$$

$$\mathfrak{g} \;=\; \mathfrak{sp}(n): \qquad \begin{cases} \mathfrak{sp}(2p) \oplus \mathfrak{sp}(2q) & (n = 2(p+q)) \\ \mathfrak{sp}(2p) \oplus \ldots \oplus \mathfrak{sp}(2p) & (l \text{ summands, } n = 2pl\,,\; l \geqslant 3) \\ \mathfrak{u}(p) & (n = 2p) \end{cases}$$

Finally, in the case of $\mathfrak{so}(n)$ with n even, we must check which of these are also $\mathbb{Z}_2$-invariant and hence $\mathbb{Z}_2$-primitive. Now if $\mathfrak{f}$ contains at least one "orthogonal block" (even a trivial one of the form $\mathfrak{so}(1) = \{0\}$ will do), then $\mathfrak{f}$ is $\mathbb{Z}_2$-invariant, because we can define the orthogonal transformation g on $\mathbb{R}^n$ of determinant -1 that implements the outer automorphism of $\mathfrak{so}(n)$ to be orthogonal of determinant -1 in that block and the identity in all other blocks. But if $\mathfrak{f}$ consists just of a single "unitary block" ($\mathfrak{f} = \mathfrak{u}(p)$ with $n = 2p$), then g is complex conjugation in $\mathbb{C}^p = \mathbb{R}^n$, and this has determinant $(-1)^p$, so $\mathfrak{f}$ will be $\mathbb{Z}_2$-invariant if and only if p is odd.

Looking at the result, we note that it includes the first statement of Theorem 7.1 above: the only reducible primitive or $\mathbb{Z}_2$-primitive subalgebras of classical Lie algebras which are simple are given by the inclusion (57) (the above case of $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ with $p = n-1$ and $q = 1$, say, taking into account that $\mathfrak{so}(1) = \{0\}$).

Passing to the case of irreducible subalgebras, we note first of all that any irreducible subalgebra $\mathfrak{s}$ of a classical Lie algebra $\mathfrak{g}$ is necessarily semisimple since its center, consisting of multiples of the identity and at the same time belonging to $\mathfrak{g}$ (whose elements are traceless), reduces to $\{0\}$. Thus if $\mathfrak{s}$ is not simple, we may decompose the irreducible representation space $V \cong \mathbb{C}^n$ of $\mathfrak{s}$ into the tensor product $V = V_1 \otimes \ldots \otimes V_r$ of irreducible representation spaces $V_i \cong \mathbb{C}^{n_i}$ of the simple ideals $\mathfrak{s}_i$ of $\mathfrak{s}$ ($1 \leqslant i \leqslant r$), where $n = n_1 \ldots n_r$ (see, e.g., the first part of Theorem 3.3 in [9, p. 272] or Proposition 3.1.8 in [15, p. 123]). Given the fact that irreducible representations map simple Lie algebras into Lie algebras of traceless matrices and applying Lemma 7.11 (for the case of sesquilinear forms), it becomes clear that for complex representations, i.e., when $\mathfrak{g} = \mathfrak{su}(n)$, and with an appropriate choice of basis, $\mathfrak{s}$ is contained in

$$\mathfrak{f}^{\times} \;=\; \mathfrak{su}(n_1) \times \ldots \times \mathfrak{su}(n_r) \quad (n = n_1 \ldots n_r)\;. \tag{90}$$

For self-conjugate representations, that is, when $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$, we can say somewhat more: since in these cases, the representation of $\mathfrak{s}$ on $\mathbb{C}^n$ preserves a bilinear form (which is symmetric when $\mathfrak{g} = \mathfrak{so}(n)$ and antisymmetric when $\mathfrak{g} = \mathfrak{sp}(n)$), Lemma 7.11 (for the case of bilinear forms) implies that the same is true for each of the representations of $\mathfrak{s}_i$ on $\mathbb{C}^{n_i}$; moreover, since the tensor product of two symmetric or two antisymmetric bilinear forms is symmetric whereas that of a symmetric and an antisymmetric bilinear form is antisymmetric, we conclude that $\mathfrak{s}$ is contained in

$$\begin{array}{rcl} \mathfrak{f}^\times & = & \mathfrak{so}(n_1) \times \ldots \times \mathfrak{so}(n_r) \times \mathfrak{sp}(n_{r+1}) \times \ldots \times \mathfrak{sp}(n_{r+s}) \\ & & (n = n_1 \ldots n_r \, n_{r+1} \ldots n_{r+s}) \end{array} \quad , \tag{91}$$

with s even when $\mathfrak{g} = \mathfrak{so}(n)$ and s odd when $\mathfrak{g} = \mathfrak{sp}(n)$ (see, e.g., the second part of Theorem 3.3 in [9, p. 272]). Moreover, once again, all these subalgebras $\mathfrak{f}^\times$ are $\mathrm{Inn}(\mathfrak{g}, \mathfrak{s})$-invariant, and when $\mathfrak{s}$ is $\mathbb{Z}_2$-invariant, they are also $\mathrm{Aut}_{\mathbb{Z}_2}(\mathfrak{g}, \mathfrak{s})$-invariant.[24] [Indeed, given $\phi \in \mathrm{Inn}(\mathfrak{g}, \mathfrak{s})$, write it in the form $\phi = \mathrm{Ad}(g)$ with $g \in G_0$. Then since ϕ preserves $\mathfrak{s}$, ϕ permutes the simple ideals of $\mathfrak{s}$ and, as can be shown by iterating the argument used in the proof of Lemma 7.7, g permutes the tensor factors of V correspondingly, i.e., there exists a permutation π of $\{1, \ldots, r\}$ such that $\phi \, \mathfrak{s}_i = \mathfrak{s}_{\pi(i)}$ and $g(v_1 \otimes \ldots \otimes v_r) = g_{1,\pi^{-1}(1)}(v_{\pi^{-1}(1)}) \otimes \ldots \otimes g_{r,\pi^{-1}(r)}(v_{\pi^{-1}(r)})$ where $\phi_{\pi(i),i}$ is a Lie algebra isomorphism from $\mathfrak{s}_i$ to $\mathfrak{s}_{\pi(i)}$ and $g_{i,\pi^{-1}(i)}$ is a linear isomorphism from $V_{\pi^{-1}(i)}$ to V_i such that $\phi_{\pi(i),i}(X_i) = g_{i,\pi^{-1}(i)} \, X_i \, g_{i,\pi^{-1}(i)}^{-1}$. But then $\phi = \mathrm{Ad}(g)$ preserves $\mathfrak{f}^\times$, since $\mathfrak{f}^\times$ consists precisely of *all* elements X of $\mathfrak{g}$ that are tensor decomposable. The same argument prevails when $\mathfrak{s}$ is $\mathbb{Z}_2$-invariant and $\phi \in \mathrm{Aut}_{\mathbb{Z}_2}(\mathfrak{g}, \mathfrak{s}) \setminus \mathrm{Inn}(\mathfrak{g}, \mathfrak{s})$, since we can still write $\phi = \mathrm{Ad}(g)$ where g is now an antiunitary transformation on $\mathbb{C}^n$ in the case of $\mathfrak{su}(n)$ and an orthogonal transformation on $\mathbb{R}^n$ of determinant -1 in the case of $\mathfrak{so}(n)$: the hypothesis that $\mathfrak{s}$ should be $\mathbb{Z}_2$-invariant merely states that the set of such automorphisms ϕ is not empty.] But this implies that if $\mathfrak{s}$ is assumed to be primitive, it must in fact be equal to $\mathfrak{f}^\times$, and the same conclusion holds if $\mathfrak{s}$ is assumed to be $\mathbb{Z}_2$-primitive, provided $\mathfrak{f}^\times$ is $\mathbb{Z}_2$-invariant. Note also that for $\mathfrak{g} = \mathfrak{su}(n)$, this additional condition of $\mathbb{Z}_2$-invariance is automatically satisfied by all the subalgebras of equation (90), provided we choose complex conjugation in $V = V_1 \otimes \ldots \otimes V_r$ to be the tensor product of complex conjugations in each tensor factor, but that for $\mathfrak{g} = \mathfrak{so}(n)$ with n even, it imposes restrictions on the subalgebras of equation (91) that are allowed; these will be derived below.

Having shown that the irreducible primitive or $\mathbb{Z}_2$-primitive subalgebras of classical Lie algebras must be sought among the subalgebras or $\mathbb{Z}_2$-invariant subalgebras $\mathfrak{f}^\times$ introduced above, our next step will be to figure out the additional constraints on the numbers n_i imposed by primitivity or $\mathbb{Z}_2$-primitivity. Beginning with the primitive case, we first note that any automorphism of $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$ which preserves a subalgebra as defined in equations (90) or (91), respectively, can only permute tensor factors of equal type and equal size. As a result, there are various situations in which such a subalgebra $\mathfrak{f}^\times$ cannot be primitive, simply because one easily finds a strictly larger proper subalgebra $\tilde{\mathfrak{f}}^\times$ of $\mathfrak{g}$ which is $\mathrm{Inn}(\mathfrak{g}, \mathfrak{f}^\times)$-invariant, namely:

- If, in the case of $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$, $\mathfrak{f}^\times$ contains "orthogonal blocks" as well as "symplectic blocks", then it must contain precisely one "orthogonal block" and precisely a pair of "symplectic blocks" of type $\mathfrak{sp}(2)$ in the case of $\mathfrak{so}(n)$ (s even) or a single "symplectic block" in the case of $\mathfrak{sp}(n)$ (s odd): otherwise, take $\tilde{\mathfrak{f}}^\times$ to be $\mathfrak{so}(p) \times \mathfrak{so}(q)$ in the case of $\mathfrak{so}(n)$ (s even) or $\mathfrak{so}(p) \times \mathfrak{sp}(q)$ in the case of $\mathfrak{sp}(n)$ (s odd), with $p = n_1 \ldots n_r$ and

$q = n_{r+1} \ldots n_{r+s}$, to conclude that $\mathfrak{f}^\times$ cannot be primitive.[25]

- If, in the case of $\mathfrak{su}(n)$ or in the case of $\mathfrak{so}(n)$ or $\mathfrak{sp}(n)$ with "orthogonal blocks" or "symplectic blocks" only, $\mathfrak{f}^\times$ contains more than two blocks whose sizes are not all equal, and arranging the blocks according to their size, say in decreasing order, with i denoting the first index for which $n_i > n_{i+1}$, take $\tilde{\mathfrak{f}}^\times$ to be as follows:

  - if $\mathfrak{g} = \mathfrak{su}(n)$, take $\tilde{\mathfrak{f}}^\times = \mathfrak{su}(p) \times \mathfrak{su}(q)$, with $p = n_1 \ldots n_i$ and $q = n_{i+1} \ldots n_r$,
  - if $\mathfrak{g} = \mathfrak{so}(n)$ and all blocks are orthogonal, take $\tilde{\mathfrak{f}}^\times = \mathfrak{so}(p) \times \mathfrak{so}(q)$, with $p = n_1 \ldots n_i$ and $q = n_{i+1} \ldots n_r$,
  - if $\mathfrak{g} = \mathfrak{so}(n)$ and all blocks are symplectic (their number s being even and $n_1 = \ldots = n_i > 2$), take

$$\begin{aligned} \tilde{\mathfrak{f}}^\times &= \mathfrak{so}(p) \times \mathfrak{so}(q) \quad \text{if } i \text{ is even} \\ \tilde{\mathfrak{f}}^\times &= \mathfrak{sp}(p) \times \mathfrak{sp}(q) \quad \text{if } i \text{ is odd} \end{aligned}$$

  with $p = n_1 \ldots n_i$ and $q = n_{i+1} \ldots n_s$,
  - if $\mathfrak{g} = \mathfrak{sp}(n)$ and all blocks are symplectic (their number s being odd and $n_1 = \ldots = n_i > 2$), take

$$\begin{aligned} \tilde{\mathfrak{f}}^\times &= \mathfrak{so}(p) \times \mathfrak{sp}(q) \quad \text{if } i \text{ is even} \\ \tilde{\mathfrak{f}}^\times &= \mathfrak{sp}(p) \times \mathfrak{so}(q) \quad \text{if } i \text{ is odd} \end{aligned}$$

  with $p = n_1 \ldots n_i$ and $q = n_{i+1} \ldots n_s$,

  to conclude that $\mathfrak{f}^\times$ cannot be primitive, except in the last two cases, provided that we choose $i = 1$ and[25]

$$\begin{array}{cl} s = 2 & \text{for } \mathfrak{so}(n) \ (s \text{ even}) \ , \\ s = 3 \ , \ n_2 = 2 = n_3 & \text{for } \mathfrak{sp}(n) \ (s \text{ odd}) \ . \end{array}$$

Thus we are left with the following candidates for irreducible primitive subalgebras:

$$\mathfrak{g} \; = \; \mathfrak{su}(n) : \qquad \left\{ \begin{array}{cc} \mathfrak{su}(p) \times \mathfrak{su}(q) & (n = pq) \\ \mathfrak{su}(p) \times \ldots \times \mathfrak{su}(p) & (l \text{ factors}, \, n = p^l, \, l \geqslant 3) \end{array} \right.$$

$$\mathfrak{g} \; = \; \mathfrak{so}(n) : \qquad \left\{ \begin{array}{cc} \mathfrak{so}(p) \times \mathfrak{so}(q) & (n = pq) \\ \mathfrak{sp}(2p) \times \mathfrak{sp}(2q) & (n = 4pq) \\ \mathfrak{so}(p) \times \ldots \times \mathfrak{so}(p) & (l \text{ factors}, \, n = p^l, l \geqslant 3) \\ \mathfrak{sp}(2p) \times \ldots \times \mathfrak{sp}(2p) & (l \text{ factors}, \, n = (2p)^l, l \geqslant 3 \text{ even}) \end{array} \right.$$

$$\mathfrak{g} \; = \; \mathfrak{sp}(n) : \qquad \left\{ \begin{array}{cc} \mathfrak{sp}(2p) \times \mathfrak{so}(q) & (n = 2pq) \\ \mathfrak{sp}(2p) \times \ldots \times \mathfrak{sp}(2p) & (l \text{ factors}, \, n = (2p)^l, l \geqslant 3 \text{ odd}) \end{array} \right.$$

[25] The possible occurrence of a pair of "symplectic blocks" of type $\mathfrak{sp}(2)$ is due to the exceptional isomorphism $\mathfrak{sp}(2) \times \mathfrak{sp}(2) \cong \mathfrak{so}(4)$, which entails that in this case, and only in this case, $\tilde{\mathfrak{f}}^\times$ is just $\mathfrak{f}^\times$ itself.

Finally, in the case of $\mathfrak{so}(n)$ with n even, we must check which of these are also $\mathbb{Z}_2$-invariant and hence $\mathbb{Z}_2$-primitive. In order to see whether it is possible to construct an orthogonal transformation g on $\mathbb{R}^n$ of determinant -1 that implements the outer automorphism of $\mathfrak{so}(n)$ in such a way as to preserve $\mathfrak{k}^\times$, we proceed case by case, making use of the fact that, as before, g can at most permute tensor factors of equal type and equal size. Moreover, if g preserves the tensor factors, then it must be the tensor product of orthogonal transformations in each tensor factor, each of which must have determinant ± 1, and we can compute its determinant from the following general formula for determinants in tensor products:

$$\det_{V\otimes W}(A\otimes B) \;=\; (\det_V A)^{\dim W}(\det_W B)^{\dim V} \;. \tag{92}$$

- $\mathfrak{k}^\times = \mathfrak{so}(p)\times\mathfrak{so}(q)$ $(n = pq)$. Note that p or q (i.e., at least one of them) must be even (since n is) and that if $p \neq q$, g must preserve the tensor factors, while if $p = q$, g may be chosen to switch them. In the first case, it follows from equation (92) that we can achieve $\det g = -1$ if and only if p or q (i.e., the other one of them) is odd, namely by taking g to be the tensor product of an orthogonal transformation of determinant -1 in the even-dimensional tensor factor with the identity in the odd-dimensional tensor factor. In the second case, the switch operator has determinant $(-1)^{p(p-1)/2}$, so we can achieve $\det g = -1$ if and only if $p/2$ is odd.

- $\mathfrak{k}^\times = \mathfrak{sp}(2p)\times\mathfrak{sp}(2q)$ $(n = 4pq)$. Note that if $p \neq q$, g must preserve the tensor factors, while if $p = q$, g may be chosen to switch them. In the first case, it follows from equation (92) that we cannot achieve $\det g = -1$. In the second case, the switch operator has determinant $(-1)^{p(2p-1)}$, so we can achieve $\det g = -1$ if and only if p is odd.

- $\mathfrak{k}^\times = \mathfrak{so}(p)\times\ldots\times\mathfrak{so}(p)$ (l factors, $n = p^l$, $l \geqslant 3$). Note that p must be even (since n is), so if g is assumed to preserve the tensor factors, it follows from equation (92) that we cannot achieve $\det g = -1$. Moreover, if g switches two of the l tensor factors but is the identity on the remaining $l-2$, then according to equation (92) it has determinant $(-1)^{(p(p-1)/2)\cdot p^{l-2}} = +1$, and since every permutation can be written as the product of transpositions, it follows that we cannot achieve $\det g = -1$.

- $\mathfrak{k}^\times = \mathfrak{sp}(2)\times\ldots\times\mathfrak{sp}(2p)$ (l factors, $n = (2p)^l$, $l \geqslant 3$). Again, if g is assumed to preserve the tensor factors, it follows from equation (92) that we cannot achieve $\det g = -1$. Similarly, if g switches two of the l tensor factors but is the identity on the remaining $l-2$, then according to equation (92) it has determinant $(-1)^{(p(p-1)/2)\cdot p^{l-2}} = +1$, and since every permutation can be written as the product of transpositions, it follows that we cannot achieve $\det g = -1$.

To summarize, we see that among the irreducible primitive subalgebras $\mathfrak{k}$ of $\mathfrak{so}(n)$ with n even, as listed above, the only ones which are $\mathbb{Z}_2$-invariant and hence also $\mathbb{Z}_2$-primitive are $\mathfrak{so}(p)\times\mathfrak{so}(q)$ $(n = pq)$ with $p \neq q$ and either p or q odd, $\mathfrak{so}(p)\times\mathfrak{so}(p)$ $(n = p^2)$ with p even and $p/2$ odd, and $\mathfrak{sp}(2p)\times\mathfrak{sp}(2p)$ $(n = 4p^2)$ with p odd.

This proves the classification of the primitive and $\mathbb{Z}_2$-primitive subalgebras of classical Lie algebras: the simple ones are determined according to Theorem 7.1 above and the results for the remaining ones are summarized in Tables 2, 3 and 4, providing the complete list of (conjugacy classes of) non-simple primitive and $\mathbb{Z}_2$-primitive subalgebras of the classical Lie algebras $\mathfrak{su}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$, respectively, with the following conventions.

The first column indicates the isomorphism type of the subalgebra. The second column indicates the conditions which are necessary for the inclusion to exist and also for avoiding repetitions, most of which are due to well-known canonical isomorphisms between classical Lie algebras of low rank:

$$\begin{gathered}
\mathfrak{so}(3) = \mathfrak{su}(2) \quad , \quad \mathfrak{sp}(2) = \mathfrak{su}(2) \quad , \quad \mathfrak{so}(2) = \mathbb{R} \\
\mathfrak{sp}(4) = \mathfrak{so}(5) \quad , \quad \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \\
\mathfrak{so}(6) = \mathfrak{su}(4)
\end{gathered}$$

The third column indicates the intrinsic nature of the subalgebra, classifying it into three types as above, namely abelian (a), truly semisimple (s) and truly reductive (r), together with the type of inclusion: reducible (red.) or irreducible (irred.). Finally, Table 3 has a fourth column, only relevant for $\mathfrak{so}(n)$ with n even, where we indicate under what additional conditions the corresponding subalgebra is also $\mathbb{Z}_2$-primitive. Note that the subalgebras of $\mathfrak{so}(2n)$ that are not $\mathbb{Z}_2$-primitive fall into two distinct conjugacy classes that are transformed into each other by an outer automorphism. (In Table 2, this additional column is superfluous since all primitive subalgebras are also $\mathbb{Z}_2$-primitive.) All tables are divided into two parts: in the upper part, we list the maximal subalgebras and in the lower part, we list the non-maximal subalgebras.

The main reason for organizing these data in this form is for better reference, for instance in the next section.

subalgebra	conditions	type
$\mathbb{R} \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$n = p + q\,,\ p \geqslant q \geqslant 1\,,\ p \geqslant 2$	$r -$ red.
$\mathbb{R}$	$n = 2$	$a -$ red.
$\mathfrak{su}(p) \times \mathfrak{su}(q)$	$n = pq\,,\ p \geqslant q \geqslant 2$	$s -$ irred.
$\mathbb{R}^{l-1} \oplus \bigoplus_{k=1}^{l} \mathfrak{su}(p)$	$n = pl\,,\ l \geqslant 3\,,\ p \geqslant 2$	$r -$ red.
$\mathbb{R}^{n-1}$	$n \geqslant 3$	$a -$ red.
$\prod_{k=1}^{l} \mathfrak{su}(p)$	$n = p^l\,,\ l \geqslant 3\,,\ p \geqslant 2$	$s -$ irred.

Table 2: Non-simple primitive and $\mathbb{Z}_2$-primitive subalgebras of $\mathfrak{su}(n)$ $(n \geqslant 2)$

subalgebra	conditions	type	$\mathbb{Z}_2$-primitive (n even)
$\mathfrak{so}(p) \oplus \mathfrak{so}(q)$	$n = p + q\,,\ p \geqslant q \geqslant 3$	$s-$red.	yes
$\mathbb{R} \oplus \mathfrak{so}(p)$	$n = p + 2\,,\ p \geqslant 3$	$r-$red.	yes
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$n = 5$	$s-$red.	–
$\mathbb{R} \oplus \mathfrak{su}(p) = \mathfrak{u}(p)$	$n = 2p\,,\ p \geqslant 3$	$r-$red.	p odd
$\mathfrak{so}(p) \times \mathfrak{so}(q)$	$n = pq\,,\ p \geqslant q \geqslant 3\,,\ p, q \neq 4$	$s-$irred.	$p \neq q$: p or q odd $p = q$: $p/2$ odd
$\mathfrak{sp}(2p) \times \mathfrak{sp}(2q)$	$n = 4pq\,,\ p \geqslant q \geqslant 1$	$s-$irred.	$p = q$: p odd
$\bigoplus_{k=1}^{l} \mathfrak{so}(p)$	$n = pl\,,\ l \geqslant 3\,,\ p \geqslant 3$	$s-$red.	yes
$\mathbb{R}^l$	$n = 2l\,,\ l \geqslant 3$	$a-$red.	yes
$\prod_{k=1}^{l} \mathfrak{so}(p)$	$n = p^l\,,\ l \geqslant 3\,,\ p \geqslant 3\,,\ p \neq 4$	$s-$irred.	no
$\prod_{k=1}^{l} \mathfrak{sp}(2p)$	$n = (2p)^l\,,\ l \geqslant 4\,,\ l$ even, $p \geqslant 1$	$s-$irred.	no
$\mathfrak{so}(p) \times \mathfrak{sp}(2) \times \mathfrak{sp}(2)$	$n = 4p\,,\ p \geqslant 3\,,\ p \neq 4$	$s-$irred.	p odd

Table 3: Non-simple primitive and $\mathbb{Z}_2$-primitive subalgebras of $\mathfrak{so}(n)$ $(n \geqslant 5)$

subalgebra	conditions	type
$\mathfrak{sp}(2p) \oplus \mathfrak{sp}(2q)$	$n = 2(p + q)\,,\ p \geqslant q \geqslant 1$	$s-$red.
$\mathbb{R} \oplus \mathfrak{su}(p) = \mathfrak{u}(p)$	$n = 2p\,,\ p \geqslant 2$	$r-$red.
$\mathfrak{sp}(2p) \times \mathfrak{so}(q)$	$n = 2pq\,,\ p \geqslant 1\,,\ q \geqslant 3\,,\ q \neq 4$	$s-$irred.
$\bigoplus_{k=1}^{l} \mathfrak{sp}(2p)$	$n = 2pl\,,\ l \geqslant 3\,,\ p \geqslant 1$	$s-$red.
$\prod_{k=1}^{l} \mathfrak{sp}(2p)$	$n = (2p)^l\,,\ l \geqslant 3\,,\ l$ odd, $p \geqslant 1$	$s-$irred.
$\mathfrak{sp}(2p) \times \mathfrak{sp}(2) \times \mathfrak{sp}(2)$	$n = 8p\,,\ p \geqslant 2$	$s-$irred.

Table 4: Non-simple primitive subalgebras of $\mathfrak{sp}(n)$ $(n$ even, $n \geqslant 4)$

8. Maximal subgroups of the classical groups

Our goal in this last section is to compute the normalizers, within the corresponding classical groups G, of the primitive subalgebras of the classical Lie algebras and thus obtain a complete list of all maximal subgroups of the classical groups.

Beginning with the case of simple primitive subalgebras, we distinguish two types: (i) the "classical" simple inclusions $\mathfrak{so}(n) \subset \mathfrak{su}(n)$, $\mathfrak{sp}(n) \subset \mathfrak{su}(n)$ and $\mathfrak{so}(n-1) \subset \mathfrak{so}(n)$ and (ii) the remaining "non-trivial" simple inclusions given by the irreducible representations that do not belong to Dynkin's list of exceptions. For the "classical" simple inclusions it is easy to find the corresponding maximal subgroups: $O(n) \subset SU(n)$, $Sp(n) \subset SU(n)$ and $O(n-1) \subset SO(n)$.

Passing to the non-simple primitive subalgebras, which have been determined in the previous section, what remains to be done is to compute the corresponding list of maximal subgroups of the classical groups. The final results are summarized in Table 5 for the group $SU(n)$, in Tables 6 and 7 for the group $SO(n)$ and in Table 8 for the group $Sp(n)$. For each maximal subgroup H of G, we list in the first column the isomorphism type of the connected one-component H_0 of H and in the second column the component group H/H_0, taking into account that H must be the normalizer $N_G(H_0)$ of H_0 in G. This information completely characterizes H.

To carry out the concrete calculations, we shall employ two tools. The first tool consists in the introduction of an "intermediate subgroup" between H_0 and H, which will be denoted by H_i and will allow us to control the size and the structure of H/H_0.

Suppose G is a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and H is a maximal subgroup of G with connected one-component H_0 and Lie algebra $\mathfrak{h}$. Assume that $\mathfrak{h}$ is not an ideal of $\mathfrak{g}$. In the particular case where $\mathfrak{g}$ is simple – which is the case of interest here – this is equivalent to assuming that $\mathfrak{h} \neq \{0\}$. According to Theorem 3.8, H is then equal to the normalizer $N_G(H_0)$ of H_0 in G.

Now consider the centralizer $Z_G(H_0)$ of H_0 in G and define H_i to be the closed subgroup of G generated by the two closed subgroups H_0 and $Z_G(H_0)$:

$$H_i = H_0\, Z_G(H_0) = Z_G(H_0)\, H_0 \ . \tag{93}$$

Then we have[26]

$$H_0 \lhd H_i \lhd H \ . \tag{94}$$

Considering the action of H on H_0 by conjugation in G (recall that $N_G(H_0)$ is defined as the set of elements g of G such that conjugation by g leaves H_0 invariant), H_i consists of those elements of H that act as inner automorphisms on H_0, while $Z_G(H_0)$ consists of those elements of H that act trivially on H_0. In other words, under the adjoint representation $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ (which maps into $\mathrm{Inn}(\mathfrak{g})$ because G is connected and maps onto $\mathrm{Inn}(\mathfrak{g})$ because G is semisimple), H is the inverse image of $\mathrm{Inn}(\mathfrak{g},\mathfrak{h})$ while H_i is the inverse image of $\mathrm{Inn}(\mathfrak{h})$, so

$$H/H_i \ \cong\ \mathrm{Inn}(\mathfrak{g},\mathfrak{h}) \,/\, \mathrm{Inn}(\mathfrak{h}) \ . \tag{95}$$

Also note that H, H_i and H_0 all have the same Lie algebra $\mathfrak{h}$ (since this is true for H_0 and H) and thus the quotient groups H/H_0, H/H_i and H_i/H_0 are discrete. Moreover, by the second isomorphism theorem of group theory, we have

$$H/H_i \ = \ \frac{H/H_0}{H_i/H_0} \ . \tag{96}$$

[26] As usual, the symbol "$\lhd$" is to be read as "is normal subgroup of".

If G is compact, the above mentioned discrete groups are in fact finite and their orders satisfy the relation

$$|H/H_0| \ = \ |H_i/H_0|\,|H/H_i| \ . \tag{97}$$

In almost all cases of interest, we may use additional properties to extract more information about the structure of H_i/H_0 and of H/H_i, and hence of H/H_0 as well.

- H_i/H_0 : Since $H_0 \cap Z_G(H_0) = Z(H_0)$, the first isomorphism theorem implies

  $$H_i/H_0 \ \cong \ Z_G(H_0)\,/\,Z(H_0) \ . \tag{98}$$

  Moreover, the inclusion $Z(G) \subset Z_G(H_0)$ is valid in general. Conversely, if G is a classical group and the inclusion of H_0 in G is an irreducible representation, then by Schur's lemma we have $Z_G(H_0) \subset (\mathbb{K}\cdot 1) \cap G \subset Z(G)$ and thus

  $$Z_G(H_0) \ = \ Z(G) \quad , \quad H_i/H_0 \ \cong \ Z(G)\,/\,Z(H_0) \ . \tag{99}$$

  More generally, by Schur's lemma, $Z_G(H_0)$ acts by scalar multiplication on each irreducible component. Finally, we note that if $\mathfrak{h}$ has maximal rank, that is, if H_0 contains a maximal torus T of G, then $Z_G(H_0) \subset T \subset H_0$ and hence $H_i = H_0$.

- H/H_i : This quotient group may identified with the group $\mathrm{Out}(\mathfrak{h}) \cap \mathrm{Inn}(\mathfrak{g})$ of those outer automorphisms of $\mathfrak{h}$ that can be extended to inner automorphisms of $\mathfrak{g}$ and thus can be implemented by conjugation by elements of the ambient group G. If $\mathfrak{h}$ is semisimple, it may also be identified with the group $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ of those automorphisms of the Dynkin diagram Γ of $\mathfrak{h}$ that can be extended to inner automorphisms of $\mathfrak{g}$ and thus can be implemented by conjugation by elements of the ambient group G:

  $$H/H_i \ \cong \ \mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) \ . \tag{100}$$

  In what follows, we shall calculate H/H_i using the results of the previous section, in particular Lemma 7.6 and Lemma 7.7.

Remark 8.1. According to the relation (96), the group H/H_0 is an upwards extension of the group H_i/H_0 by the group H/H_i, but simple examples show that this extension is not necessarily split, that is, a semi-direct product.

The second tool is the following lemma which, in combination with Lemma 7.6, allows to handle the reducible inclusions.

Lemma 8.2. *Assume that G is one of the classical groups $SU(n)$, $SO(n)$ or $Sp(n)$ and consider the standard embeddings*

$$\mathfrak{g}(p) \oplus \mathfrak{g}(q) \subset \mathfrak{g}(n) \quad (n = p + q)$$

by block diagonal matrices, where $\mathfrak{g}(r)$ is a generic symbol standing for the corresponding type of classical Lie algebra (i.e., $\mathfrak{su}(r)$, $\mathfrak{so}(r)$ or $\mathfrak{sp}(r)$, respectively). Then if $g \in G$ normalizes both $\mathfrak{g}(p)$ and $\mathfrak{g}(q)$, g must itself be block diagonal. The same statement holds for more than two direct summands.

Proof. Writing $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and assuming that g normalizes matrices of the form $\left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)$ as well as matrices of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & Y \end{smallmatrix}\right)$, it follows immediately that B and C must vanish, since the X's and Y's form irreducible sets of matrices. ■

With these generalities out of the way, we proceed to the case by case analysis of each of the entries of Tables 5–8.

8.1. Maximal subgroups of $SU(n)$.

1. The inclusion

$$S\big(U(p) \times U(q)\big) \ \subset \ SU(n) \qquad (n = p + q)$$

is given by the direct sum of the defining representations of $U(p)$ and $U(q)$. Explicitly, it is obtained as the restriction of the inclusion of $U(p) \times U(q)$ into $U(n)$ by block diagonal matrices. In fact, the inverse image of $SU(n) \subset U(n)$ under this inclusion is the subgroup

$$H_0 \ = \ S\big(U(p) \times U(q)\big) \ = \ \{\, (A, B) \in U(p) \times U(q) \mid \det(A) \det(B) = 1 \,\}$$

consisting of the block diagonal matrices belonging to $SU(n)$, which is connected. It follows from Lemma 7.6 that the centralizer $Z_G(H_0)$ of H_0 in G is contained in H_0 ; hence $H_i = H_0$. Similarly, we compute $H = N_G(H_0)$ by combining Lemma 7.6 and Lemma 8.2 to conclude that the elements h of H must be of the form $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ or $h = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$, with $p = q$ in the second case. In the first case, it follows that $h \in H_0$, and thus $H = H_0$ if $p \neq q$. In the second case, we may write h as the product of an element of H_0 with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in SU(n)$ $(n = 2p)$, whose square is -1_n and belongs to H_0, so we may conclude that $H/H_0 = \mathbb{Z}_2$ if $p = q$.

2. The inclusion

$$SU(p) \times_{\mathbb{Z}_d} SU(q) \ \subset \ SU(n) \qquad (n = pq)$$

is given by the tensor product of the defining representations of $U(p)$ and $U(q)$, and $d = \gcd(p, q)$ is the greatest common divisor of p and q. Explicitly, it is induced by the homomorphism

$$\begin{array}{ccc} U(p) \times U(q) & \longrightarrow & U(n) \\ (A, B) & \longmapsto & A \otimes B \end{array}$$

which, in contrast to the situation encountered in the first item, is not injective but rather has a non-trivial kernel given by

$$\{\, (\exp(i\alpha)1_p \,, \exp(-i\alpha)1_q) \mid \alpha \in \mathbb{R} \,\} \ \cong \ U(1) \ .$$

Using the formula

$$\det(A \otimes B) \;=\; (\det(A))^q \, (\det(B))^p \;,$$

we see that the inverse image of $SU(n) \subset U(n)$ under this homomorphism is the subgroup

$$S'\big(U(p) \times U(q)\big) \;=\; \{\, (A,B) \in U(p) \times U(q) \mid \det(A)^q \det(B)^p = 1 \,\} \;.$$

and the restricted homomorphism

$$\begin{array}{ccc} S'\big(U(p) \times U(q)\big) & \longrightarrow & SU(n) \\ (A,B) & \longmapsto & A \otimes B \end{array}$$

still has the same kernel

$$\{\, (\exp(i\alpha) 1_p \,, \exp(-i\alpha) 1_q) \mid \alpha \in \mathbb{R} \,\} \;\cong\; U(1) \;.$$

Its intersection with the connected subgroup $SU(p) \times SU(q)$ is

$$\{\, (\exp(2\pi i k/d) 1_p \,, \exp(-2\pi i k/d) 1_q) \mid 0 \leqslant k < d \,\}$$

and is isomorphic to $\mathbb{Z}_d$. Factoring this out, we obtain the desired inclusion. Note also that the center $Z(G)$ of G is contained in the quotient group $S'\big(U(p) \times U(q)\big)/U(1)$ which is generated by $Z(G)$ and by the quotient group $\big(SU(p) \times SU(q)\big)/\mathbb{Z}_d$; hence

$$H_0 \;\cong\; \big(SU(p) \times SU(q)\big)/\mathbb{Z}_d \;,$$

and

$$H_i \;\cong\; S'\big(U(p) \times U(q)\big)/U(1) \;,$$

so according to equation (99),

$$H_i/H_0 \;\cong\; Z(G)/Z(H_0) \;\cong\; \mathbb{Z}_n/\mathbb{Z}_{n/d} \;\cong\; \mathbb{Z}_d \;.$$

Explicitly, a representative of the connected component of H_i corresponding to $k \mod d$ is given by the matrix $\exp(2\pi i k/n)\, 1_n$, since when k is a multiple of d, k/n will be a multiple of d/n which can be written in the form

$$\frac{d}{n} \;=\; \frac{r}{p} + \frac{s}{q}$$

(where r and s are chosen such that $rq' + sp' = 1$ with $p' = p/d$ and $q' = q/d$, which is possible since p' and q' are relatively prime) and therefore $\exp(2\pi i k/n)\, 1_n \in H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we first note that (for $p \geqslant 3$) the Dynkin diagram of $\mathfrak{su}(p)$ admits only one automorphism which can be implemented by an antilinear involution σ_p in $\mathbb{C}^p$. Since the tensor product of linear / antilinear maps is a linear / antilinear map and the tensor product between a linear map and an antilinear map is not well defined, we see that there is only one non-trivial automorphism of the Dynkin diagram Γ of $\mathfrak{su}(p) \times \mathfrak{su}(q)$ induced by automorphisms of the Dynkin diagrams of the factors which can be extended to an automorphism of $\mathfrak{su}(n)$, namely the one implemented by an anti-linear involution of the form $\sigma_p \otimes \sigma_q$, but this will always be an outer automorphism of $\mathfrak{su}(n)$. Thus it becomes clear that for $p \neq q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = \{1\}$, while for $p = q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ can at most be equal to the group $\mathbb{Z}_2$, corresponding to the possibility of switching the factors in the tensor product that exists in this case. Explicitly, a representative of the corresponding connected component of H is given by the transformation

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ z = x \otimes y & \longmapsto & z^\tau = \exp(i\phi_p)\, y \otimes x \end{array} ,$$

which belongs to $SU(n)$ $(n = p^2)$ if we choose the phase $\exp(i\phi_p)$ according to

$$\exp(i\phi_p) \;=\; \left\{ \begin{array}{ccc} +1 \text{ or } -1 & \text{if} & p = 0 \mod 4 \\ +1 & \text{if} & p = 1 \mod 4 \\ \exp(i\pi/4) & \text{if} & p = 2 \mod 4 \\ -1 & \text{if} & p = 3 \mod 4 \end{array} \right. ,$$

since the permutation which maps $x \otimes y$ to $y \otimes x$, when written in a basis of $\mathbb{C}^n$ provided by a basis of $\mathbb{C}^p$ by taking tensor products between vectors of the latter, is the product of $\binom{p}{2} = p(p-1)/2$ transpositions

$$e_i \otimes e_j \longleftrightarrow e_j \otimes e_i \qquad (1 \leqslant i < j \leqslant p)$$

with p fixed points $e_i \otimes e_i$ $(1 \leqslant i \leqslant p)$ and hence is an involution with determinant $(-1)^{p(p-1)/2}$. Therefore, we conclude that for $p = q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = \mathbb{Z}_2$. Finally, the structure of the group H/H_0 in this case can be deduced by observing that for $p \neq 2 \mod 4$, the map $.^\tau$ has order 2 and hence H/H_0 is the direct product $\mathbb{Z}_p \times \mathbb{Z}_2$, whereas for $p = 2 \mod 4$, i.e., $p = 2r$ with r odd, it has order 8, but its square coincides with the matrix $\exp(2\pi i r^2/p^2)\, 1_n = i\, 1_n$ which represents one of the non-trivial connected components of H_i, and the square of the latter belongs to H_0, so it follows that H/H_0 is a non-split upward extension of the group $\mathbb{Z}_p$ by the group $\mathbb{Z}_2$ $(H/H_0 = \mathbb{Z}_p \,.\, \mathbb{Z}_2)$ which can be explicitly constructed as the quotient group $\mathbb{Z}_p \times_{\mathbb{Z}_2} \mathbb{Z}_4$.

3. Generalizing the procedure of the first item, we define the inclusion

$$S\big(U(p) \times \ldots \times U(p)\big) \;\subset\; SU(n) \qquad (n = pl)$$

by the direct sum of l copies of the defining representation of $U(p)$, which realizes $U(p) \times \ldots \times U(p)$ by block diagonal matrices in $U(n)$, with l blocks of size p along the diagonal. The inverse image of $SU(n) \subset U(n)$ under this inclusion is the subgroup

$$H_0 \;=\; S\big(U(p) \times \ldots \times U(p)\big)$$

consisting of the block diagonal matrices belonging to $SU(n)$, which is connected. As before, it follows from Lemma 7.6 that the centralizer $Z_G(H_0)$ of H_0 in G is contained in H_0; hence $H_i = H_0$. Similarly, we compute $H = N_G(H_0)$ by combining Lemma 7.6 and Lemma 8.2 to conclude that the elements h of H must be such that, when represented as block $(l \times l)$-matrices (with entries that are themselves $(p \times p)$-matrices), they contain precisely one nonvanishing entry in each line and each column: obviously, any such matrix h defines a permutation $\sigma(h)$ of $\{1, \ldots, l\}$. Thus we obtain a group homomorphism from H to the permutation group S_l which has kernel H_0 and is surjective (we can use the same argument as in item 1 above to show that its image contains all transpositions), so we may conclude that $H/H_0 = S_l$.

4. Generalizing the procedure of the second item, we define the inclusion

$$\big(SU(p) \times \ldots \times SU(p)\big)/\mathbb{Z}_p^{l-1} \subset SU(n) \qquad (n = p^l)$$

by the tensor product of l copies of the defining representation of $U(p)$. Explicitly, it is induced by the homomorphism

$$\begin{array}{ccc} U(p) \times \ldots \times U(p) & \longrightarrow & U(n) \\ (A_1, \ldots, A_l) & \longmapsto & A_1 \otimes \ldots \otimes A_l \end{array}$$

which, once again, is not injective but rather has a non-trivial kernel given by

$$\{\, (\exp(i\alpha_1)\, 1_p\,, \ldots, \exp(i\alpha_l)\, 1_p) \mid \exp(i(\alpha_1 + \ldots + \alpha_l)) = 1 \,\} \;\cong\; U(1)^{l-1} \,.$$

Using the formula

$$\det\big(A_1 \otimes \ldots \otimes A_l\big) \;=\; \big(\det(A_1)\, \ldots\, \det(A_l)\big)^{p^{\,l-1}} ,$$

we see that the inverse image of $SU(n) \subset U(n)$ under this homomorphism is the subgroup

$$S'\big(U(p) \times \ldots \times U(p)\big) \;=\; \begin{array}{r} \{\, (A_1, \ldots, A_l) \in U(p) \times \ldots \times U(p) \mid \\ \big(\det(A_1)\, \ldots\, \det(A_l)\big)^{p^{\,l-1}} = 1 \,\} \,, \end{array}$$

and the restricted homomorphism

$$\begin{array}{ccc} S'\big(U(p) \times \ldots \times U(p)\big) & \longrightarrow & SU(n) \\ (A_1, \ldots, A_l) & \longmapsto & A_1 \otimes \ldots \otimes A_l \end{array}$$

still has the same kernel

$$\{\, (\exp(i\alpha_1)\, 1_p\,, \ldots, \exp(i\alpha_l)\, 1_p) \mid \exp(i(\alpha_1 + \ldots + \alpha_l)) = 1 \,\} \;\cong\; U(1)^{l-1} \,.$$

Its intersection with the connected subgroup $SU(p) \times \ldots \times SU(p)$ is

$$\{(\exp(2\pi i k_1/p)\, 1_p\,, \ldots, \exp(2\pi i k_l/p)\, 1_p) \mid \exp(2\pi i(k_1 + \ldots + k_l)/p) = 1\}$$

and is isomorphic to $\mathbb{Z}_p^{l-1}$. Factoring this out, we obtain the desired inclusion. Note also that the center $Z(G)$ of G is contained in the quotient group $S'\big(U(p) \times \ldots \times U(p)\big)/U(1)^{l-1}$ which is generated by $Z(G)$ and by the quotient group $\big(SU(p) \times \ldots \times SU(p)\big)/\mathbb{Z}_p^{l-1}$; hence

$$H_0 \;\cong\; \big(SU(p) \times \ldots \times SU(p)\big)/\mathbb{Z}_p^{l-1} \;,$$

and

$$H_i \;\cong\; S'\big(U(p) \times \ldots \times U(p)\big)/U(1)^{l-1} \;,$$

so according to equation (99),

$$H_i/H_0 \;\cong\; Z(G)/Z(H_0) \;\cong\; \mathbb{Z}_n/\mathbb{Z}_p \;\cong\; \mathbb{Z}_{p^{l-1}} \;.$$

Explicitly, a representative of the connected component of H_i corresponding to $k \mod p^{\,l-1}$ is given by the matrix $\exp(2\pi ik/n)\, 1_n$, since when k is a multiple of $p^{\,l-1}$, $\exp(2\pi ik/n)\, 1_n \in H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we note, as before, that there is no automorphism of the Dynkin diagram of $\mathfrak{su}(p) \times \ldots \times \mathfrak{su}(p)$ induced by automorphisms of the Dynkin diagrams of the factors that can be extended to an inner automorphism of $\mathfrak{su}(n)$. Thus it becomes clear that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ can at most be equal to the symmetric group S_l, corresponding to the possibility of permuting the l factors in the tensor product. But for $l \geqslant 3$, we can implement any transposition through an involution of determinant $+1$ since, for example, the determinant of the transformation

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ z = x_1 \otimes x_2 \otimes x_3 \otimes \ldots \otimes x_l & \longmapsto & z^{\tau_{12}} = \pm\, x_2 \otimes x_1 \otimes x_3 \otimes \ldots \otimes x_l \end{array} ,$$

which represents the transposition τ_{12} ($\tau_{12}(1) = 2,\ \tau_{12}(2) = 1,\ \tau_{12}(i) = i$ for $i \geqslant 3$) is $(-1)^{p^{\,l-1}(p\mp 1)/2}$, and this is equal to $+1$ when p is even and, with an appropriate choice of sign, also when p is odd. Therefore, it follows that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = S_l$ and that H/H_0 is the direct product $\mathbb{Z}_{p^{\,l-1}} \times S_l$.

8.2. Maximal subgroups of $SO(n)$.

1. The inclusion

$$SO(p) \times SO(q) \;\subset\; SO(n) \qquad (n = p + q)$$

is given by the direct sum of the defining representations of $O(p)$ and $O(q)$. Explicitly, it is obtained as the restriction of the inclusion of $O(p)\times O(q)$ into $O(n)$ by block diagonal matrices. In fact, the inverse image of $SO(n) \subset O(n)$ under this inclusion is the subgroup

$$H_+ \;=\; S\big(O(p) \times O(q)\big) \;=\; \{\, (A, B) \in O(p) \times O(q) \;|\; \det(A)\det(B) = 1 \,\}$$

connected component H_0	component group H/H_0		
$S\big(U(p) \times U(q)\big)$ (reducible) (direct sum)	$\{1\}$ $\mathbb{Z}_2$	for for	$n = p+q\,,\ p > q \geqslant 1$ $n = p+q\,,\ p = q \geqslant 1$
$SU(p) \times_{\mathbb{Z}_d} SU(q)$ (irreducible) (tensor product) $(d = \gcd(p,q))$	$\mathbb{Z}_d$	for	$n = pq\,,\ p > q \geqslant 2$
$SU(p) \times_{\mathbb{Z}_p} SU(p)$ (irreducible) (tensor product)	$\mathbb{Z}_p \times \mathbb{Z}_2$ $\mathbb{Z}_p \times_{\mathbb{Z}_2} \mathbb{Z}_4$	for for	$n = p^2\,,\ p \geqslant 2\,,\ p \neq 2 \mod 4$ $n = p^2\,,\ p \geqslant 2\,,\ p = 2 \mod 4$
$S\left(\prod_{k=1}^{l} U(p)\right)$ (reducible) (direct sum)	S_l	for	$n = pl\,,\ l \geqslant 3\,,\ p \geqslant 1$
$\prod_{k=1}^{l} SU(p)\,/\,\mathbb{Z}_p^{l-1}$ (irreducible) (tensor product)	$\mathbb{Z}_{p^{l-1}} \times S_l$	for	$n = p^l\,,\ l \geqslant 3\,,\ p \geqslant 2$

Table 5: Non-simple maximal subgroups H of $SU(n)$ $(n \geqslant 2)$

consisting of the block diagonal matrices belonging to $SO(n)$, which has two connected components: its one-component is the subgroup

$$H_0 \;=\; SO(p) \times SO(q) \;=\; \{\,(A,B) \in O(p) \times O(q) \mid \det(A) = 1 = \det(B)\,\}\ ,$$

while the other component is the set

$$\{\,(A,B) \in O(p) \times O(q) \mid \det(A) = -1 = \det(B)\,\}\ .$$

It follows from Lemma 7.6 that the centralizer $Z_G(H_0)$ of H_0 in G is contained in H_+ ; more precisely, we have $H_i = H_+$ if p and q are both odd and $H_i = H_0$ otherwise. Similarly, we compute $H = N_G(H_0)$ by combining Lemma 7.6 and Lemma 8.2 to conclude that, as in the $SU(n)$ case, the elements h of H must be of the form $h = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ or $h = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$, with $p = q$ in the second case. In the first case, it follows that $h \in H_+$, and thus $H = H_+$, $H/H_0 = \mathbb{Z}_2$ if $p \neq q$. In the second case, we may write

h as the product of an element of H_+ with the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in SO(n)$ $(n = 2p)$, whose square is -1_n and belongs to H_+ but belongs to H_0 only when p is even, so we may conclude that $H/H_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$ if $p = q$ is even and $H/H_0 = \mathbb{Z}_4$ if $p = q$ is odd.

2. The inclusion

$$SO(4) \subset SO(5)$$

is a special case of the inclusion $SO(n-1) \subset SO(n)$ and is the only one to appear here because $SO(n-1)$ is simple if $n \geqslant 6$. But whether simple or not, $\mathfrak{so}(n-1)$ is a maximal subalgebra of $\mathfrak{so}(n)$, and the normalizer of the corresponding connected subgroup $H_0 = \{\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right) \,|\, A \in SO(n-1)\} \cong SO(n-1)$ in $SO(n)$ is the subgroup $H_+ = \{\left(\begin{smallmatrix} A & 0 \\ 0 & \det A \end{smallmatrix}\right) \,|\, A \in O(n-1)\} \cong O(n-1)$. Therefore, $H/H_0 = \mathbb{Z}_2$.

3. The inclusion

$$U(p) \subset SO(n) \qquad (n = 2p)$$

is given by

$$A + iB \longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Note that $\mathfrak{u}(p)$ has maximal rank in $\mathfrak{so}(n)$ and hence $H_i = H_0$. Moreover, $\mathrm{Aut}(\Gamma) = Z_2$ where the non-trivial diagram automorphism is given by complex conjugation $A + iB \mapsto A - iB$, and this can be implemented by conjugating $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$ with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which belongs to $SO(n)$ if and only if $n = 2p$ with p even. Therefore, $H/H_0 = \mathbb{Z}_2$ if $n = 2p$ with p even and $H/H_0 = \{1\}$ if $n = 2p$ with p odd.

4. The inclusions

$$\begin{array}{rcll} SO(p) \times SO(q) & \subset & SO(n) & \qquad (n = pq\,, p \text{ or } q \text{ odd}) \\ SO(p) \times_{\mathbb{Z}_2} SO(q) & \subset & SO(n) & \qquad (n = pq\,, p \text{ and } q \text{ even}) \end{array}$$

are given by the tensor product of the defining representations of $O(p)$ and $O(q)$. Explicitly, they are induced by the homomorphism

$$\begin{array}{ccc} O(p) \times O(q) & \longrightarrow & O(n) \\ (A, B) & \longmapsto & A \otimes B \end{array}$$

which, in contrast to the situation encountered in the first item, is not injective but rather has a non-trivial kernel given by

$$\{(1_p\,, 1_q), (-1_p\,, -1_q)\} \;\cong\; \mathbb{Z}_2\,.$$

Using the formula

$$\det(A \otimes B) \;=\; (\det(A))^q\,(\det(B))^p\,,$$

we see that the inverse image of $SO(n) \subset O(n)$ under this homomorphism is the subgroup

$$S'\big(O(p) \times O(q)\big) \;=\; \left\{ \begin{array}{ccccc} S\big(O(p) \times O(q)\big) & \text{if} & p \text{ odd} & , & q \text{ odd} \\ O(p) \times SO(q) & \text{if} & p \text{ odd} & , & q \text{ even} \\ SO(p) \times O(q) & \text{if} & p \text{ even} & , & q \text{ odd} \\ O(p) \times O(q) & \text{if} & p \text{ even} & , & q \text{ even} \end{array} \right. .$$

In the first three cases, the group $S'\big(O(p) \times O(q)\big)$ has two connected components such that the kernel of the homomorphism introduced above meets both components in exactly one element (since $(1_p\,, 1_q)$ and $(-1_p\,, -1_q)$ belong to different connected components), so the restriction of this homomorphism to the connected one-component provides the desired inclusion. In the last case, the group $S'\big(O(p) \times O(q)\big)$ has four connected components, but the kernel of the homomorphism introduced above is contained in the connected one-component (since $(1_p\,, 1_q)$ and $(-1_p\,, -1_q)$ belong to the same connected component), so it is necessary to factor this out in order to obtain the following sequence of inclusions:

$$SO(p) \times_{\mathbb{Z}_2} SO(q) \;\subset\; O(p) \times_{\mathbb{Z}_2} O(q) \;\subset\; SO(n)\;.$$

Note also that the center $Z(G)$ of G is contained in H_0; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we first note that (for p even, $p \geqslant 4$) the Dynkin diagram of $\mathfrak{so}(p)$ admits only one non-trivial automorphism which can be implemented by a reflection σ_p in $\mathbb{R}^p$.[27] Computing determinants, we see that the automorphisms of the Dynkin diagram Γ of $\mathfrak{so}(p) \times \mathfrak{so}(q)$ induced by automorphisms of the Dynkin diagrams of the factors can always be extended to automorphisms of $\mathfrak{so}(n)$ and that these will be outer automorphisms (implemented by a matrix in $O(n)$ that does not belong to $SO(n)$) if p or q is odd but will be inner automorphisms (implemented by a matrix in $SO(n)$) if p and q are even. Thus it becomes clear that for $p \neq q$,

$$\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) \;=\; \left\{ \begin{array}{ccl} \{1\} & \text{if} & p \neq q,\, p \text{ or } q \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if} & p \neq q,\, p \text{ and } q \text{ even} \end{array} \right. ,$$

while for $p = q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ will contain an additional factor, corresponding to the possibility of switching factors in the tensor product that exists in this case. Explicitly, a representative of the corresponding connected component of H is given by the transformation

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ z = x \otimes y & \longmapsto & z^\tau = \pm\, y \otimes x \end{array} ,$$

[27] The argument can also be applied in the case of $\mathfrak{so}(4)$, even though its Dynkin diagram is not connected. In the case of $\mathfrak{so}(8)$, there are other non-trivial automorphisms of the Dynkin diagram, but these cannot be implemented by linear maps on $\mathbb{R}^8$.

which, by the same argument employed in the $SU(n)$ case, is an involution with determinant $(-1)^{p(p\mp 1)/2}$ and thus belongs to $SO(n)$, provided we make an adequate choice of sign, except when $p = 2 \bmod 4$. Therefore, we conclude that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ is equal to $\mathbb{Z}_2$ if p is odd, to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if $p = 2 \bmod 4$, to $(\mathbb{Z}_2 \times \mathbb{Z}_2) : \mathbb{Z}_2$ (or $\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$) if $p = 0 \bmod 4$ with $p > 4$ and to the full permutation group S_4 if $p = 4$ (this case is also covered in item 8 below). Note that, in the penultimate case, the structure of this group as a semi-direct product is given by conjugation with the transformation $.^\tau$ which maps $A \otimes B$ to $B \otimes A$ and thus acts on $\mathbb{Z}_2 \times \mathbb{Z}_2$ by switching the factors.

5. The inclusion

$$Sp(2p) \times_{\mathbb{Z}_2} Sp(2q) \ \subset \ SO(n) \qquad (n = 4pq)$$

is given by the tensor product of the defining representations of $Sp(2p)$ and $Sp(2q)$. Indeed, noting that the tensor product of two pseudo-real/quaternionic representations is real, we can proceed in the same way as in the previous item, though with some simplifications: the homomorphism

$$\begin{array}{ccc} Sp(2p) \times Sp(2q) & \longrightarrow & O(n) \\ (A, B) & \longmapsto & A \otimes B \end{array}$$

always has image contained in $G = SO(n)$ and the same kernel as before:

$$\{(1_{2p}, 1_{2q}), (-1_{2p}, -1_{2q})\} \ \cong \ \mathbb{Z}_2 \ .$$

Therefore, it provides the desired inclusion of the corresponding quotient group. Note also that the center $Z(G)$ of G is contained in H_0; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we note that for $p \neq q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = \{1\}$, while for $p = q$, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ can at most be equal to the group $\mathbb{Z}_2$, corresponding to the possibility of switching the factors in the tensor product that exists in this case. The argument to decide whether this switch operator has determinant $+1$ or -1 is the same as before and leads to the conclusion that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ is trivial if $p = q$ is odd and is equal to $\mathbb{Z}_2$ if $p = q$ is even.

6. Generalizing the procedure of item 1, we define the inclusion

$$SO(p) \times \ldots \times SO(p) \ \subset \ SO(n) \qquad (n = pl)$$

by the direct sum of l copies of the defining representation of $O(p)$, which realizes $O(p) \times \ldots \times O(p)$ by block diagonal matrices in $O(n)$, with l blocks of size p along the diagonal. The inverse image of $SO(n) \subset O(n)$ under this inclusion is the subgroup

$$H_+ \ = \ S\big(O(p) \times \ldots \times O(p)\big)$$

consisting of the block diagonal matrices belonging to $SO(n)$, which has 2^{l-1} connected components, its one-component being the subgroup

$$H_0 = SO(p) \times \ldots \times SO(p) .$$

As before, it follows from Lemma 7.6 that the centralizer $Z_G(H_0)$ of H_0 in G is contained in H_+ ; more precisely, we have $H_i = H_+$ if p is odd and $H_i = H_0$ if p is even. Similarly, we compute $H = N_G(H_0)$ by combining Lemma 7.6 and Lemma 8.2 to conclude that, as in the $SU(n)$ case, the elements h of H must be such that, when represented as block $(l \times l)$-matrices (with entries that are themselves $(p \times p)$-matrices), they contain precisely one nonvanishing entry in each line and each column: obviously, any such matrix h defines a permutation $\sigma(h)$ of $\{1, \ldots, l\}$. Thus we obtain a group homomorphism from H to the permutation group S_l which has kernel H_+ and is surjective (we can use the same argument as in item 1 above to show that its image contains all transpositions, their preimages being matrices whose square is -1_p in two of the l diagonal blocks and 1_p in the remaining $l-2$ ones, so these belong to H_+ but belong to H_0 only when p is even), so we may conclude that H/H_0 is an upwards extension of the group $\mathbb{Z}_2^{l-1}$ by the symmetric group S_l, which is a split extension, that is, a semi-direct product $H/H_0 = \mathbb{Z}_2^{l-1} : S_l$ (or $S_l \ltimes \mathbb{Z}_2^{l-1}$), if p is even and is a non-split extension $H/H_0 = \mathbb{Z}_2^{l-1} . S_l$ if p is odd.

7. Generalizing the procedure of item 4, we define the inclusions

$$\begin{array}{lll} SO(p) \times \ldots \times SO(p) \ \subset \ SO(n) & & (n = p^l \,,\ p \text{ odd}) \\ \big(SO(p) \times \ldots \times SO(p)\big)/\mathbb{Z}_2^{l-1} \ \subset \ SO(n) & & (n = p^l \,,\ p \text{ even}) \end{array}$$

by the tensor product of l copies of the defining representation of $O(p)$. Explicitly, they are induced by the homomorphism

$$\begin{array}{ccc} O(p) \times \ldots \times O(p) & \longrightarrow & O(n) \\ (A_1, \ldots, A_l) & \longmapsto & A_1 \otimes \ldots \otimes A_l \end{array}$$

which, once again, is not injective but rather has a non-trivial kernel given by

$$\{ (\epsilon_1 1_p \,, \ldots, \epsilon_l 1_p) \mid \epsilon_1, \ldots, \epsilon_l = \pm 1 \,,\ \epsilon_1 \ldots \epsilon_l = 1 \} \ \cong \ \mathbb{Z}_2^{l-1} .$$

Using the formula

$$\det\big(A_1 \otimes \ldots \otimes A_l\big) = \big(\det(A_1) \ldots \det(A_l)\big)^{p^{l-1}} ,$$

we see that the inverse image of $SO(n) \subset O(n)$ under this homomorphism is the subgroup

$$S'\big(O(p) \times \ldots \times O(p)\big) = \left\{ \begin{array}{ll} S\big(O(p) \times \ldots \times O(p)\big) & \text{if} \quad p \text{ odd} \\ O(p) \times \ldots \times O(p) & \text{if} \quad p \text{ even} \end{array} \right. .$$

In the first case, the group $S'\big(O(p) \times \ldots \times O(p)\big)$ has 2^{l-1} connected components such that the kernel of the homomorphism introduced above meets each of them in exactly one element, so the restriction of this homomorphism to the connected one-component provides the desired inclusion. In the second case, the group $S'\big(O(p) \times \ldots \times O(p)\big)$ has 2^l connected components, but the kernel of the homomorphism introduced above is contained in the connected one-component, so it is necessary to factor this out in order to obtain the following sequence of inclusions:

$$\big(SO(p) \times \ldots \times SO(p)\big)/\mathbb{Z}_2^{l-1} \ \subset\ \big(O(p) \times \ldots \times O(p)\big)/\mathbb{Z}_2^{l-1} \ \subset\ SO(n)\ .$$

Note also that the center $Z(G)$ of G is contained in H_0 ; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we note, as before, that every non-trivial automorphism of the Dynkin diagram Γ of $\mathfrak{so}(p) \times \ldots \times \mathfrak{so}(p)$ induced by non-trivial automorphisms of the Dynkin diagrams of the factors (of course these exist only when p is even) can be extended to an inner automorphism of $\mathfrak{so}(n)$. Moreover, $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ will contain a permutation group as an additional factor, corresponding to the possibility of switching the factors in the tensor product. But for $l \geqslant 3$, we can proceed as in the $SU(n)$ case to implement any transposition through an involution of determinant $+1$. Therefore we conclude that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ is equal to the permutation group S_l if p is odd, to $\mathbb{Z}_2^l : S_l$ (or $S_l \ltimes \mathbb{Z}_2^l$) if p is even with $p > 4$ and to the permutation group S_{2l} if $p = 4$ (this case is also covered in item 8 below). In the penultimate case, the structure of this group is that of a semi-direct product since S_l acts on $\mathbb{Z}_2^l$ by switching the factors.

8. Generalizing the procedure of item 5, we define the inclusion

$$\big(Sp(2p) \times \ldots \times Sp(2p)\big)/\mathbb{Z}_2^{l-1} \ \subset\ SO(n) \qquad (n = (2p)^l,\ l \text{ even})$$

by the tensor product of l copies of the defining representation of $Sp(2p)$. Indeed, noting that the tensor product of l copies of a pseudo-real/ quaternionic representation is real if l is even, we can proceed as in the previous item, though with some simplifications: the homomorphism

$$\begin{array}{ccc} Sp(2p) \times \ldots \times Sp(2p) & \longrightarrow & O(n) \\ (A_1, \ldots, A_l) & \longmapsto & A_1 \otimes \ldots \otimes A_l \end{array}$$

always has image contained in $G = SO(n)$ and has the same kernel as before:

$$\{\, (\epsilon_1 1_{2p}\,, \ldots, \epsilon_l 1_{2p}) \mid \epsilon_1, \ldots, \epsilon_l = \pm 1\,,\ \epsilon_1 \ldots \epsilon_l = 1 \,\} \ \cong\ \mathbb{Z}_2^{l-1}\ .$$

Therefore, it provides the required inclusion of the corresponding quotient group. Note also that the center $Z(G)$ of G is contained in H_0 ; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we note, as before, that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ can at most be equal to the symmetric group S_l, corresponding to the possibility of permuting the factors in the tensor product. But for $l \geqslant 3$, we can proceed as in the $SU(n)$ case to implement any transposition through an involution of determinant $+1$. Therefore, we conclude that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = S_l$.

connected component H_0	component group H/H_0		
$SO(p) \times SO(q)$ (reducible) (direct sum)	$\mathbb{Z}_2$	for	$n = p + q\,,\ p > q \geqslant 2$
	$\mathbb{Z}_4$	for	$n = p + q\,,\ p = q \geqslant 3$ odd
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	for	$n = p + q\,,\ p = q \geqslant 4$ even
$SO(4)$	$\mathbb{Z}_2$	for	$n = 5$
$U(p)$	$\{1\}$	for	$n = 2p\,,\ p \geqslant 3\,,\ p$ odd
	$\mathbb{Z}_2$	for	$n = 2p\,,\ p \geqslant 4\,,\ p$ even
$SO(p) \times SO(q)$ (irreducible) (tensor product)	$\{1\}$	for	$n = pq\,,\ p > q \geqslant 3$ p or q odd
	$\mathbb{Z}_2$	for	$n = pq\,,\ p = q \geqslant 3$ $p = q$ odd
$SO(p) \times_{\mathbb{Z}_2} SO(q)$ (irreducible) (tensor product)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	for	$n = pq\,,\ p > q \geqslant 4$ p and q even
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	for	$n = pq\,,\ p = q \geqslant 6$ $p = q = 2 \mod 4$
	$(\mathbb{Z}_2 \times \mathbb{Z}_2) : \mathbb{Z}_2$	for	$n = pq\,,\ p = q \geqslant 8$ $p = q = 0 \mod 4$
$Sp(2p) \times_{\mathbb{Z}_2} Sp(2q)$ (irreducible) (tensor product)	$\{1\}$	for	$n = 4pq\,,\ p > q \geqslant 1$
	$\{1\}$	for	$n = 4pq\,,\ p = q$ odd
	$\mathbb{Z}_2$	for	$n = 4pq\,,\ p = q$ even

Table 6: Non-simple maximal subgroups H of $SO(n)$ $(n \geqslant 5)$, part 1

8.3. Maximal subgroups of $Sp(n)$ (n even).

1. The inclusion

$$Sp(2p) \times Sp(2q) \ \subset \ Sp(n) \qquad (n = 2(p+q))$$

is given by the direct sum of the defining representations of $Sp(2p)$ and $Sp(2q)$. Again, it is obtained by taking block diagonal matrices, and the inverse image of $Sp(n)$ under this inclusion is the subgroup $H_0 = Sp(2p) \times Sp(2q)$ consisting of the block diagonal matrices belonging to $Sp(n)$, which is connected. The remainder of the argument is the same as in item 1 of the $SU(n)$ case, leading to the conclusion that $H = H_0$ if $p \neq q$ and $H/H_0 = \mathbb{Z}_2$ if $p = q$.

connected component H_0	component group H/H_0
$\prod_{k=1}^{l} SO(p)$ (reducible) (direct sum)	$\mathbb{Z}_2^{l-1}.\,S_l$ for $n = pl\,,\, l \geqslant 3\,,\, p \geqslant 3\,,\, p$ odd $\mathbb{Z}_2^{l-1} : S_l$ for $n = pl\,,\, l \geqslant 3\,,\, p \geqslant 2\,,\, p$ even
$\prod_{k=1}^{l} SO(p)$ (irreducible) (tensor product)	S_l for $n = p^l,\, l \geqslant 3\,,\, p \geqslant 3\,,\, p$ odd
$\prod_{k=1}^{l} SO(p)\,/\,\mathbb{Z}_2^{l-1}$ (irreducible) (tensor product)	$\mathbb{Z}_2^l : S_l$ for $n = p^l,\, l \geqslant 3\,,\, p \geqslant 6\,,\, p$ even
$\prod_{k=1}^{l} Sp\,(2p)\,/\,\mathbb{Z}_2^{l-1}$ (irreducible) (tensor product)	S_l for $n = (2p)^l,\, l \geqslant 4\,,\, l$ even, $p \geqslant 1$

Table 7: Non-simple maximal subgroups H of $SO(n)$ $(n \geqslant 5)$, part 2

2. The inclusion
$$U(p) \ \subset \ Sp\,(n) \qquad (n = 2p)$$
is given by
$$A + iB \ \longmapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$
Note that $\mathfrak{u}(p)$ has maximal rank in $\mathfrak{sp}(n)$ and hence $H_i = H_0$. Moreover, $\mathrm{Aut}(\Gamma) = Z_2$ where the non-trivial diagram automorphism is given by complex conjugation $A + iB \mapsto A - iB$, and this can be implemented by conjugating $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$ with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which belongs to $Sp\,(n)$. Therefore, $H/H_0 = \mathbb{Z}_2$.

3. The inclusions
$$Sp\,(2p) \times SO(q) \ \subset \ Sp\,(n) \qquad (n = 2pq\,,\ q \text{ odd})$$
$$Sp\,(2p) \times_{\mathbb{Z}_2} SO(q) \ \subset \ Sp\,(n) \qquad (n = 2pq\,,\ q \text{ even})$$
are given by the tensor product of the defining representations of $Sp\,(2p)$ and $O(q)$. Indeed, noting that the tensor product of a pseudo-real/quaternionic representation and a real representation is pseudo-real/quaternionic, we can

proceed in the same way as in items 4 and 5 of the $SO(n)$ case: the homomorphism

$$\begin{array}{ccc} Sp(2p) \times O(q) & \longrightarrow & Sp(n) \\ (A, B) & \longmapsto & A \otimes B \end{array}$$

always has image contained in $G = Sp(n)$ and has the same kernel as before:

$$\{(1_{2p}, 1_q), (-1_{2p}, -1_q)\} \cong \mathbb{Z}_2 .$$

On the other hand, the group $Sp(2p) \times O(q)$ has two connected components. If q is odd, the kernel of the above homomorphism meets both components in exactly one element (since $(1_{2p}, 1_q)$ and $(-1_{2p}, -1_q)$ belong to different connected components), so the restriction of this homomorphism to the connected one-component provides the desired inclusion. If q is even, the kernel of the above homomorphism is contained in the connected one-component (since $(1_{2p}, 1_q)$ and $(-1_{2p}, -1_q)$ belong to the same connected component), so it is necessary to factor this out in order to obtain the following sequence of inclusions:

$$Sp(2p) \times_{\mathbb{Z}_2} SO(q) \subset Sp(2p) \times_{\mathbb{Z}_2} O(q) \subset Sp(n) .$$

Note also that the center $Z(G)$ of G is contained in H_0 ; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we can proceed in the same way as in items 4 and 5 of the $SO(n)$ case to conclude that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g})$ is trivial if q is odd [28] and is equal to $\mathbb{Z}_2$ if q is even, except when $p = 1$ and $q = 4$, in which case it is equal to the full permutation group S_3 (this case is also covered in item 5 below).

4. Generalizing the procedure of the first item, we define the inclusion

$$Sp(2p) \times \ldots \times Sp(2p) \subset Sp(n) \qquad (n = 2pl)$$

by the direct sum of l copies of the defining representation of $Sp(2p)$. Again, it is obtained by taking block diagonal matrices, and the inverse image of $Sp(n)$ under this inclusion is the subgroup $H_0 = Sp(2p) \times \ldots \times Sp(2p)$ consisting of the block diagonal matrices belonging to $Sp(n)$, which is connected. The remainder of the argument is the same as in item 3 of the $SU(n)$ case, leading to the conclusion that $H/H_0 = S_l$.

5. Generalizing the procedure of previous items, we define the inclusion

$$\big(Sp(2p) \times \ldots \times Sp(2p)\big)/\mathbb{Z}_2^{l-1} \subset Sp(n) \qquad (n = (2p)^l,\ l \text{ odd})$$

by the tensor product of l copies of the defining representation of $Sp(2p)$.

[28] Here, we have to exclude the possibility of a switch operator between the tensor factors when $p = 2$ and $q = 5$, since $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$, but this follows from Lemma 7.7, taking into account that the two fundamental representations of this simple Lie algebra used to construct the embedding into $\mathfrak{sp}(20)$ are not quasiequivalent, since they have different dimensions (4 and 5, respectively).

connected component H_0	component group H/H_0		
$Sp(2p) \times Sp(2q)$ (reducible) (direct sum)	$\{1\}$ $\mathbb{Z}_2$	for for	$n = 2(p+q)\,,\ p > q \geqslant 1$ $n = 2(p+q)\,,\ p = q \geqslant 1$
$U(p)$	$\mathbb{Z}_2$	for	$n = 2p\,,\ p \geqslant 2$
$Sp(2p) \times SO(q)$ (irreducible) (tensor product)	$\{1\}$	for	$n = 2pq\,,\ p \geqslant 1\,,\ q \geqslant 3\,,\ q$ odd
$Sp(2p) \times_{\mathbb{Z}_2} SO(q)$ (irreducible) (tensor product)	$\mathbb{Z}_2$	for	$n = 2pq\,,\ p \geqslant 1\,,\ q \geqslant 4\,,\ q$ even except when $p = 1$ and $q = 4$
$\prod_{k=1}^{l} Sp(2p)$ (reducible) (direct sum)	S_l	for	$n = 2pl\,,\ l \geqslant 3\,,\ p \geqslant 1$
$\prod_{k=1}^{l} Sp(2p)\,/\,\mathbb{Z}_2^{l-1}$ (irreducible) (tensor product)	S_l	for	$n = (2p)^l\,,\ l \geqslant 3\,,\ l$ odd$, p \geqslant 1$

Table 8: Non-simple maximal subgroups H of $Sp(n)$ (n even, $n \geqslant 4$)

Indeed, noting that the tensor product of l copies of a pseudo-real/quaternionic representation is pseudo-real/quaternionic if l is odd, we can proceed as in item 8 of the $SO(n)$ case: the homomorphism

$$\begin{array}{ccc} Sp(2p) \times \ldots \times Sp(2p) & \longrightarrow & Sp(n) \\ (A_1, \ldots, A_l) & \longmapsto & A_1 \otimes \ldots \otimes A_l \end{array}$$

always has image contained in $G = Sp(n)$ and the same kernel as before:

$$\{\, (\epsilon_1 1_{2p}\,, \ldots, \epsilon_l 1_{2p}) \mid \epsilon_1, \ldots, \epsilon_l = \pm 1\,,\ \epsilon_1 \ldots \epsilon_l = 1 \,\} \;\cong\; \mathbb{Z}_2^{l-1}\ .$$

Therefore, it provides the desired inclusion of the corresponding quotient group. Note also that the center $Z(G)$ of G is contained in H_0 ; hence $H_i = H_0$.

In order to compute the normalizer H of the connected subgroup H_0 so defined, we note that $\mathrm{Aut}(\Gamma)\cap\mathrm{Inn}(\mathfrak{g})$ can at most be equal to the symmetric group S_l, corresponding to the possibility of switching the factors in the tensor product. But for $l \geqslant 3$, we can, once more, proceed as in the $SU(n)$ case to implement any transposition through an involution of determinant $+1$. Therefore, we conclude that $\mathrm{Aut}(\Gamma) \cap \mathrm{Inn}(\mathfrak{g}) = S_l$.

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