# New and improved results for packing identical unitary radius circles within triangles, rectangles and strips*

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**Abstract**

The focus of study in this paper is the class of packing problems. More specifically, it deals with the placement of a set of $N$ circular items of unitary radius inside an object with the aim of minimizing its dimensions. Differently shaped containers are considered, namely circles, squares, rectangles, strips and triangles. By means of the resolution of nonlinear equations systems through the Newton-Raphson method, the herein presented algorithm succeeds in improving the accuracy of previous results attained by continuous optimization approaches up to numerical machine precision. The computer implementation and the data sets are available at `http://www.ime.usp.br/~egbirgin/packing/`.

**Keywords:** packing, nonlinear equations system, Newton's method, nonlinear programming.

# 1 Introduction

Packing problems commonly arise in practical life. Hence, strategies capable of efficiently solving them are of great interest, not only from a purely mathematical but also from an economical standpoint. Some of the techniques available in the published literature consist of reasonably fast discrete heuristics [21, 22], but which are not guaranteed to converge to global optima. Others employ nonlinear models [5–7, 12–15, 17–20], which can be solved by nonlinear programming algorithms. Of special relevance to this paper, such iterative methods generate linearly convergent sequences to whose set of accumulation points the looked-for answer is expected to belong.

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In particular, in a recently published work [8], twice differentiable models for both two and three-dimensional packing problems were introduced and further solved with the aid of Algencan [1–4], a modern Augmented Lagrangian routine for optimization of smooth minimization problems with general constraints. Although feasible answers were obtained for all discussed cases, they were of poor precision in comparison to the known optimal results [16]. Motivated by these achievements, this study aspired to develop a method of *quadratic* convergence rate that would improve their accuracy and hopefully also provide optimal solutions not yet reported in the literature.

This paper is organized as follows. In Section 2 the problem is formally stated and the proposed approach is fully described. Section 3 details the most significant challenges faced in its implementation. In Section 4 the numerical experiments are delivered. Finally, Section 5 concludes the paper.

## 2 Nonlinear model and suggested approach

The problem at hand is that of packing a set of $N$ identical circles of radius $r = 1$ (hereinafter called *items*) in a fixed-shaped figure (denominated *object*) while minimizing the latter's dimensions. Throughout this work, several geometric forms were treated, namely circles, squares, rectangles, triangles and strips, defined as a rectangle that has got one of its dimensions fixed.

In order for a setup to be accepted as valid, all items must obviously not overlap or violate the object's boundaries. Being solely dependent upon items' positions, the non-overlapping condition can be expressed, irrespective of the object's form, as:

$$\sqrt{(c_i^x - c_j^x)^2 + (c_i^y - c_j^y)^2} \geq r_i + r_j \text{ for all } i \neq j \tag{2.1}$$

where $c_i = (c_i^x, c_i^y)$ and $r_i$ denote the $i$-th item's centre and radius, respectively, for $1 \leq i \leq N$.

On the other hand, both the objective function and the non-boundaries-violation constraints vary according to the object and therefore must be individually studied.

**Circular object** Without loss of generality, assume the object's centre $C$ to be located at the coordinate system's origin and let $R$ represent its radius. It may be readily seen that all items will be completely contained within the object's boundaries if and only if the inequality $R \geq r_i + \sqrt{(c_i^x)^2 + (c_i^y)^2}$ holds for every $i = 1, \ldots, N$. Thus, the following nonlinear model can be written:

$$\begin{aligned} &\text{minimize } R \\ &\text{subject to } (c_i^x)^2 + (c_i^y)^2 \leq (R - r_i)^2 \text{ for all } i \\ &\qquad\qquad \text{non-overlapping constraints (2.1)} \end{aligned} \tag{2.2}$$

**Squared object** Under the assumption that the coordinate system's origin coincides with the lower left-hand vertex of the square, the nonlinear model results naturally:

$$\begin{aligned} &\text{minimize } L \\ &\text{subject to } r_i \le c_i^x \le L - r_i \text{ for all } i \\ &\qquad\qquad r_i \le c_i^y \le L - r_i \text{ for all } i \\ &\qquad\qquad \text{non-overlapping constraints (2.1)} \end{aligned} \tag{2.3}$$

**Strip object** Indicating by $W$ the variable width of an strip with fixed height $L$ and asserting as true the same assumption concerning the system's origin, one can derive the model below:

$$\begin{aligned} &\text{minimize } W \\ &\text{subject to } r_i \le c_i^x \le L - r_i \text{ for all } i \\ &\qquad\qquad r_i \le c_i^y \le W - r_i \text{ for all } i \\ &\qquad\qquad \text{non-overlapping constraints (2.1)} \end{aligned} \tag{2.4}$$

**Rectangular object** Based on whether the objective function is intended to minimize the rectangle's perimeter or its area, two different and equally interesting models may be formulated:

$$\begin{aligned} &\text{minimize } L + W \text{ or } L \times W \\ &\text{subject to } r_i \le c_i^x \le L - r_i \text{ for all } i \\ &\qquad\qquad r_i \le c_i^y \le W - r_i \text{ for all } i \\ &\qquad\qquad \text{non-overlapping constraints (2.1)} \end{aligned} \tag{2.5}$$

**Triangular object** For the (equilateral) triangle of side length $L$, the coordinate system's origin is taken as the base's midpoint:

$$\begin{aligned} &\text{minimize } L \\ &\text{subject to } r_i \le c_i^y \text{ for all } i \\ &\qquad\qquad 2\sqrt{3}c_i^y - 6c_i^x \le 3L - 4\sqrt{3}r_i \text{ for all } i \\ &\qquad\qquad 2\sqrt{3}c_i^y + 6c_i^x \le 3L - 4\sqrt{3}r_i \text{ for all } i \\ &\qquad\qquad \text{non-overlapping constraints (2.1)} \end{aligned} \tag{2.6}$$

The explicit resolution of problems (2.2) through (2.6) by utilizing an Augmented Lagrangian nonlinear solver was the technique attempted in [8]. Two main concerns were raised by the paper authors: (i) the quadratic relation between the number of items and the number of constraints, which makes their evaluation a costly task, and (ii) the difficulty of achieving high precision results with the employed routine.

The central idea of this study stems from the observation that in an optimal configuration (i.e. one that realizes the global minimum of the above stated optimization problems) a number of items are placed in contact with each other and with the object's boundaries (see Figure 1), making active the matching constraints. Consequently, if those contacts were known a priori, an overdetermined system of nonlinear equations could then be constructed, to whose solution set an optimal arrangement must belong. Moreover, it will be shown that the number of such equations is linear with respect to the number of items (in contrast to the quadratic number of constraints in the nonlinear optimization models).

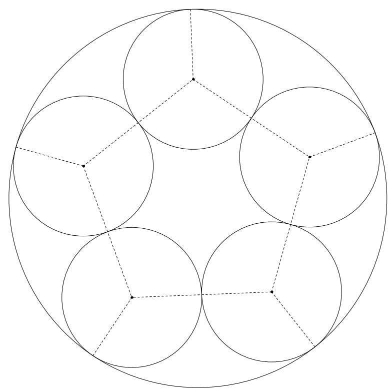


Figure 1: Optimal setup of 5 items in a circle

More formally, let $\Phi$ and $\Psi$ designate the supposedly known sets of contacts, in an optimal solution, of items with the object and with other items, respectively.

$$\begin{aligned} i \in \Phi &\iff \text{item } i \text{ makes contact with object,} \\ (i,j) \in \Psi &\iff \text{item } i \text{ makes contact with item } j. \end{aligned}$$

Besides, allow $G$ to be the undirected graph whose vertices are the items' centres and such that two vertices are adjacent if and only if the corresponding items are in contact. Note that, since $G$ is planar (i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints; see Figure 1), it holds true that $|\Psi| = O(N)$ (see, for example, [9]). (The same conclusion stems from the fact that each circular item cannot touch more than six others, so that $6N$ serves as an upper bound on the number of such contacts.) Likewise, $|\Phi| \leq N$.

Now, consider a nonlinear system $F : \mathbb{R}^n \to \mathbb{R}^m$ where the number of equations $m$ equals the number of contacts — $|\Phi| + |\Psi|$, already known to belong to $O(N)$ — and the number of variables $n$ may assume one of the values below (coordinates $(x, y)$ of each item's centre plus the number of variable dimensions of the containing object):

$$n = \begin{cases} 2N + 1 & \text{for the circle, square, strip and triangle,} \\ 2N + 2 & \text{for the rectangle.} \end{cases}$$

To each $(i,j) \in \Psi$ corresponds an active constraint $\sqrt{(c_i^x - c_j^x)^2 + (c_i^y - c_j^y)^2} = 2r$ or, equivalently, $(c_i^x - c_j^x)^2 + (c_i^y - c_j^y)^2 = (2r)^2$. This translates into the addition of the following equation to the system $F$:

$$f_{ij}^{\psi}(\cdot) = (c_i^x - c_j^x)^2 + (c_i^y - c_j^y)^2 - 4r^2. \tag{2.7}$$

Clearly, the equation $f_i^{\phi}(\cdot)$ that shall be included in $F$ for each $i \in \Phi$ depends on the form of the object. In the circular case, for instance, to each $i \in \Phi$ corresponds an active constraint $R = r_i + \sqrt{(c_i^x)^2 + (c_i^y)^2}$ or, equivalently, $(R - r_i)^2 = (c_i^x)^2 + (c_i^y)^2$. This translates into the addition of the following equation to the system $F$:

$$f_i^{\phi}(\cdot) = (R - r_i)^2 - (c_i^x)^2 + (c_i^y)^2. \tag{2.8}$$

The analogous procedure for differently shaped containers, being trivially deducible from the appropriate nonlinear problem, will be omitted herein for the sake of brevity.

It should be noted, however, that while overdetermined (as a rule, $m \gg n$), the system deduced above will be compatible as long as the contacts are assumed to be known in advance. That is because many of the equations it comprises are redundant. This observation is

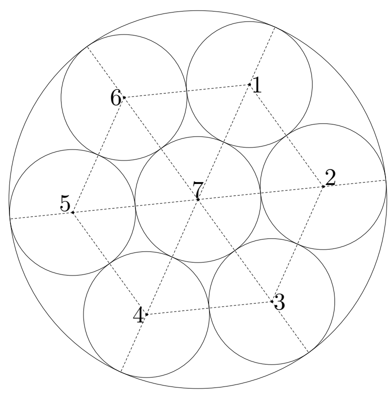



Figure 2: Optimal configuration for 7 items in a circle (overdetermined compatible system with 15 variables and 18 equations)

illustrated by Figure 2, where the central item's position is uniquely dictated, for example, by its contacts (depicted by dashed lines) with items 2 and 5, rendering the equations relative to its contacts with other surrounding items superfluous.

One critical question that arises from this strategy is how the information about the contacts made in an optimal solution could be learned. It occurs that, by way of a straightforward analysis of the poor-precision answers found by [8], such knowledge may be acquired to a high degree of confidence. For that purpose, it must be adopted a value $\varepsilon \in \mathbb{R}_+$ amounting to the minimum distance between the border of two items that should not be regarded as adjacent and the inequality below has to be tested for all $i, j$ satisfying $1 \le i < j \le N$:

$$\sqrt{(c_i^x - c_j^x)^2 + (c_i^y - c_j^y)^2} - 2r \le \varepsilon, \tag{2.9}$$

where the values of $c_i$ and $c_j$ are taken from the output of [8]. If it holds, then the related equation $f^{\psi}_{ij}$ is incorporated into the nonlinear system $F$. Similarly, the equations $f^{\phi}_i$ are originated by an analogous mechanism conducted for the detection of contacts between each item and the object.

Bear in mind that the value of $\varepsilon$ is crucial to the success of the method and cannot be dissociated from the quality of the answers given by ALGENCAN. In such configuration, three are the possibilities for each pair of items: (i) they overlap, (ii) they do not overlap and are far away from each other, or (iii) they do not overlap but are very close to each other. Hence, considering a poor-precision result acquired via the resolution of the corresponding nonlinear model (2.2)–(2.6), contacts between pairs of items will be forced in cases (i) and (iii). In order to distinguish between cases (ii) and (iii), $\varepsilon$ plays a vital role.

Let $\omega$ be the maximum overlapping between any pair of items in a solution to the NLP model. If $\omega$ is such that it were considered a "reasonable" overlapping for the underlying packing problem by the employed solver, then it is safe to assume that any pair of items that do not overlap but whose borders are at a distance less than $\omega$ may be in contact. In this manner, $\omega$ is a satisfactory initial candidate for $\varepsilon$.

Starting from this knowledge, as a large number of optimal solutions for packing problems within circles and squares had already been made publicly available, the value of $\varepsilon$ was empirically adjusted around $\omega$ in a way that would correctly identify the contacts in those instances.

Once this task has been accomplished, a root of $F$ is looked for utilizing the Newton-Raphson method. The rationale behind this choice is the expectation that, if initially fed with a guess $x^{(0)}$ sufficiently close to the true solution $x^*$, the algorithm will produce a sequence $\{x^{(k)}\}$ quadratically convergent to $x^*$ (see, for example, [10]). The equations that describe the iterative process are:

$$\begin{aligned} J_F(x^{(k)})d &= -F(x^{(k)}) \\ x^{(k+1)} &= x^{(k)} + d, \end{aligned} \tag{2.10}$$

where $J_F$ is the Jacobian matrix of $F$.

We consider two different ways of solving the overdetermined (albeit compatible, as long as all contacts have been detected correctly) linear system (2.10):

**QR decomposition of the Jacobian matrix** Due to the absolute lack of information on the rank of $J_F(x^{(k)})$, a variant of the $QR$ method, the $QR$ decomposition with column pivoting [11], is calculated. Thanks to its highly desirable numerical stability characteristics when $J_F(x^{(k)})$ is not well conditioned, this is the strategy of choice in the algorithm developed.

**Cholesky's method applied to the normal equations** In spite of its inferior numerical properties, the normal equations approach has been more successful in computing $x^{(k+1)}$ whenever $J_F(x^{(k)})$ is found to be rank deficient. Such phenomenon may be justified by the realization that, in this situation, there are infinitely many solutions to (2.10), among which only one interest us — the one that minimizes the object's dimensions.

For that reason, a sensible approach is to solve the least squares problem, with the new linear system becoming:

$$J_F^T(x^{(k)})J_F(x^{(k)})d = -J_F^T(x^{(k)})F(x^{(k)}) \tag{2.11}$$

It should be remarked that $M = J_F^T(x^{(k)})J_F(x^{(k)}) \in \mathbb{R}^{n\times n}$ is both symmetric and positive semidefinite. More importantly, it is singular, seeing that $\mathrm{rank}(M) = \mathrm{rank}(J_F(x^{(k)}))$ and also that this path is used only when $\mathrm{rank}(J_F(x^{(k)})) < n$. For that reason, to solve the linear system (2.11), the Modified Cholesky decomposition [10] is preferred, which actually factorizes a slight perturbation of the matrix $J_F^T(x^{(k)})J_F(x^{(k)})$. The vector $d$ easily follows by forward and back substitution.

After each iteration, Newton's method is checked for convergence, which is characterized by the verification of $x^{(k+1)} = x^{(k)}$ and constitutes the main stopping criterion. Still, if the initial value $x^{(0)}$ is too far from the true zero, the method might fail to converge, and a cap on the number of iterates is made necessary as a secondary stopping criterion.

# 3 Implementation aspects

In this section the most important practical implementation details are discussed.

## 3.1 System indetermination

Contrary to the logical intuition that the constructed nonlinear system would be typically overdetermined, its indetermination was of one the earliest challenges that had to be coped with. A plausible explanation is the fact that, even on an optimal configuration, there can be items taking part in less than two contacts, thus contributing with the addition of more variables than equations to the system (causing it to become undetermined). Those which exhibit such attribute are named *loose items* (see Figure 3(a)).

As a means to overcome this obstacle, a preprocessing routine detects and temporarily removes all loose items from the set to be packed. The proposed technique is then ordinarily carried out for the remaining items and only when their optimal placement is determined (see Figure 3(b)) are the "rattlers" reintroduced into the problem.

Such job may be thought of as a further optimization problem, just with a largely decreased number of variables (proportional to the number of loose items). Let $L \subseteq \{1, \ldots, N\}$ be the set of indices of loose items and $l$ its cardinality. The variables of the aforementioned optimization problem are the centre $c_i \in \mathbb{R}^2$ and the radius $r_i \in \mathbb{R}$ for all $i \in L$. The underneath model therefore follows:

$$\begin{aligned}
\text{maximize } & D \\
\text{subject to } & r_i \geq D \text{ for all } i \in L && (3.1)\\
& d(c_i, c_j)^2 \geq (r_i + r_j)^2 \text{ for all } i, j \in L,\, i \neq j && (3.2)\\
& d(c_i, c_j)^2 \geq (r_i + r)^2 \text{ for all } i \in L, j \notin L && (3.3)\\
& \text{corresponding non-violation constraints (see Section 2)}
\end{aligned}$$

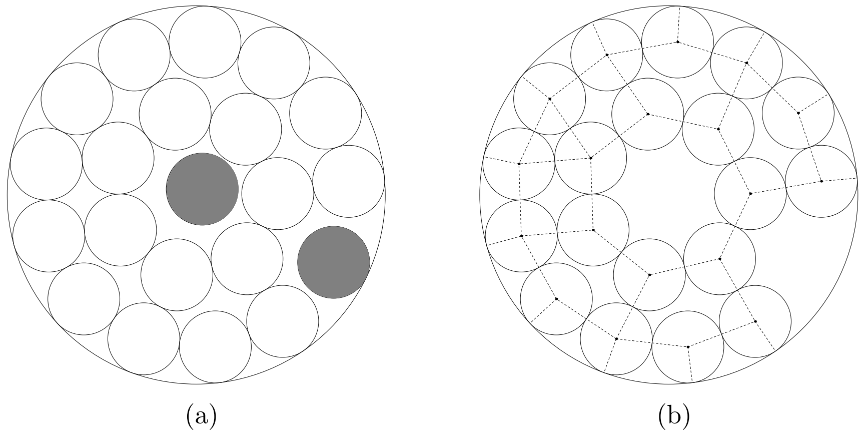


Figure 3: Example of arrangement exhibiting system indetermination: (a) greyed items making less contacts (0 and 1) than the number of equations they introduce; (b) optimal layout after removing those items.

where $d(\cdot,\cdot)$ stands for the Euclidean distance and the values of $r$ and $c_j$ for each $j \notin L$ are considered constant and are taken from the output of the method for the contracted set (see Figure 3(b)).

By solving it with ALGENCAN, two are the possible outcomes: either an optimal packing of the original set is found (see Figure 4) or, should the reincorporation of the once withdrawn items fail, it can be concluded that the contacts have been erroneously detected in the first place and that the method must be re-executed with a more properly estimated $\varepsilon$ parameter.

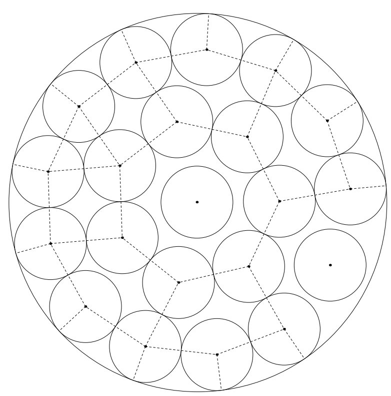
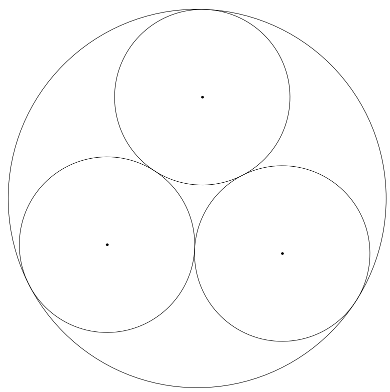


Figure 4: Layout after reintegration of loose items with the loose items centrally placed in the room available

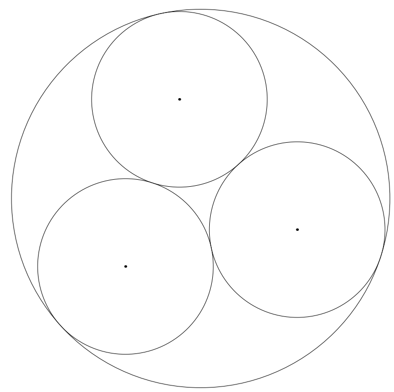
If the optimal value $D^*$ obtained is such that $D^* \geq r$, then all loose items have been fit into the object. It should also be noted that whenever $D^* > r$, their radius is taken as being equal to $r$ and, as a result, they will be centrally placed in the room available.

## 3.2 Loss of convergence

It has been verified that, for circular objects, there usually exists a neighbourhood of arbitrarily close optimal solutions, derived from the mere rotation of the whole set of items in the interior of the object (see Figure 5).



(a)



(b)

Figure 5: Distinct optimal setups for 3 items in a circle, where (b) is attainable by rotating the items in (a).

Since it severely impairs the convergence of Newton's method, such ill behaviour had to be avoided by selecting one of the items to have one of its centre's coordinates unchanged by the algorithm. Unfortunately, though, none of the heuristics assessed for the selection of that item showed themselves to be consistently satisfactory. Because of that, each of the $n - l$ non-loose items is successively iterated to assume this role and the best solution gets saved.

## 3.3 Post-optimization overlapping and violation elimination

In order to guarantee that the given answers are eligible for comparison to the best ever published [16], it is mandatory to first eliminate any overlapping or violations that might remain. In other words, it should be guaranteed that the arrangements associated with them are correct under the analytical rigour. The accomplishment of this requirement is achieved by subjecting the solution to the nonlinear system of equations given by Newton's method to scaling.

Taking the items in pairs, the distance between their centres is evaluated. Let $\delta$ be the minimum of all such values. If $\delta \geq 2r$, then there are no overlapping constraints being disobeyed and no adjustments to be made. On the other hand, if $\delta < 2r$, then the items must be spread out so that $\delta \geq 2r$ holds. A new placement such that $d(c'_i, c'_j) \geq 2r$ for every $i \neq j$ can be attained by simply making

$$c'_i = \frac{2r}{\delta} \cdot c_i \tag{3.4}$$

for each item $i$.

In fact, since it holds true that $d(c_i, c_j) \geq \delta$ for every $i \neq j$, it easily follows that

$$d(c_i', c_j') = \frac{2r}{\delta} \cdot d(c_i, c_j) \geq \frac{2r}{\delta} \cdot \delta = 2r, \tag{3.5}$$

which is the intended result.

As for the eventual violation of the object's boundaries, there is no option other than enlarging the object's dimensions until all items are entirely contained within its boundaries. For the circular case, for instance, it suffices to make

$$R' = \max_{\{i=1,\dots,N\}} \sqrt{(c_i^x)^2 + (c_i^y)^2} + r. \tag{3.6}$$

After those post-optimization corrections have been made, it is safe to assert that a solution with the maximum machine precision has been found. It finally turns out practicable to draw all the desired comparisons, which are the subject of the next section.

To end this section, Algorithm 1 provides an overview of the methods introduced for the circular case.

**Algorithm 1** Pseudocode for the circular case

**Require:** $N :=$ number of items
**Require:** $r :=$ item's radius (if other than 1.0)
**Require:** $T :=$ maximum processing time allotted
**Require:** $K :=$ cap on the number of Newton steps

1: $R^* \leftarrow +\infty$
2: **while** elapsed time $\leq T$ **do**
3: Run ALGENCAN with its default parameters to solve the nonlinear model (2.2) with feasibility and optimality tolerances equal to $10^{-4}$ and consider its answer $(x_{\text{NLP}}, R_{\text{NLP}})$ as an initial guess for Newton's method.
4: Temporarily remove all loose items, as detailed in Subsection 3.1, and redefine $(x_{\text{NLP}}, R_{\text{NLP}})$ accordingly.
5: Detect all contacts between items and with the object, as explained in Section 2, and construct an appropriate system of nonlinear equations $F$.
6: $\hat{R} \leftarrow +\infty$
7: **for all** non-loose items $i$ **do**
8: $k \leftarrow 0$, $(x^{(0)}, R^{(0)}) \leftarrow (x_{\text{NLP}}, R_{\text{NLP}})$
9: Remove variable $c_i^x$ from $(x^{(0)}, R^{(0)})$ and replace it in $F$ with its value from $(x_{\text{NLP}}, R_{\text{NLP}})$.
10: **while** $k < K$ **do**
11: $k \leftarrow k + 1$
12: Solve the Newtonian system (2.10) for $F$ and obtain $(x^{(k)}, R^{(k)})$.
13: **if** $(x^{(k)}, R^{(k)}) = (x^{(k-1)}, R^{(k-1)})$ **then**
14: Break. {Newton's method converged}
15: **end if**
16: **end while**
17: Reintroduce variable $c_i^x$ to $(x^{(0)}, R^{(0)})$, taking its value from $(x_{\text{NLP}}, R_{\text{NLP}})$.
18: Update the best computed result $(\hat{x}, \hat{R})$ (i.e. if $R^{(k)} < \hat{R}$).
19: **end for**
20: Reintroduce all loose items previously removed, as explained in Subsection 3.1.
21: Evaluate the minimum distance $\delta$ between the centres of each pair of items.
22: **if** $\delta < 2r$ **then**
23: **for all** items $i$ **do**
24: Redefine $(\hat{c}_i^x, \hat{c}_i^y)$ as $2r/\delta \cdot (\hat{c}_i^x, \hat{c}_i^y)$.
25: **end for**
26: Redefine $\hat{R}$ as $\max_{\{i=1,\ldots,N\}} \sqrt{(\hat{c}_i^x)^2 + (\hat{c}_i^y)^2} + r$.
27: **end if**
28: Update the best computed result $(x^*, R^*)$ (i.e. if $\hat{R} < R^*$).
29: **end while**
30: **return** $(x^*, R^*)$

**Remark.** The most noteworthy difference between Algorithm 1 and the method developed for differently shaped containers is the absence of lines 6, 7, 18 and 19 in the latter variant, since there is no neighbourhood of arbitrarily close optimal solutions to be tackled in the first place.

# 4 Numerical results

All tests were conducted on a 2.4GHz Intel Core 2 Quad with 4GB of RAM memory and running GNU/Linux operating system. The code, fully written in Fortran 77, was compiled by the G77 Fortran compiler of GCC with the `-O3` optimization directive enabled. The values of $T$ and $K$ that yielded the results presented later in this section are 5h and 1000, respectively.

We solved instances with the number of items varying from 1 to 50. Tables 1 and 2 show the resulting values for the circular object's radius and the squared object's side, respectively, and their confrontation with the best ever reported [16]. Their first column holds the number of items; the second, the solution found by the developed method; the third, the reference values from [16]; the fourth, the difference between them; the fifth, the elapsed CPU time (in seconds).

It should be noticed that the answers obtained coincide with the values of reference. More precisely, in 48 of the cases of packing in a circle and in 44 of those of packing in a square, the results matched to all decimal places, i.e. up to the machine precision, with an absolute error of less than $10^{-16}$ for circles and $10^{-12}$ for squares (the difference being due to the number of digits in the data sets available in [16]). In the other 8 instances where maximal precision has not been achieved, the absolute error is always of the order of $10^{-6}$.

Tables 3 and 4 present unpublished results for the packing problems of minimizing the area and the perimeter of an enclosing rectangle, while Table 5 exhibits unprecedented solutions for the area minimization of triangular and strip objects. Lastly, Figure 6 illustrates a few selected solutions.

# 5 Concluding remarks

This study addressed itself to the problem of packing unitary radius circles within differently shaped containers with the aim of minimizing its dimensions. The approach developed builds upon approximate solutions provided by continuous optimization techniques formerly developed. By pursuing the zero of a nonlinear equations system properly deduced from the contacts established in a candidate solution, refinement of those approximate results up to the machine precision were made possible.

For all studied problems, feasible solutions comparable to the best results already known were achieved. However, the treatment of packing problems in triangles, rectangles and strips, whose answers had not been published in the literature before, can be regarded as an even more remarkable contribution.

The Fortran 77 source code of the routines implemented during this work and a complete description of all solutions, each of which being composed of its items' positions and a graphical representation of the contacts they make, are available at `http://www.ime.usp.br/~egbirgin/packing/`, as well as the best results given by them.

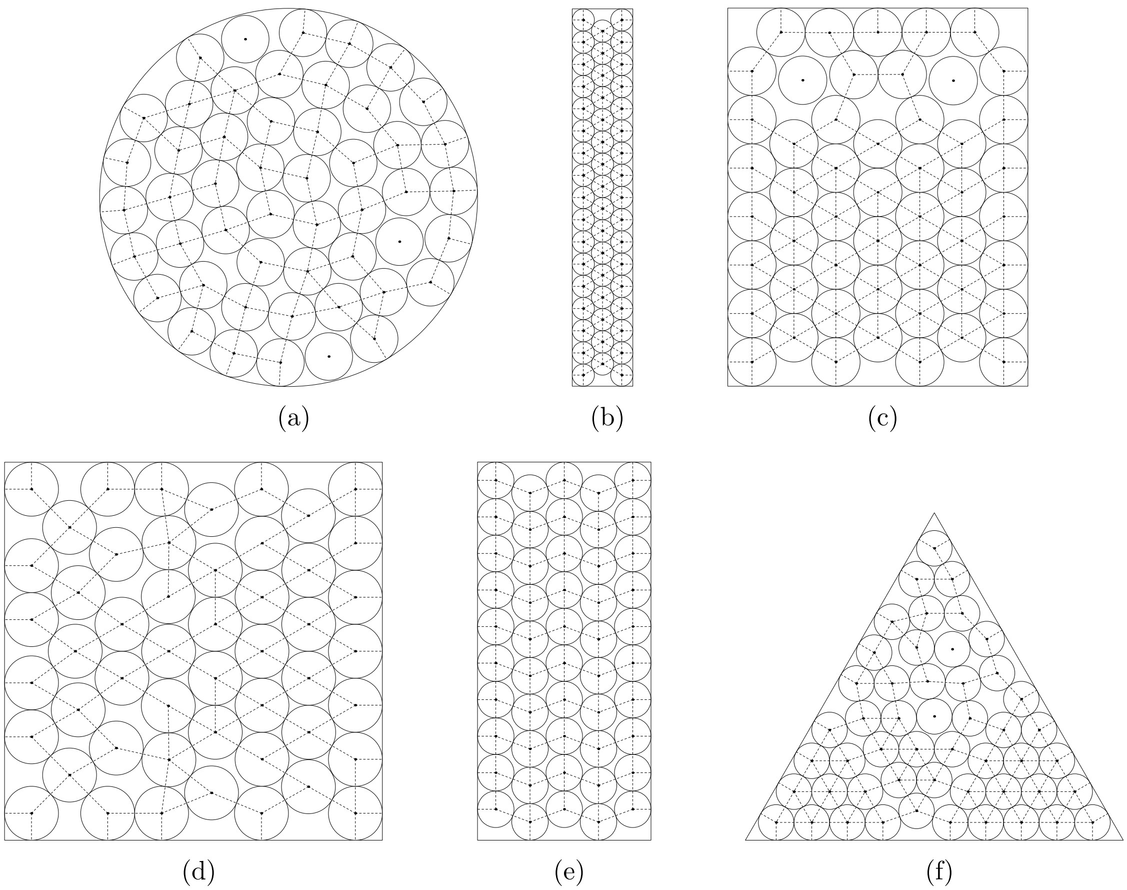


Figure 6: Optimal layouts for 50 items in a: (a) circle; (b) rectangle, minimizing its area; (c) rectangle, minimizing its perimeter; (d) square; (e) strip, with fixed length $L = 9.5$; and (f) equilateral triangle.

| # of items | Circle radius | Reference | Difference | Time |
|---|---|---|---|---|
| 1 | 1.0000000000000000 | 1.0000000000000000 | 0.0E+00 | 0.00 |
| 2 | 2.0000000000000000 | 2.0000000000000000 | 0.0E+00 | 0.11 |
| 3 | 2.1547005383792515 | 2.1547005383792515 | 0.0E+00 | 0.00 |
| 4 | 2.4142135623730949 | 2.4142135623730949 | 0.0E+00 | 0.00 |
| 5 | 2.7013016167040798 | 2.7013016167040798 | 0.0E+00 | 0.09 |
| 6 | 3.0000000000000000 | 3.0000000000000000 | 0.0E+00 | 0.10 |
| 7 | 3.0000000000000000 | 3.0000000000000000 | 0.0E+00 | 0.00 |
| 8 | 3.3047648709624866 | 3.3047648709624866 | 0.0E+00 | 0.05 |
| 9 | 3.6131259297527527 | 3.6131259297527532 | -4.4E−16 | 271.82 |
| 10 | 3.8130256313981246 | 3.8130256313981246 | 0.0E+00 | 0.13 |
| 11 | 3.9238044001630872 | 3.9238044001630872 | 0.0E+00 | 29.90 |
| 12 | 4.0296019301161836 | 4.0296019301161836 | 0.0E+00 | 16.43 |
| 13 | 4.2360679774997898 | 4.2360679774997898 | 0.0E+00 | 0.90 |
| 14 | 4.3284285548608370 | 4.3284285548608370 | 0.0E+00 | 9.67 |
| 15 | 4.5213569647061647 | 4.5213569647061647 | 0.0E+00 | 28.79 |
| 16 | 4.6154255948731944 | 4.6154255948731944 | 0.0E+00 | 72.64 |
| 17 | 4.7920337483105797 | 4.7920337483105788 | 8.9E−16 | 6.42 |
| 18 | 4.8637033051562728 | 4.8637033051562728 | 0.0E+00 | 9.62 |
| 19 | 4.8637033051562728 | 4.8637033051562728 | 0.0E+00 | 12.20 |
| 20 | 5.1223207369915293 | 5.1223207369915285 | 8.9E−16 | 8.94 |
| 21 | 5.2523174750102433 | 5.2523174750102424 | 8.9E−16 | 36.89 |
| 22 | 5.4397189590722146 | 5.4397189590722137 | 8.9E−16 | 86.94 |
| 23 | 5.5452042225748590 | 5.5452042225748581 | 8.9E−16 | 147.69 |
| 24 | 5.6516610917654377 | 5.6516610917654369 | 8.9E−16 | 29.95 |
| 25 | 5.7528243308571163 | 5.7528243308571154 | 8.9E−16 | 464.85 |
| 26 | 5.8281765369429506 | 5.8281765369429497 | 8.9E−16 | 44.52 |
| 27 | 5.9063978473941567 | 5.9063978473941550 | 1.8E−15 | 2535.80 |
| 28 | 6.0149380973715152 | 6.0149380973715125 | 2.7E−15 | 14142.00 |
| 29 | 6.1385979039781473 | 6.1385979039781455 | 1.8E−15 | 100.01 |
| 30 | 6.1977410708792275 | 6.1977410708792258 | 1.8E−15 | 267.66 |
| 31 | 6.2915026221291814 | 6.2915026221291814 | 0.0E+00 | 71.14 |
| 32 | 6.4294629709501150 | 6.4294629709501132 | 1.8E−15 | 790.47 |
| 33 | 6.4867381281037089 | 6.4867031235604333 | 3.5E−05 | 14955.90 |
| 34 | 6.6109570900010040 | 6.6109570900010031 | 8.9E−16 | 13744.63 |
| 35 | 6.6971710917902456 | 6.6971710917902438 | 1.8E−15 | 3260.99 |
| 36 | 6.7467537934241761 | 6.7467537934241752 | 8.9E−16 | 259.48 |
| 37 | 6.7587704831436346 | 6.7587704831436337 | 8.9E−16 | 1092.96 |
| 38 | 6.9618869652281514 | 6.9618869652281488 | 2.7E−15 | 641.64 |
| 39 | 7.0578841626240081 | 7.0578841626240045 | 3.6E−15 | 245.79 |
| 40 | 7.1238464359431282 | 7.1238464359431246 | 3.6E−15 | 273.25 |
| 41 | 7.2600123286770479 | 7.2600123286770462 | 1.8E−15 | 637.95 |
| 42 | 7.3469494647914715 | 7.3467964069427687 | 1.5E−04 | 5287.91 |
| 43 | 7.4199448563412131 | 7.4199448563412114 | 1.8E−15 | 3610.53 |
| 44 | 7.4980366829952523 | 7.4980366829952487 | 3.6E−15 | 1232.29 |
| 45 | 7.5729123263675255 | 7.5729123263675211 | 4.4E−15 | 3264.80 |
| 46 | 7.6501799146936715 | 7.6501799146936680 | 3.6E−15 | 856.59 |
| 47 | 7.7241700525980201 | 7.7241700525980184 | 1.8E−15 | 2506.63 |
| 48 | 7.7912714305586714 | 7.7912714305586679 | 3.6E−15 | 12961.80 |
| 49 | 7.8868709588028691 | 7.8868709588028647 | 4.4E−15 | 14312.55 |
| 50 | 7.9475152747835143 | 7.9475152747835107 | 3.6E−15 | 1605.08 |

Table 1: Optimal results for the packing of unitary circles in a circle

| # of items | Square side | Reference | Difference | Time |
|---|---|---|---|---|
| 1 | 2.0000000000000000 | 2.0000000000000000 | 0.0E+00 | 0.00 |
| 2 | 3.4142135623730949 | 3.4142135623783694 | -5.3E−12 | 0.00 |
| 3 | 3.9318516525781364 | 3.9318516525819986 | -3.9E−12 | 0.01 |
| 4 | 4.0000000000000000 | 4.0000000000000000 | 0.0E+00 | 0.00 |
| 5 | 4.8284271247461898 | 4.8284271247356418 | 1.1E−11 | 0.01 |
| 6 | 5.3282011773513762 | 5.3282011773649129 | -1.4E−11 | 0.48 |
| 7 | 5.7320508075688776 | 5.7320508075691876 | -3.1E−13 | 2.29 |
| 8 | 5.8637033051562746 | 5.8637033051581451 | -1.9E−12 | 4.56 |
| 9 | 6.0000000000000000 | 5.9999999999879998 | 1.2E−11 | 0.69 |
| 10 | 6.7474415232381135 | 6.7474415232485301 | -1.0E−11 | 43.44 |
| 11 | 7.0225095034303822 | 7.0225095034205376 | 9.8E−12 | 456.84 |
| 12 | 7.1449575542752672 | 7.1449575542971164 | -2.2E−11 | 126.30 |
| 13 | 7.4631768820241113 | 7.4630478288597386 | 1.3E−04 | 2.96 |
| 14 | 7.7320508075688776 | 7.7320508075709107 | -2.0E−12 | 13.44 |
| 15 | 7.8637033051562764 | 7.8637033051639973 | -7.7E−12 | 135.58 |
| 16 | 8.0000000000000000 | 8.0000000000000000 | 0.0E+00 | 0.00 |
| 17 | 8.5326603474980978 | 8.5326603474943603 | 3.7E−12 | 109.89 |
| 18 | 8.6564023547027524 | 8.6564023547027134 | 3.9E−14 | 13.93 |
| 19 | 8.9074609393260822 | 8.9074609393257855 | 3.0E−13 | 5.78 |
| 20 | 8.9780833528217379 | 8.9780833528604074 | -3.9E−11 | 5.56 |
| 21 | 9.3580199588727577 | 9.3580199588783994 | -5.6E−12 | 758.56 |
| 22 | 9.4639295431339381 | 9.4638450909735710 | 8.4E−05 | 12.73 |
| 23 | 9.7274066103125492 | 9.7274066102921175 | 2.0E−11 | 4447.51 |
| 24 | 9.8637033051562764 | 9.8637033051186727 | 3.8E−11 | 233.46 |
| 25 | 10.0000000000000000 | 10.0000000000000000 | 0.0E+00 | 0.84 |
| 26 | 10.3774982039134294 | 10.3774982039012666 | 1.2E−11 | 96.25 |
| 27 | 10.4799830400508842 | 10.4799830400439067 | 7.0E−12 | 974.71 |
| 28 | 10.6754536943453164 | 10.6754536943208187 | 2.4E−11 | 80.53 |
| 29 | 10.8151200175936939 | 10.8151200176298907 | -3.6E−11 | 21.42 |
| 30 | 10.9085683308339956 | 10.9085683308326153 | 1.4E−12 | 28.21 |
| 31 | 11.1934033520469818 | 11.1934033520763752 | -2.9E−11 | 155.25 |
| 32 | 11.3819824265232441 | 11.3819824264966716 | 2.7E−11 | 438.73 |
| 33 | 11.4641016151377571 | 11.4639440323935258 | 1.6E−04 | 115.67 |
| 34 | 11.7274066103125492 | 11.7274066102475238 | 6.5E−11 | 6428.40 |
| 35 | 11.8637033051562764 | 11.8637033052067267 | -5.0E−11 | 1802.77 |
| 36 | 12.0000000000000000 | 12.0000000000480007 | -4.8E−11 | 29.29 |
| 37 | 12.1818588307319349 | 12.1817863967843891 | 7.2E−05 | 6577.27 |
| 38 | 12.2389635913615287 | 12.2384376438652254 | 5.3E−04 | 6889.47 |
| 39 | 12.2899151085505363 | 12.2899151085466070 | 3.9E−12 | 112.60 |
| 40 | 12.6283749264972833 | 12.6283749265423863 | -4.5E−11 | 2133.86 |
| 41 | 12.7469384531873313 | 12.7469384531744172 | 1.3E−11 | 198.40 |
| 42 | 12.8532221454500828 | 12.8532221454774298 | -2.7E−11 | 67.83 |
| 43 | 13.0994720835212828 | 13.0993251411339831 | 1.5E−04 | 2055.75 |
| 44 | 13.1958675394493774 | 13.1957481262430427 | 1.2E−04 | 8522.40 |
| 45 | 13.3819824265232477 | 13.3819824265206115 | 2.6E−12 | 510.51 |
| 46 | 13.4641016151377588 | 13.4639878881361117 | 1.1E−04 | 151.07 |
| 47 | 13.6775877082279198 | 13.6774298825312073 | 1.6E−04 | 4731.81 |
| 48 | 13.8059970535441412 | 13.8059970536389738 | -9.5E−11 | 2214.69 |
| 49 | 13.9484250865937067 | 13.9484250865204586 | 7.3E−11 | 1334.68 |
| 50 | 14.0124815721935416 | 14.0100949163104573 | 2.4E−03 | 12245.42 |

Table 2: Optimal results for the packing of unitary circles in a square

| # of items | Rectangle length | Rectangle width | Rectangle area | Time |
|---|---|---|---|---|
| 1 | 2.0000000000000000 | 2.0000000000000000 | 4.0000000000000000 | 0.00 |
| 2 | 4.0000000000000000 | 2.0000000000000000 | 8.0000000000000000 | 0.00 |
| 3 | 6.0000000000000000 | 2.0000000000000000 | 12.0000000000000000 | 0.00 |
| 4 | 8.0000000000000000 | 2.0000000000000000 | 16.0000000000000000 | 0.00 |
| 5 | 10.0000000000000000 | 2.0000000000000000 | 20.0000000000000000 | 0.00 |
| 6 | 4.0000000000000000 | 6.0000000000000000 | 24.0000000000000000 | 0.34 |
| 7 | 2.0000000000000000 | 14.0000000000000000 | 28.0000000000000000 | 0.66 |
| 8 | 8.0000000000000000 | 4.0000000000000000 | 32.0000000000000000 | 1.50 |
| 9 | 6.0000000000000000 | 6.0000000000000000 | 36.0000000000000000 | 0.01 |
| 10 | 10.0000000000000000 | 4.0000000000000000 | 40.0000000000000000 | 0.00 |
| 11 | 8.0000000000000018 | 5.4641016151377553 | 43.7128129211020493 | 11.20 |
| 12 | 8.0000000000000000 | 6.0000000000000000 | 48.0000000000000000 | 4.40 |
| 13 | 26.0000000000000000 | 2.0000000000000000 | 52.0000000000000000 | 7.24 |
| 14 | 5.4641016151377553 | 10.0000000000000018 | 54.6410161513775634 | 12.46 |
| 15 | 3.7320508075688776 | 16.0000000000000036 | 59.7128129211020564 | 8.70 |
| 16 | 3.7320508075688776 | 17.0000000000000036 | 63.4448637286709314 | 15.01 |
| 17 | 5.4641016151377553 | 12.0000000000000018 | 65.5692193816530704 | 31.35 |
| 18 | 3.7320508075688776 | 19.0000000000000036 | 70.9089653438086884 | 61.10 |
| 19 | 7.4641016151377562 | 10.0000000000000018 | 74.6410161513775705 | 8.64 |
| 20 | 14.0000000000000036 | 5.4641016151377553 | 76.4974226119285987 | 0.01 |
| 21 | 15.0000000000000036 | 5.4641016151377553 | 81.9615242270663487 | 0.02 |
| 22 | 3.7320508075688776 | 23.0000000000000036 | 85.8371685740842025 | 26.52 |
| 23 | 5.4641016151377553 | 16.0000000000000036 | 87.4256258422040986 | 8.07 |
| 24 | 5.4641016151377553 | 17.0000000000000036 | 92.8897274573418628 | 35.01 |
| 25 | 3.7320508075688776 | 26.0000000000000071 | 97.0333209967908488 | 23.36 |
| 26 | 5.4641016151377553 | 18.0000000000000036 | 98.3538290724796127 | 114.36 |
| 27 | 5.4641016151377553 | 19.0000000000000036 | 103.8179306876173627 | 10.10 |
| 28 | 12.0000000000000036 | 8.9282032302755123 | 107.1384387633061834 | 80.71 |
| 29 | 20.0000000000000036 | 5.4641016151377553 | 109.2820323027551268 | 82.37 |
| 30 | 5.4641016151377553 | 21.0000000000000036 | 114.7461339178928768 | 10.10 |
| 31 | 3.7320508075688776 | 32.0000000000000071 | 119.4256258422041128 | 58.06 |
| 32 | 5.4641016151377553 | 22.0000000000000036 | 120.2102355330306409 | 176.39 |
| 33 | 14.0000000000000053 | 8.9282032302755123 | 124.9948452238572258 | 1687.60 |
| 34 | 7.1961524227066338 | 18.0000000000000036 | 129.5307436087194333 | 140.46 |
| 35 | 24.0000000000000036 | 5.4641016151377553 | 131.1384387633061408 | 53.59 |
| 36 | 25.0000000000000071 | 5.4641016151377553 | 136.6025403784439334 | 185.56 |
| 37 | 5.4641016151377553 | 25.8612097182042078 | 141.3082777906558363 | 2353.71 |
| 38 | 26.0000000000000071 | 5.4641016151377553 | 142.0666419935816691 | 38.55 |
| 39 | 27.0000000000000071 | 5.4641016151377553 | 147.5307436087194333 | 301.99 |
| 40 | 7.1961524227066338 | 21.0000000000000036 | 151.1192008768393293 | 333.47 |
| 41 | 5.4641016151377553 | 28.0000000000000071 | 152.9948452238571974 | 244.77 |
| 42 | 22.0000000000000036 | 7.1961524227066338 | 158.3153532995459614 | 285.47 |
| 43 | 18.0000000000000071 | 8.9282032302755123 | 160.7076581449592823 | 389.22 |
| 44 | 5.4641016151377553 | 30.0000000000000071 | 163.9230484541326973 | 1447.03 |
| 45 | 5.4641016151377553 | 31.0000000000000071 | 169.3871500692704615 | 503.78 |
| 46 | 7.1961524227066338 | 24.0000000000000036 | 172.7076581449592254 | 485.91 |
| 47 | 5.4641016151377553 | 32.0000000000000071 | 174.8512516844081972 | 520.40 |
| 48 | 20.0000000000000071 | 8.9282032302755123 | 178.5640646055103105 | 3034.99 |
| 49 | 5.4641016151377553 | 33.8612097182042078 | 185.0210907117578643 | 5101.41 |
| 50 | 5.4641016151377553 | 34.0000000000000071 | 185.7794549146837255 | 83.78 |

Table 3: Optimal results for the packing of unitary circles in a rectangle (min. $L \times W$)

| # of items | Rectangle length | Rectangle width | Rectangle semiperimeter | Time |
|---|---|---|---|---|
| 1 | 2.0000000000000000 | 2.0000000000000000 | 4.0000000000000000 | 0.00 |
| 2 | 4.0000000000000000 | 2.0000000000000000 | 6.0000000000000000 | 0.00 |
| 3 | 3.7320508075688776 | 4.0000000000000000 | 7.7320508075688776 | 0.00 |
| 4 | 4.0000000000000000 | 4.0000000000000000 | 8.0000000000000000 | 0.17 |
| 5 | 4.0000000000000000 | 5.4641016151377544 | 9.4641016151377535 | 0.24 |
| 6 | 4.0000000000000000 | 6.0000000000000000 | 10.0000000000000000 | 0.32 |
| 7 | 5.8612097182041998 | 5.4641016151377553 | 11.3253113333419542 | 1.06 |
| 8 | 5.4641016151377553 | 6.0000000000000009 | 11.4641016151377571 | 1.50 |
| 9 | 6.0000000000000000 | 6.0000000000000000 | 12.0000000000000000 | 0.01 |
| 10 | 7.1961524227066338 | 6.0000000000000009 | 13.1961524227066356 | 44.93 |
| 11 | 6.0000000000000009 | 7.4641016151377562 | 13.4641016151377571 | 3.62 |
| 12 | 6.0000000000000000 | 8.0000000000000000 | 14.0000000000000000 | 3.79 |
| 13 | 7.4626564857803901 | 7.4632672693142670 | 14.9259237550946580 | 1944.55 |
| 14 | 7.1961524227066338 | 8.0000000000000018 | 15.1961524227066356 | 42.44 |
| 15 | 8.0000000000000018 | 7.4641016151377562 | 15.4641016151377571 | 14.93 |
| 16 | 8.0000000000000000 | 8.0000000000000000 | 16.0000000000000000 | 24.32 |
| 17 | 8.9282032302755123 | 7.9427193491449888 | 16.8709225794205011 | 737.93 |
| 18 | 8.0000000000000036 | 8.9282032302755123 | 16.9282032302755141 | 287.59 |
| 19 | 7.4641016151377562 | 10.0000000000000018 | 17.4641016151377571 | 7.37 |
| 20 | 8.9282032302755123 | 9.0000000000000036 | 17.9282032302755141 | 556.65 |
| 21 | 9.4337452285295686 | 9.2209018981307658 | 18.6546471266603362 | 113.34 |
| 22 | 9.9427193491449888 | 8.9282032302755123 | 18.8709225794205011 | 979.18 |
| 23 | 8.9282032302755123 | 10.0000000000000036 | 18.9282032302755141 | 421.24 |
| 24 | 9.4641016151377553 | 10.0000000000000000 | 19.4641016151377571 | 86.33 |
| 25 | 11.0000000000000036 | 8.9282032302755123 | 19.9282032302755141 | 213.47 |
| 26 | 10.6602540378443891 | 9.9427193491449888 | 20.6029733869893761 | 3528.82 |
| 27 | 10.6602540378443891 | 10.0000000000000036 | 20.6602540378443926 | 88.10 |
| 28 | 12.0000000000000036 | 8.9282032302755123 | 20.9282032302755141 | 74.90 |
| 29 | 9.4641016151377571 | 12.0000000000000018 | 21.4641016151377571 | 20.58 |
| 30 | 10.6602540378443891 | 11.0000000000000036 | 21.6602540378443926 | 436.24 |
| 31 | 10.9282032302755141 | 11.4265717909344140 | 22.3547750212099281 | 2237.26 |
| 32 | 11.9427193491449870 | 10.6602540378443891 | 22.6029733869893761 | 1417.69 |
| 33 | 12.0000000000000036 | 10.6602540378443891 | 22.6602540378443926 | 214.35 |
| 34 | 10.9282032302755123 | 12.0000000000000036 | 22.9282032302755141 | 731.71 |
| 35 | 12.3923048454132676 | 11.0000000000000036 | 23.3923048454132712 | 133.13 |
| 36 | 10.6602540378443891 | 13.0000000000000071 | 23.6602540378443962 | 82.16 |
| 37 | 12.3923048454132676 | 11.8612097182042024 | 24.2535145636174718 | 7399.74 |
| 38 | 11.9841557353269081 | 12.3924132707237522 | 24.3765690060506586 | 2891.91 |
| 39 | 12.3923048454132676 | 12.0000000000000036 | 24.3923048454132712 | 3329.17 |
| 40 | 12.9282032302755123 | 12.0000000000000036 | 24.9282032302755141 | 155.32 |
| 41 | 12.3923048454132676 | 12.9427193491449888 | 25.3350241945582582 | 6254.69 |
| 42 | 12.3923048454132676 | 13.0000000000000071 | 25.3923048454132747 | 1650.45 |
| 43 | 13.6029140930960750 | 12.3923268915991969 | 25.9952409846952719 | 1035.43 |
| 44 | 13.8844501917489893 | 12.3923048454132712 | 26.2767550371622605 | 13131.52 |
| 45 | 13.9841229965638760 | 12.3923048454132676 | 26.3764278419771436 | 1069.14 |
| 46 | 14.0000000000000071 | 12.6602540378443891 | 26.6602540378443962 | 1879.21 |
| 47 | 14.0000000000000071 | 12.9282032302755123 | 26.9282032302755212 | 1281.24 |
| 48 | 13.0000000000000071 | 14.1243556529821479 | 27.1243556529821532 | 1709.87 |
| 49 | 12.3923048454132676 | 15.0000000000000071 | 27.3923048454132747 | 7546.82 |
| 50 | 12.3923048454132712 | 15.6028097181778964 | 27.9951145635911658 | 6366.25 |

Table 4: Optimal results for the packing of unitary circles in a rectangle (min. $L+W$)

| # of items | Triangle side | Time | # of items | Strip length | Time |
|---|---|---|---|---|---|
| 1 | 3.4641016151377545 | 0.00 | 1 | 2.0000000000000000 | 0.00 |
| 2 | 5.4641015260409098 | 0.00 | 2 | 1.9999999999999998 | 0.04 |
| 3 | 5.4641015582188635 | 0.05 | 3 | 1.9999999999999998 | 0.02 |
| 4 | 6.9282031370137620 | 0.03 | 4 | 1.9999999999999998 | 0.04 |
| 5 | 7.4641015438602771 | 54.72 | 5 | 2.6959705453537524 | 0.02 |
| 6 | 7.4641015587429127 | 206.36 | 6 | 3.3228756555322954 | 0.00 |
| 7 | 8.9282031100914168 | 0.18 | 7 | 3.5612494995995996 | 0.01 |
| 8 | 9.2938099467443216 | 12.95 | 8 | 3.6887986310766987 | 0.07 |
| 9 | 9.4641015510046618 | 25.109 | 9 | 3.9996673748986948 | 9.17 |
| 10 | 9.4641015666630892 | 0.37 | 10 | 4.6959705453537524 | 0.04 |
| 11 | 10.7300878190524358 | 13.38 | 11 | 5.1224989991991992 | 0.05 |
| 12 | 10.9282031596736786 | 4.22 | 12 | 5.3775972621533974 | 2.42 |
| 13 | 11.4064957458284422 | 80.55 | 13 | 5.8538814987957917 | 840.02 |
| 14 | 11.4641015604162533 | 1122.57 | 14 | 5.9993347497973915 | 1112.34 |
| 15 | 11.4641015695434394 | 24.63 | 15 | 6.6959705453537559 | 0.02 |
| 16 | 12.7136286310567250 | 2186.82 | 16 | 7.0663958932300979 | 5.33 |
| 17 | 12.9282031457004436 | 49.64 | 17 | 7.4525364626094142 | 7.98 |
| 18 | 13.2937904231493249 | 121.93 | 18 | 7.8539155052528402 | 687.63 |
| 19 | 13.4480543405474720 | 223.02 | 19 | 7.9996778073991033 | 2515.58 |
| 20 | 13.4641015644155306 | 20.57 | 20 | 8.6959705453537559 | 40.77 |
| 21 | 13.4641015778907409 | 42.47 | 21 | 9.1732901304367189 | 3185.98 |
| 22 | 14.6125656032291964 | 413.39 | 22 | 9.4524415753023714 | 31.15 |
| 23 | 14.8826696938712466 | 2117.64 | 23 | 9.8537288511462791 | 2537.63 |
| 24 | 14.9282031609796402 | 373.50 | 24 | 9.9993623469257145 | 161.81 |
| 25 | 15.2938099693721306 | 1.18 | 25 | 10.6959705453537577 | 0.66 |
| 26 | 15.4589390002039853 | 1067.00 | 26 | 11.1733168632638993 | 2878.17 |
| 27 | 15.4641015656802985 | 6.78 | 27 | 11.4522265853969465 | 267.37 |
| 28 | 15.4641015817250107 | 782.69 | 28 | 11.8539308197359858 | 94.54 |
| 29 | 16.6056026842964179 | 2708.93 | 29 | 11.9993600450532973 | 857.99 |
| 30 | 16.7300878617292312 | 24.34 | 30 | 12.6959705453537595 | 1.32 |
| 31 | 16.9282031492725551 | 158.16 | 31 | 13.1733140056536673 | 10490.33 |
| 32 | 17.2474929494386764 | 179.30 | 32 | 13.4519215824209422 | 7042.79 |
| 33 | 17.4064957212102627 | 128.48 | 33 | 13.8515722749965562 | 663.65 |
| 34 | 17.4635536344819791 | 568.62 | 34 | 13.9993894685688876 | 2806.64 |
| 35 | 17.4641015734708560 | 82.36 | 35 | 14.6959705453537612 | 8.50 |
| 36 | 17.4641015898910545 | 48.32 | 36 | 15.1731171516082881 | 5723.33 |
| 37 | 18.5312410691358664 | 1463.95 | 37 | 15.4520220872811613 | 1503.56 |
| 38 | 18.7298248517387407 | 395.28 | 38 | 15.8330120530395764 | 504.22 |
| 39 | 18.9160916884815435 | 1210.17 | 39 | 15.9999999998085887 | 1014.63 |
| 40 | 18.9282031752999664 | 658.87 | 40 | 16.6959705453537595 | 14.86 |
| 41 | 19.2938099551359308 | 639.66 | 41 | 17.1729034249762442 | 6632.41 |
| 42 | 19.4064957825474025 | 26.92 | 42 | 17.4516358985406050 | 2548.92 |
| 43 | 19.4635536523988257 | 1459.48 | 43 | 17.8140606308452512 | 2577.70 |
| 44 | 19.4641015803674549 | 1370.20 | 44 | 17.9993696642869736 | 12212.56 |
| 45 | 19.4641015941498345 | 10.92 | 45 | 18.6959705453537630 | 3.66 |
| 46 | 20.5275000891198900 | 3295.17 | 46 | 19.1726994816344884 | 1476.64 |
| 47 | 20.7032882042547612 | 1837.24 | 47 | 19.4518331606560650 | 14077.38 |
| 48 | 20.8825408318815278 | 1526.27 | 48 | 19.7495596428938782 | 6813.48 |
| 49 | 20.9282031663479380 | 501.34 | 49 | 19.9993724839986804 | 3527.02 |
| 50 | 21.2464302653662145 | 569.27 | 50 | 20.6959705453537630 | 4.96 |

Table 5: Optimal results for the packing of unitary circles in a triangle and a strip ($L = 9.5$)

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