The Complete Pre-order Polytope: Facets and Separation Problem [1]

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1996

Abstract. Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes. A subset $A \subseteq A_n$ is called a *complete pre-order* of D_n if $H = (V_n, A)$ is a total transitive subdigraph of D_n (i.e., uv or $vu \in A$, for any two distinct vertices u, v in V_n; and if uv and $vw \in A$ then $uw \in A$ for any three distinct vertices u, v, w in V_n). Given weights $w_a \in \mathbb{R}$ for all $a \in A_n$, the *complete pre-order problem* is to find a complete pre-order $A \subseteq A_n$ such that $w(A) := \sum_{a \in A} w_a$ is as large as possible. This problem is known to be $\mathcal{NP}$-hard. Let $\mathcal{P}^n$ denote the *complete pre-order polytope*, i.e., $\mathcal{P}^n$ is the convex hull of the incidence vectors of all complete pre-orders of D_n. We mention how $\mathcal{P}^n$ relates to other known polytopes such as the linear ordering and the partial ordering polytopes. We present a lifting theorem and various classes of facet-defining inequalities for $\mathcal{P}^n$: the triangle-, pentagon-, heptagon- and semi-k-fence inequalities. We also show that the separation problem for the class of semi-k-fences is $\mathcal{NP}$-complete.

Keywords: Facets of polytopes, complete pre-order, transitive digraph, separation problem, $\mathcal{NP}$-complete.

1. Introduction and basic notation

A binary relation R on a set S, $R \subseteq S \times S$, is called *total* if $(a, b) \in R$ or $(b, a) \in R$ for all $a, b \in S$, $a \neq b$, *anti-symmetric* if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in S$ and *transitive* if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in S$, $a \neq b \neq c \neq a$.

Let $D = (V, A)$ be a digraph and $B \subseteq A$. We say that B is a *linear order* (*complete pre-order*) of D if B is an anti-symmetric, transitive and total (a transitive and total) binary relation on V.

If $D = (V, A)$ is a digraph with n nodes and $B \subseteq A$ is a linear order of D, then there is precisely one directed n-path (a path of length n) in D whose arcs are in B (and whose transitive closure is B). The sequence of nodes of this path, say $O := \langle u_1, \ldots, u_n \rangle$, is another notation for this linear order. We also denote by $A(O)$ the set B. Thus, for the sequence O above, $A(O) = \{(u_i, u_j) \mid 1 \leq i < j \leq n\}$.

If $S \subseteq \mathbb{R}^m$ then $conv\, S$ denotes the convex hull of S, that is, the set of elements in $\mathbb{R}^m$ obtained by a convex combination of finitely many elements in S.

Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes, and set

$$\mathcal{A} := \{A \subseteq A_n \mid A \text{ is a complete pre-order of } D_n\} \ .$$

[1]This work has been partially supported by FAPESP (Proc. 96/04505-2) and CNPq (Proc. 680065/94-6, 304527/89-0).

Given weights w_a for any arc $a \in A_n$, the *complete pre-order problem* (CPO, for short) is to solve $\max\{w(A) \mid A \in \mathcal{A}\}$, that is, to find a complete pre-order of maximum weight.

This problem—known to be $\mathcal{NP}$-hard (Wakabayashi [1992])—will be studied here from a polyhedral point of view. For a more detailed and complete study of this problem the reader should refer to Gurgel [1992].

Let $\mathbb{R}^m$, $m = |A_n| = n(n-1)$, denote the real vector space where every component of a vector $x \in \mathbb{R}^m$ is indexed by an arc $uv \in A_n$. For every arc set $A \subseteq A_n$, we denote by x^A the *incidence vector of* A (defined as $x_a^A = 1$ if $a \in A$ and $x_a^A = 0$ if $a \in A_n \backslash A$). The notation $x(A)$ stands for $\sum_{a \in A} x_a$.

The *complete pre-order polytope* $\mathcal{P}^n$ is the convex hull of the incidence vectors x^A of the complete pre-orders A of D_n, i.e,

$$\mathcal{P}^n = \text{conv}\left\{x^A \in \{0,1\}^{A_n} \mid A \text{ is a complete pre-order of } D_n\right\} .$$

In the next section we show how $\mathcal{P}^n$ relates to other known polytopes such as the linear ordering and partial ordering polytopes. We also show that $\mathcal{P}^n$ is full-dimensional and present a useful lifting theorem. In Section 3 we describe some facet-defining inequalities for $\mathcal{P}^n$, and in Section 4 we study the separation problem for these facet-defining inequalities. In particular, we show that for the semi-k-fence inequalities the separation problem is $\mathcal{NP}$-complete.

The basic concepts in graph theory and in polyhedral theory we use in this paper can be found in Bondy and Murty [1976], Grünbaum [1967] and Schrijver [1986]. We only mention here a few concepts with the purpose of establishing our notation.

A *directed graph* or *digraph* $D = (V, A)$ consists of a finite nonempty set V of *nodes* and a set A of *arcs* which are ordered pairs of distinct elements of V. An arc $\alpha = (u, v)$ will also be denoted by uv or u, v.

If H is a subdigraph of a digraph $G = (V, A)$, we use the notation $H \subseteq G$. For $X \subseteq V$ ($X \subseteq A$), $G[X]$ denotes the subdigraph of G spanned by X. If $X \subseteq V$, $G - X$ denotes the subdigraph obtained from G, by deleting the set X. To simplify the notation, $D - v$ denotes the subdigraph $D - X$, if $X = \{v\}$, $v \in V$.

If H is a digraph (graph) sometimes we may denote its node set by VH and its arc set by AH. A *triangle* in a digraph G is a set consisting of three arcs of the form uv, vw and uw.

The complete digraph on n nodes is denoted by $D_n = (V_n, A_n)$. If $\alpha = (u, v)$ is an arc of D_n, $\overline{\alpha}$ denotes the *reverse arc* of α, i.e., $\overline{\alpha} = (v, u)$. If $B \subseteq A_n$, $\overline{B}$ denotes the set $\{\overline{\alpha} \mid \alpha \in B\}$.

A nonempty sequence of arcs $P = \langle (v_1, v_2), (v_2, v_3) \ldots, (v_{k-1}, v_k) \rangle$ in a digraph $D = (V, A)$ such that $v_i \neq v_j$ for $i \neq j$ is called a $(v_1 - v_k)$-*directed path*. If P is a $(v_1 - v_k)$-directed path and $(v_k, v_1) \in A$ then $C = \langle (v_1, v_2), (v_2, v_3) \ldots, (v_{k-1}, v_k), (v_k, v_1) \rangle$ is called a *directed cycle*. The *length* of a path (cycle) is the number of its arcs. A *directed k-path* (*k-cycle*) is a directed path (directed cycle) of length k; it will be sometimes denoted by the sequence of its nodes, such as

$\langle v_1, \ldots, v_k \rangle$. For convenience, if C is a cycle we also denote by C the set of arcs in C and by VC the set of nodes in C.

Let $P \subseteq \mathbb{R}^m$ be a polyhedron. An inequality $a^T x \le a_0$ ($a \in \mathbb{R}^m$, $a_0 \in \mathbb{R}$) is *valid* for P if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \le a_0\}$. If $a^T x \le a_0$ is valid for P, then $F_a = \{x \in P \mid a^T x = a_0\}$ is the *face* (of P) *induced* or *defined* by $a^T x \le a_0$. A valid inequality $a^T x \le a_0$ defines a *facet* of P if $F_a \ne \emptyset$ and $\dim(F_a) = \dim(P) - 1$. If $a^T x \le a_0$ is valid and F_a defines a *facet* of P, we say that $a^T x \le a_0$ is *facet-defining* (for P).

2. The complete pre-order polytope $\mathcal{P}^n$

Consider the polytope Q^n defined as

$$
\begin{aligned}
Q^n := \Big\{ & x \in \mathbb{R}^{A_n} \mid 0 \le x_a \le 1 && \text{for all } a \in A_n \ , \\
& x_{uv} + x_{vu} \ge 1 && \text{for all } u, v \text{ in } V_n \ , \\
& x_{uv} + x_{vw} - x_{uw} \le 1 && \text{for all } u, v, w \text{ in } V_n \Big\} \ .
\end{aligned}
$$

It is easy to see that $\mathcal{P}^n \subseteq Q^n$ and that the integral points of Q^n coincide with the incidence vectors of the complete pre-orders of D_n. Therefore, $\mathcal{P}^n = \text{conv}\,\{x \in Q_n \mid x \text{ integral}\}$.

In order to relate $\mathcal{P}^n$ with some known polytopes associated with transitive orderings, let $\mathcal{P}^n_{LO}$, $\mathcal{P}^n_{PO}$ and $\mathcal{P}^n_{PRE}$ be the polytopes defined as the convex hull of all incidence vectors of the linear orders, the partial orders and the pre-orders of D_n, respectively. The polytope $\mathcal{P}^n_{LO}$ were studied by Reinelt [1985] and Gröestchel, Jünger and Reinelt [1985]. Classes of facets of $\mathcal{P}^n_{PO}$ were obtained by Müller [1993, 1996] and independently by Gurgel [1992]. It is easy to observe that $\mathcal{P}^n_{LO} \subseteq \mathcal{P}^n_{PO} \subseteq \mathcal{P}^n_{PRE}$, and that for $n \ge 3$, all these inclusions are proper. In fact, $\mathcal{P}^n_{PO} \cup \mathcal{P}^n \subseteq \mathcal{P}^n_{PRE}$; the inclusion being an equality if $n = 2$ and a proper inclusion if $n > 2$. Furthermore, it is not difficult to prove that $\mathcal{P}^n_{LO} = \mathcal{P}^n_{PO} \cap \mathcal{P}^n$ and $\mathcal{P}^n_{LO}$ is a face of $\mathcal{P}^n$ and also a face of $\mathcal{P}^n_{PO}$ (Gurgel [1992]).

The next result shows that $\mathcal{P}^n$ is full-dimensional. This implies that the facet-defining inequalities are unique up to a multiplication by a positive constant.

Theorem 1. For $n \ge 2$, $\dim \mathcal{P}^n = n(n-1)$.

Proof. Let $a^T x = b$ be a linear equation such that $\mathcal{P}^n \subseteq conv\{x \in \{0,1\}^{A_n} \mid a^T x = b\}$. Let $\alpha = (u, v)$ be an arc of A_n and $O = \langle v, u, w_1, \ldots, w_s \rangle$ be a linear order of D_n, where $\langle w_1, \ldots, w_s \rangle$ is any linear order of $D_n - \{u, v\}$. Since $A(O)$ and $A(O) \cup \{\alpha\}$ are complete pre-orders of D_n, it follows that

$$\sum_{\beta \in A(O)} a_\beta x_\beta = \sum_{\beta \in A(O)} a_\beta x_\beta + a_\alpha x_\alpha = b \ .$$

Thus, $a_\alpha = 0$ and therefore a is a zero vector. ∎

In what follows we present a lifting theorem which shows that inequalities that are facet-defining for $\mathcal{P}^n$ and satisfy a certain condition can be 'lifted' to define a facet of $\mathcal{P}^m$, $m > n$.

In the next theorem we assume that the node set of D_n is $V_n = \{1, 2, \ldots, n\}$.

Theorem 2 (Lifting Theorem). Let $a^T x \leq a_0$ be a valid inequality for $\mathcal{P}^n$, $n \geq 2$. Define $\widehat{a} \in \mathbb{R}^{n(n+1)}$ as

$$\begin{aligned} \widehat{a}_\alpha &= a_\alpha \ , \quad \text{if } \alpha \in A_n \ ; \\ \widehat{a}_{i,n+1} &= \widehat{a}_{n+1,i} = 0 \ , \quad \text{for } i = 1, \ldots, n \ . \end{aligned}$$

Then,

(i) $\widehat{a}^T x \leq a_0$ is valid for $\mathcal{P}^{n+1}$;

(ii) If $a^T x \leq a_0$ is facet-defining for $\mathcal{P}^n$ and there exists a linear order $A \subseteq A_n$ such that $a^T x^A = a_0$, then $\widehat{a}^T x \leq a_0$ is facet-defining for $\mathcal{P}^{n+1}$.

Proof. (i) It is not difficult to conclude that the inequality $\widehat{a}^T x \leq a_0$ is valid for $\mathcal{P}^{n+1}$.
(ii) Let $F_{\widehat{a}}$ be the face of $\mathcal{P}^{n+1}$ defined as

$$F_{\widehat{a}} = \{x \in \mathcal{P}^{n+1} \mid \widehat{a}^T x = a_0\} \ .$$

Suppose that $\widehat{b}^T x \leq b_0$ defines a facet of $\mathcal{P}^{n+1}$ such that

$$F_{\widehat{a}} \subseteq F_{\widehat{b}} = \{x \in \mathcal{P}^{n+1} \mid \widehat{b}^T x = b_0\} \ . \tag{1}$$

Let $O = \langle i_1, i_2, \ldots, i_n \rangle$ be a linear order of D_n such that $a^T x^{A(O)} = a_0$. For each $j \in \{1, \ldots, n\}$, let O_1^j and O_2^j be the following linear orders of D_{n+1}:

$$\begin{aligned} O_1^j &= \langle i_1, \ldots, i_j, n+1, i_{j+1}, \ldots, i_n \rangle \quad \text{and} \\ O_2^j &= \langle i_1, \ldots, i_{j-1}, n+1, i_j, \ldots, i_n \rangle \ . \end{aligned}$$

Consider now the arc sets $A_i^j = A(O_i^j)$, $i = 1, 2$. Since $a^T x^{A(O)} = a_0$ and A_i^j $(1 \leq i \leq 2$, $1 \leq j \leq n)$ is obtained from $A(O)$ by adding arcs adjacent to $n+1$, it follows that

$$\widehat{a}^T x^{A_1^j} = \widehat{a}^T x^{A_2^j} = a_0 \quad \text{for all } j \in \{1, \ldots, n\} \ .$$

Since $F_{\widehat{a}} \subseteq F_{\widehat{b}}$, then

$$\widehat{b}^T x^{A_1^j} = \widehat{b}^T x^{A_2^j} = b_0 \ ,$$

and therefore $\widehat{b}_{n+1,\,i_j} = \widehat{b}_{i_j,n+1}$ for all $j \in \{1, \ldots, n\}$.

In order to show that $\widehat{b}_{n+1,i_j} = 0$, suppose that $j \in \{1, \ldots, n\}$ and consider $A_3^j = A_1^j \cup \{(n+1, i_j)\}$. Then A_1^j and A_3^j are complete pre-orders of D_{n+1} which satisfy $\widehat{a}^T x^{A_1^j} = \widehat{a}^T x^{A_3^j} = a_0$.

Since $F_{\widehat{a}} \subseteq F_{\widehat{b}}$, it follows that $\widehat{b}^T x^{A_1^j} = \widehat{b}^T x^{A_3^j} = b_0$ and therefore $\widehat{b}_{n+1,i_j} = 0$. Thus $\widehat{b}_{i_j,n+1} = \widehat{b}_{n+1,i_j} = 0$ for all $j \in \{1, \ldots, n\}$. This implies that $b^T x \leq b_0$ is valid for $\mathcal{P}^n$, where b is the restriction of $\widehat{b}$ to the arc set A_n.

Now using (1), we conclude that

$$\{x \in \mathcal{P}^n \mid a^T x = a_0\} \subseteq \{x \in \mathcal{P}^n \mid b^T x = b_0\} .$$

Since $a^T x \leq a_0$ induces a facet of $\mathcal{P}^n$ and this polytope is full-dimensional, it follows that $b^T x \leq b_0$ induces a facet of $\mathcal{P}^n$. Then, there exists a real non-negative μ such that $a^T = \mu b^T$. Thus $\widehat{a}^T = \mu \widehat{b}^T$, since $\widehat{a}_{i,n+1} = \widehat{a}_{n+1,i} = \widehat{b}_{i,n+1} = \widehat{b}_{n+1,i} = 0$ for $i \in \{1, \ldots, n\}$. Therefore $\widehat{a}^T x \leq a_0$ induces a facet of $\mathcal{P}^{n+1}$. ∎

3. Facets of $\mathcal{P}^n$

Let us consider first the trivial inequalities. Using the lifting theorem, it is easy to prove that

$$\begin{aligned} x_\alpha &\leq 1, && \text{for any } \alpha \in A_n, \\ x_\alpha + x_{\overline{\alpha}} &\geq 1, && \text{for any } \alpha \in A_n, \\ x_\alpha + x_\beta - x_\gamma &\leq 1, && \text{for any triangle } \{\alpha, \beta, \gamma\} \text{ of } D_n, \end{aligned}$$

are facet-defining inequalitites for $\mathcal{P}^n$, $n \geq 3$. The latter are called *total-* and *triangle-inequality.* As for the non-negativity inequalities,

$$x_\alpha \geq 0, \qquad \text{for any } \alpha \in A_n,$$

we note that they do not define facets, as they are the sum of the inequalities $-x_{\overline{\alpha}} \geq -1$ and $x_\alpha + x_{\overline{\alpha}} \geq 1$.

3.1 Semi-k-fence inequalities

We introduce in this section the so-called semi-k-fence inequalities. These inequalities were derived from the k-fence inequalities, which were first introduced by Grötschel, Jünger and Reinelt [1985] in the study of the linear ordering polytope $\mathcal{P}_{LO}^n$. We define below these inequalities.

Definition 3. A digraph D is a *k-fence* if $|VD| = 2k$, $k \geq 3$, and there is a partition of VD into two subsets $\mathcal{S} = \{s_1, \ldots, s_k\}$ and $I = \{i_1, \ldots, i_k\}$ such that $AD = \{(s_j, i_j) \mid j = 1, \ldots, k\} \cup \{(i_j, s_l) \mid 1 \leq j \neq l \leq k\}$. The inequality $x(AD) \leq k^2 - k + 1$ induced by D is called a *simple k-fence inequality.* The arcs (s_j, i_j) are called *pales.*

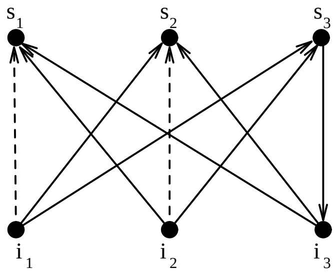

Figure 1: A semi-3-fence with pivot (s_3, i_3)

These authors proved that the simple k-fence inequalities are facet-defining for the polytope $\mathcal{P}^n_{LO}$. It is easy to see that they are not valid for $\mathcal{P}^n$. Nevertheless, by adding up simple k-fence inequalities with some facet-defining inequalities for $\mathcal{P}^n$ we obtain facet-defining inequalities for $\mathcal{P}^n$. More precisely, the inequalities obtained in this way, called semi-k-fences, are defined as follows.

Definition 4. A *semi-k-fence* is a digraph that can be obtained from a k-fence by changing the direction of precisely $k-1$ pales. In other words, it is a digraph G such that:

(i) $|VG| = 2k$, $k \geq 2$;

(ii) VG can be partitioned into two subsets, each of order k, say $S = \{s_1, \ldots, s_k\}$ and $I = \{i_1, \ldots, i_k\}$, and there is an index t, $1 \leq t \leq k$, such that

$$AG = \{(s_t, i_t)\} \cup \{(i_l, s_j) \mid 1 \leq j, l \leq k\} \backslash \{(i_t, s_t)\} \ .$$

The set S is called the set of *superior nodes* of G, I is the set of *inferior nodes* and (s_t, i_t) is the *pivot* of the semi-k-fence (see Figure 1). The arcs (i_j, s_j) such that $j \neq t$ are referred as *negative arcs* of G. The set of these arcs is denoted by AG_-, and the complementary set $AG_+ = AG \backslash AG_-$ is the set of the *positive arcs* of G. The inequality

$$x(AG_+) - x(AG_-) \leq k^2 - 2k + 2$$

induced by G is called a *semi-k-fence inequality.*

Theorem 5. The semi-k-fence inequality (induced by a semi-k-fence G),

$$x(AG_+) - x(AG_-) \leq k^2 - 2k + 2 \ ,$$

is facet-defining for $\mathcal{P}^n$, $n \geq 2k$, $k \geq 3$.

Proof. Let $a^T x \leq a_0$ denote the semi-k-fence inequality. We first show that $a^T x \leq a_0$ is valid for $\mathcal{P}^n$, $n \geq 2k$. Suppose that the pivot of G is (s_k, i_k). Thus $AG_- = \{(i_l, s_l) \mid 1 \leq l \leq k-1\}$. Let

H be a subdigraph of D_n such that x^{AH} is a vertex of $\mathcal{P}^n$ and suppose that $A^+ = AG_+ \cap AH$ and $A^- = AG_- \cap AH$. Therefore $|A^+| \leq k^2 - k + 1$ and $|A^-| \leq k - 1$. To prove that $x(A^+) - x(A^-) \leq k^2 - 2k + 2$ we analyse three cases:

Case 1: $|A^-| \geq k - 2$

If $|A^+| = k^2 - k + 1$, that is, H has all the positive arcs of G, it follows from the transitivity of AH that $|A^-| = k - 1$ and $x(A^+) - x(A^-) \leq k^2 - 2k + 2$. If $|A^+| \leq k^2 - k$ the last inequality is immediate.

Case 2: $|A^-| = k - 3$

Let $\alpha = (i_l, s_l)$ and $\beta = (i_p, s_p)$ be two arcs in $AG_- \backslash AH$. Since $\overline{\alpha}, \overline{\beta} \in AH$, then $(i_l, s_p) \notin A^+$ or $(i_p, s_l) \notin A^+$. If $(s_k, i_k) \notin A^+$ then $|A^+| \leq k^2 - k - 1$. Otherwise, either $(i_l, s_k) \notin A^+$ or $(i_k, s_l) \notin A^+$. Thus $|A^+| \leq k^2 - k - 1$ and $|A^+| - |A^-| \leq k^2 - 2k + 2$.

Case 3: $|A^-| < k - 3$

Let α and β be two arcs in $AG_- \backslash AH$, $\alpha = (i_l, s_l)$ and $\beta = (i_p, s_p)$, $l \neq p$. Since $x^{AH} \in \mathcal{P}^n$, then $\overline{\alpha}$ and $\overline{\beta}$ are in H. In this case, either $(i_l, s_p) \notin AH$ or $(i_p, s_l) \notin AH$, otherwise α and β would be in H, a contradiction. We can conclude here that for each two negative arcs of G that do not belong to H, at least one positive arc of G is not in H. Clearly, all these positive arcs are distinct. Since $(k - 1 - |A^-|)$ arcs are in $AG^- \backslash AH$, at least $\binom{k-1-|A^-|}{2}$ positive arcs are not in H. Thus

$$|A^+| \leq k^2 - k + 1 - \binom{k - 1 - |A^-|}{2} .$$

Since $|A^-| < k - 3$, then $k - 1 - |A^-| > 2$ and $\binom{k-1-|A^-|}{2} \geq k - 1 - |A^-|$. Therefore $|A^+| \leq k^2 - k + 1 - (k - 1 - |A^-|)$ and $|A^+| - |A^-| \leq k^2 - 2k + 2$.

Let us prove now that the semi-k-fence inequality is facet-defining for $\mathcal{P}^{2k}$. Assume that $F_b = \{x \in \mathcal{P}^{2k} \mid b^T x = b_0\}$ is a facet of $\mathcal{P}^{2k}$ such that $F_b \supseteq F_a = \{x \in \mathcal{P}^{2k} \mid a^T x = a_0\}$.

Let us prove that exists $\lambda \in \mathbb{R}^+$ such that $b = \lambda a$, in the following order:

(a) $b_\alpha = 0$ for all $\alpha \in \overline{AG}$;

(b) $b_\alpha = 0$ for all $\alpha \in AD_{2k} \backslash (AG \cup \overline{AG}\ )$;

(c) $b_\alpha = \lambda a_\alpha$ for all $\alpha \in AG$.

(a) Let O be the following linear order of D_{2k}:

$$O = \langle i_1, i_2, \ldots, i_{k-1}, s_k, i_k, s_1, s_2, \ldots, s_{k-1} \rangle .$$

Then $A(O)$ is a complete pre-order of D_{2k} such that $x^{A(O)} \in F_a \subseteq F_b$. We leave to the reader the proof that $b_\alpha = 0$ for all $\alpha \in \overline{A(O)}$,which can be shown by considering different pre-orders. Since $AG \subseteq A(O)$, it follows that $b_\alpha = 0$ for all $\alpha \in \overline{AG}$.

(b) Consider $S' = S \backslash \{s_k\}$ and $I' = I \backslash \{i_k\}$. Let us show first that $b_{u,v} = 0$ if $u, v \in S'$.

Suppose w.l.o.g. that $u = s_i$ and $v = s_j$, $1 \le i \ne j \le k-1$. Consider the following linear order of D_{2k}:

$$\widehat{O} = \langle i_1, i_2, \ldots, i_{k-1}, s_k, i_k, s_j, s_i, O' \rangle \ ,$$

where O' is any sequence of nodes in $S' \backslash \{s_j, s_i\}$. Then $A(\widehat{O})$ and $A(\widehat{O}) \cup \{(s_i, s_j)\}$ are complete pre-orders whose incidence vectors are in F_a. Since $F_a \subseteq F_b$, these vectors are in F_b and

$$0 = b^T x^{A(\widehat{O})} - b^T x^{A(\widehat{O}) \cup \{(s_i,s_j)\}} \ .$$

Thus $b_{s_i,s_j} = 0$.

With similar arguments one can verify that $b_{u,v} = 0$ if $u, v \in I'$. Thus to complete the proof of (b), it remains to verify the following assertions, for all $r \ne k$ and $l \ne k$:

$$(\mathrm{b}_1) \quad b_{s_r,s_k} = 0 \ , \qquad (\mathrm{b}_2) \quad b_{s_k,s_r} = 0 \ ,$$

$$(\mathrm{b}_3) \quad b_{i_k,i_l} = 0 \ , \qquad (\mathrm{b}_4) \quad b_{i_l,i_k} = 0 \ .$$

Since (s_k, s_r) and (i_l, i_k) are arcs of $A(O)$, where O is the linear order defined in (a), it follows that $b_{s_r,s_k} = b_{i_k,i_l} = 0$. This verifies ($\mathrm{b}_1$) and ($\mathrm{b}_3$). To verify ($\mathrm{b}_2$), let us construct the digraph H from G, by changing the orientation of the arcs $\alpha = (s_k, i_k)$ and $\beta = (i_j, s_j)$, where $j \ne r$. Note that such an arc β exists, since $k \ge 3$. Thus,

$$|AH \cap AG_+| = k^2 - k \quad \text{and} \quad |AH \cap AG_-| = k - 2 \ ,$$

and therefore,

$$x(AH \cap AG_+) - x(AH \cap AG_-) = k^2 - 2k + 2 \ .$$

The set AH can be extended to the following linear order:

$$\widehat{O} = \langle O', s_j, i_j, O'', s_r, s_k \rangle \ ,$$

where O' is any sequence of nodes in $I \backslash \{i_j\}$ and O'' is any sequence of nodes in $S \backslash \{s_j, s_r, s_k\}$. Note that $A(\widehat{O})$ and $A(\widehat{O}) \cup \{s_k, s_r\}$ are complete pre-orders whose incidence vectors are in F_a and therefore in F_b. Thus,

$$0 = b^T x^{A(\widehat{O})} - b^T x^{A(\widehat{O}) \cup \{(s_k,s_r)\}} \ ,$$

which implies that $b_{s_k,s_r} = 0$. With similar arguments one can verify (b_4) completing the proof of (b). This last proof is left to the reader.

(c) Suppose that $b_{s_k,i_k} = \lambda$, $\lambda \in \mathbb{R}$ and $\mathrm{IND} = \{1, 2, \ldots, k-1\}$. Let us prove that:

$$(\mathrm{c}_1) \quad b_{i_j,s_j} = -\lambda \ , \quad \forall j \in \mathrm{IND} \ ; \qquad (\mathrm{c}_2) \quad b_{i_j,s_k} = \lambda \ , \quad \forall j \in \mathrm{IND} \ ;$$

$$(\mathrm{c}_3) \quad b_{i_k,s_r} = \lambda \ , \quad \forall r \in \mathrm{IND} \qquad \text{and} \qquad (\mathrm{c}_4) \quad b_{i_j,s_r} = \lambda \ , \quad \forall r, j \in \mathrm{IND}, \ r \ne j \ .$$

(c_1) Consider the digraph H obtained from G by changing the orientation of (s_k, i_k) and (i_j, s_j). Then AG and AH can be extended to the linear orders O_G and O_H, respectively.

$$\begin{aligned} O_G &= \langle i_1, \ldots, i_{k-1}, s_k, i_k, s_1, \ldots, s_{k-1} \rangle \\ O_H &= \langle i_1, \ldots, i_{j-1}, i_k, i_{j+1}, \ldots, i_{k-1}, s_j, i_j, s_1, \ldots, s_{j-1}, s_k, s_{j+1}, \ldots, s_{k-1} \rangle \ . \end{aligned}$$

Then $A(O_G)$ and $A(O_H)$ are complete pre-orders and $x^{A(O_H)}$, $x^{A(O_G)} \in F_a \subseteq F_b$. From (a) and (b), we can conclude that $b_{i_j,s_j} + b_{s_k,i_k} = O$. Thus $b_{i_j,s_j} = -\lambda$.

(c_2) Consider the digraph H obtained from G by changing the orientations of (i_j, s_k) and (i_j, s_j). Like in case (c_1), consider the linear order O_H obtained by extending the set AH:

$$O_H = \langle i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{k-1}, s_k, i_k, s_j, i_j, s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{k-1} \rangle \ .$$

It is not difficult to see that $x^{A(O_H)} \in F_a \subseteq F_b$. Thus $b^T x^{A(O_H)} - b^T x^{A(O_G)} = 0$, where O_G is the linear order defined in (c_1). Using previous results we can conclude that $b_{i_j,s_k} + b_{i_j,s_j} = 0$. From ($c_1$), it follows that $b_{i_j,s_k} = \lambda$.

(c_3) Consider now the digraph H obtained from G by changing the orientations of (i_k, s_r) and (i_r, s_r). As in (c_2), we can have the linear order:

$$O_H = \langle i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{k-1}, s_r, i_r, s_k, i_k, s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{k-1} \rangle \ .$$

Analogously to (c_2), we obtain $b_{i_r,s_r} + b_{i_k,s_r} = 0$. From ($c_1$), it follows that $b_{i_k,s_r} = \lambda$.

(c_4) Suppose w.l.o.g that $r < j$ and consider the digraph H obtained from G changing the orientations of the arcs (s_k, i_k), (i_j, s_r), (i_j, s_j) and (i_r, s_r). The set AH can be extended to the following linear order:

$$\begin{aligned} O_H = \langle & i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k, s_r, i_r, s_j, i_j, \\ & s_1, \ldots, s_{r-1}, s_{r+1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_k \rangle \ . \end{aligned}$$

Note that $A(O_H)$ is a complete pre-order and that all the positive arcs of G, with exception of the arcs (i_j, s_r) and (s_k, i_k), are in $A(O_H)$. Since only the negative arcs (i_j, s_j) and (i_r, s_r) of G are not in $A(O_H)$, it follows that

$$x(A(O_H) \cap AG_+) + x(A(O_H) \cap AG_-) = k^2 - 2k + 2 \ .$$

Thus $x^{A(O_H)} \in F_A$. Since $F_a \subseteq F_b$, it follows that $b^T x^{A(O_G)} - b^T x^{A(O_H)} = 0$, where O_G is the linear order defined in (c_1). From (a) and (b), we can conclude that $b_{s_k,i_k} + b_{i_r,s_r} + b_{i_j,s_j} + b_{i_j,s_r} = 0$. Since $b_{i_r,s_r} = b_{i_j,s_j} = -\lambda$ by (c_1) and $b_{s_k,i_k} = \lambda$ by hypothesis, it follows that $b_{i_j,s_r} = \lambda$, completing the proof of (c).

From (c_1), (c_2), (c_3) and (c_4), we conclude that $b_\alpha = \lambda a_\alpha$, if $\alpha \in AG$, $\lambda \in \mathbb{R}$. From (a), (b) and (c), we conclude that $b = \lambda a$, for $\lambda \in \mathbb{R}^+$. Therefore $a^T x \leq a_0$ is facet-defining for

$\mathcal{P}^{2k}$. Since the hypotheses of the lifting theorem are satisfied by this inequality, it follows that $x(AG_+) - x(AG_-) \leq k^2 - 2k + 2$ is a facet-defining for $\mathcal{P}^n$, $n \geq 2k$. ■

Note that for $n \geq 6$ the class of semi-k-fence inequalities, where $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, is very large. Its cardinality is $\sum_{k=3}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{2k} \binom{2k}{k} k \right]$.

3.2 Pentagon and Heptagon-inequalities

Let C be the arc set of a directed cycle of D_n, say $C = \langle a_1, \dots, a_k \rangle$, where $a_i = v_i v_{i+1}$ for $i = 1, \dots, k-1$ and $a_k = v_k v_1$. Then the set

$$CO := \{v_i v_{i+2} \in A_n \mid i = 1, \dots, k-2\} \cup \{v_{k-1} v_1, v_k v_2\}$$

is called the set of *2-chords* of C. For any directed cycle $C \subset A_n$ of length at least 5,

$$x(C) - x(CO) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

is called the *2-chorded cycle inequality* (induced by C) and

$$x(C) - x(CO) + x(\overline{C}) - x(\overline{CO}) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

is called the *double 2-chorded cycle inequality* (induced by C).

Lemma 6. Let C be a directed k-cycle of D_n, $5 \leq k \leq n$. Then the 2-chorded cycle inequality induced by C,

$$x(C) - x(CO) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

and the double 2-chorded cycle inequality induced by C,

$$x(C) - x(CO) + x(\overline{C}) - x(\overline{CO}) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

are valid for $\mathcal{P}^n$, $n \geq 5$.

Proof. It suffices to prove for $k = n$, as the lifting theorem gives the desired result. Suppose that $C = \langle (1,2), (2,3), \dots, (n-1,n), (n,1) \rangle$. Then $CO = \{(1,3), (2,4), \dots, (n-1,1), (n,2)\}$. For each $\alpha \in CO$, let β and γ be the two arcs in C such that $\{\alpha, \beta, \gamma\}$ is a triangle in D_n. Consider the associated triangle-inequality $x_\beta + x_\gamma - x_\alpha \leq 1$. Adding up, for all $\alpha \in CO$, these triangle-inequalities we obtain

$$\sum_{\alpha \in CO} (x_\beta + x_\gamma - x_\alpha) = 2x(C) - x(CO) \leq |CO| = |C| \ .$$

Since $-x(CO) \leq 0$ is valid for $\mathcal{P}^n$, adding this inequality to the latter one and then dividing by 2, we obtain that $x(C) - x(CO) \leq \frac{1}{2}|C|$ is valid for $\mathcal{P}^n$. Since for every vertex of $\mathcal{P}^n$ the left-hand side of this inequality is an integer, we can round the right-hand side down to the next integer and obtain the validity of the 2-chorded cycle inequality induced by C.

In the same way as we obtained the inequality $2x(C) - x(CO) \leq |C|$, adding up triangle-inequalities we obtain the the inequality $2x(\overline{C}) - x(\overline{CO}) \leq |C|$.

From the total-inequalities we obtain the validity of $-x(CO) - x(\overline{CO}) \leq -|C|$.

Adding up the three last inequalities, we have

$$2x(C) + 2x(\overline{C}) - 2x(CO) - 2x(\overline{CO}) \leq |C| \ .$$

Thus the validity of the double 2-chorded inequality induced by C can be obtained by dividing both sides of this last inequality by 2 and round the right-hand side down to the next integer. ■

Theorem 7. The pentagon-inequality

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 4 \ ,$$

induced by a directed 5-cycle C of D_n, as defined above, is facet-defining for $\mathcal{P}^n$, $n \geq 5$.
Proof. Let $a^T x \leq a_0$ denote the pentagon-inequality induced by C. We prove first the following assertions:

(i) $a^T x \leq a_0$ is valid for $\mathcal{P}^5$;

(ii) $a^T x \leq a_0$ is facet-defining for $\mathcal{P}^5$.

(i) This proof will be done by combining (and rounding down) valid inequalities for $\mathcal{P}^5$. From the total-inequalities, we obtain the validity of

$$-x(CO) - x(\overline{CO}) \leq -5 \ . \tag{1}$$

From the triangle-inequalities, we obtain the validity of

$$2x(C) - x(CO) \leq 5 \qquad \text{and} \tag{2}$$
$$x(\overline{C}) + x(\overline{CO}) - x(CO) \leq 5 \ . \tag{3}$$

To simplify, denote by (4) and (5) the 2-chorded 5-cycle and the double 2-chorded 5-cycle inequalities induced by C.

Considering the following combination of these inequalities, 2.(1) + 3.(2) + (3) + (4) + 2.(5), we obtain the inequality $9x(C) - 9x(CO) + 3x(\overline{C}) - 3x(\overline{CO}) \leq 16$. By dividing this inequality by 3 and rounding down we obtain

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 5 \ . \tag{6}$$

By dividing (5) + (6) by 2 and rounding down we obtain

$$2x(C) - 2x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 3 \ . \tag{7}$$

The sum 2.(1) + 3.(2) + (3) + (4) + (6) + 4.(7) results in the inequality $18x(C) - 18x(CO) + 6x(\overline{C}) - 6x(\overline{CO}) \leq 29$. The validity of the pentagon-inequality for $\mathcal{P}^5$ is obtained by dividing both sides of this last inequality by 6 and rounding down.

(ii) Suppose, w.l.o.g., that $C = \langle(1,2),(2,3),(3,4),(4,5),(5,1)\rangle$. The symbol $\oplus$ indicates that all the additions (for nodes) are made modulo 5. Denote by F_a the face defined by $a^T x \leq a_0$ and let F_b be a facet of $\mathcal{P}^5$ such that

$$F_a = \{x \in \mathcal{P}^5 \mid a^T x = a_0\} \subseteq F_b = \{x \in \mathcal{P}^5 \mid b^T x = b_0\} \ .$$

We will prove that there exists $\lambda \in \mathbb{R}^+$ such that $b = \lambda a$.

First, for each $i \in VC$, let O_i be the linear order of D_5 given by $\langle i, i\oplus 1, i\oplus 2, i\oplus 3, i\oplus 4\rangle$. Consider the following sets of arcs:

$$\begin{aligned} M_i &= A(O_i) \cup \{(i\oplus 1, i), (i\oplus 3, i\oplus 2)\} \quad \text{and} \\ M'_i &= A(O_i) \cup \{(i\oplus 1, i), (i\oplus 4, i\oplus 3)\} \ . \end{aligned}$$

It is easy to see that

$$x^{M_i}\,,\ x^{M'_i} \in F_a \subseteq F_b \quad \text{for every } i \in VC \ .$$

Therefore

$$b^T x^{M_i} = b^T x^{M'_i} \quad \text{for every } i \in VC \ .$$

Since the symmetric difference of $M_i \Delta M'_i$ is $\{(i\oplus 3, i\oplus 2), (i\oplus 4, i\oplus 3)\}$, we can conclude from the above results that $b_{i\oplus 3, i\oplus 2} = b_{i\oplus 4, i\oplus 3}$ for $i = 1, \ldots, 5$. Thus there exists $\lambda \in \mathbb{R}$ such that $b_\alpha = \lambda$ for every $\alpha \in \overline{C}$.

Consider now, for each $i \in VC$, the following arc set:

$$M''_i = M_i \cup \{(i\oplus 4, i\oplus 3), (i\oplus 4, i\oplus 2)\} \ .$$

From $x^{M''_i} \in F_a \subseteq F_b$, we obtain $b^T x^{M_i} = b^T x^{M''_i}$. Thus $b_{i\oplus 4, i\oplus 2} = -b_{i\oplus 4, i\oplus 3} = -\lambda$. From this, we can easily conclude that $b_\alpha = -\lambda$ for all $\alpha \in \overline{CO}$.

Analogously, we can obtain $b_\beta = \mu$ for all $\beta \in C$ and $b_\beta = -\mu$ for all $\beta \in CO$, for some $\mu \in \mathbb{R}$.

The inequality $b^T x \leq b_0$ can be written as

$$\mu x(C) - \mu x(CO) + \lambda x(\overline{C}) - \lambda x(\overline{CO}) \leq b_0 = \lambda + \mu \ . \tag{8}$$

Consider now the linear order $O = \langle 2,5,3,1,4\rangle$ and $A = A(O)$. It is easy to verify that $x^A \in F_a$. Since $F_a \subseteq F_b$, from (8) we can conclude that $2\mu - 2\lambda = \lambda + \mu$. Thus $\mu = 3\lambda$ and $b^T x \leq b_0$ can be written as $3\lambda x(C) - 3\lambda x(CO) + \lambda x(\overline{C}) - \lambda x(\overline{CO}) \leq 4\lambda$. Therefore $b = \lambda a$ for $\lambda \in \mathbb{R}^+$ and $a^T x \leq a_0$ is a facet of $\mathcal{P}^5$.

Now to conclude the proof it suffices to use the lifting theorem. Note that in the last paragraph we exhibited a linear order $A \subseteq A_5$ such that $a^T x^A = a_0$. ■

In the next example we show an instance of the complete pre-order problem, whose optimum solution could be obtained only after adding a pentagon-inequality to a relaxation of the problem.

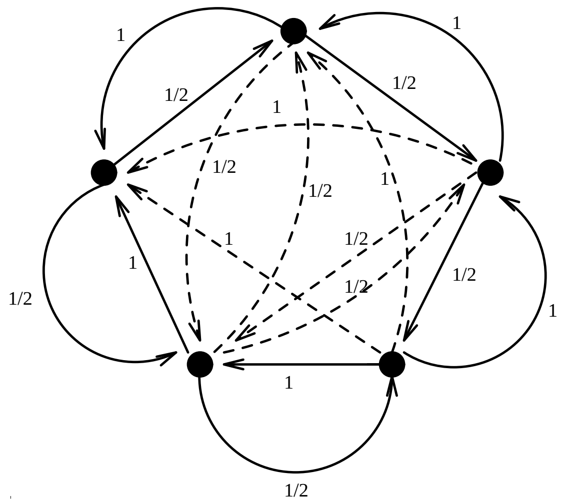

Figure 2:

Let

$$Q^5 := \Big\{ x \in \mathbb{R}^{A_5} \mid \begin{array}{ll} 0 \le x_\alpha \le 1 & \text{for all } \alpha \in A_5 \ , \\ x_{uv} + x_{vu} \ge 1 & \text{for all } u, v \text{ in } V_5 \ , \\ x_{uv} + x_{vw} - x_{uw} \le 1 & \text{for all } u, v, w \text{ in } V_5 \Big\} \ , \end{array}$$

and consider the weight function w given by $w^T x = 3x(C) - 3x(CO) + x(\overline{C}) + x(\overline{CO})$, where C is a directed hamiltonian circuit of D_5. The optimum solution x^* to the problem $max\{w^T x : x \in Q^5\}$ is shown in Figure 2, where $x^*_\alpha = 0$ if α is not drawn.

After adding the pentagon-inequality induced by the circuit C (mentioned above) to the system that defines Q^5, we obtained an integer solution x^* which is a vertex of $\mathcal{P}^5$. The solution x^* is shown in Figure 3, where $x^*_\alpha = 1$, if α is drawn, and $x^*_\alpha = 0$, if α is not drawn.

A straigtforward generalization of the pentagon-inequality to a directed circuit of length seven does not lead to a facet-defining inequality. We were able to find facet-defining inequalities induced by circuits of length seven, called here heptagon-inequalities, but they are rather complicated. They can be seen as generalizations of the pentagon-inequalities, but it is not clear to us how to generalize them further, at least in a way we are able to prove.

In what follows the symbol $\oplus$ indicates that the addition is taken modulo 7.

Definition 8. Let G be a digraph with $VG = \{1, 2, \ldots, 7\}$ and $AG = C \cup CO \cup \overline{C} \cup \overline{CO} \cup D \cup \overline{D}$, where $C = \langle (1,2), (2,3), \ldots, (6,7), (7,1) \rangle$, $CO = \{(i, i \oplus 2) \mid i = 1, \ldots, 7\}$ and $D = \{(i, i \oplus 3) : i = 1, \ldots, 7\}$. Then the inequality

$$4x(C) - 4x(CO) + 2x(\overline{C}) - 2x(\overline{CO}) + x(D) - x(\overline{D}) \le 9$$

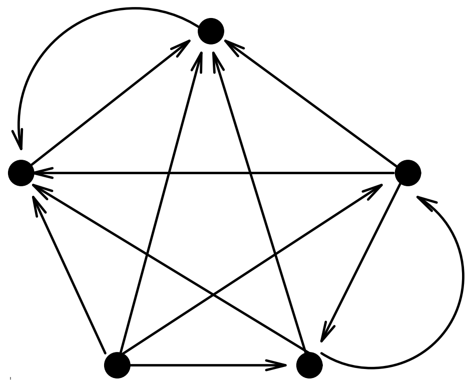

Figure 3:

induced by G is called *heptagon-inequality.*

Owing to its length (about 20 pages), the proof of the next theorem will be omitted, but the interested reader can find it in Gurgel [1992].

Theorem 9. The heptagon-inequality induced by the digraph G, as in Definition 8,

$$4x(C) - 4x(CO) + 2x(\overline{C}) - 2x(\overline{CO}) + x(D) - x(\overline{D}) \leq 9$$

is facet-defining for $\mathcal{P}^n$, $n \geq 7$.

4. Complexity of the separation problem for the facets of $\mathcal{P}^n$

It is easy to see that the separation problem for the triangle-, the pentagon- and the heptagon-inequalities is polynomially solvable. We are referring here to the straightforward algorithm where one has to check all subsets of 3, resp. 5 and 7 nodes of D_n (and all possible different inequalities induced by them). We shall prove now that the separation problem for the semi-k-fence inequalities is $\mathcal{NP}$-complete. To this end consider the following problem, which we call SFENCE.

Instance: A complete digraph $D_n = (V_n, A_n)$, a weight function $x : A_n \to [0,1]$ and an integer k, $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Question: Does D_n contain a semi-k-fence D' such that $x(AD'_+) - x(AD'_-) > k^2 - 2k + 2$?

Inspired on a result obtained by Müller [1996] — who has shown the $\mathcal{NP}$-completeness of the separation problem for the simple t-reinforced k-fence inequalities for the acyclic subgraph polytope — we could prove that the problem CLIQUE can be reduced to SFENCE.

The problem CLIQUE, defined below, is well-known to be $\mathcal{NP}$-complete (see Karp [1972]).

Instance: A simple connected graph $G = (V, E)$ and an integer k, $n \geq k \geq 3$.

Question: Does G contain a clique with at least k nodes?

Theorem 10. The problem SFENCE is $\mathcal{NP}$-complete.

Proof. It is immediate that SFENCE is in $\mathcal{NP}$. We show that CLIQUE can be reduced to SFENCE.

Let $G = (V, E)$ and k be given as an instance of CLIQUE. Suppose that $VG = \{v_1, \ldots, v_{n-1}\}$, $n \geq 4$, and $k \geq 3$. The corresponding instance of SFENCE is constructed as follows: a complete digraph D_p of order $p = 2n$ with node set $VD_p = \{i, \widehat{i} : v_i \in VG\} \cup \{n, \widehat{n}\}$; a weight function $x : AD_p \to [0, 1]$ defined as

$$x(a) = \begin{cases} \alpha & \text{if } a = (\widehat{i}, j) \text{ and } v_iv_j \in AG \ , \\ \alpha & \text{if } a \in \{(\widehat{i}, n), (\widehat{n}, i) : i = 1, \ldots, n-1\} \ , \\ 1 & \text{if } a = (n, \widehat{n}) \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where $\alpha = (k^2 - k + 1)/k^2$.

Note that if $0 < \alpha < 1$ and $v_iv_j \in AG$ then $x(\widehat{i}, j) = x(\widehat{j}, i) = \alpha$.

We claim that G contains a clique with at least $k-1$ nodes if and only if D_p contains a semi-k-fence D' such that

$$x(AD'_+) - x(AD'_-) > k^2 - 2k + 2 \ .$$

Suppose first that G contains a clique with $k-1$ nodes $v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}$. Then the node sets $\{i_1, \ldots, i_{k-1}, n\}$, $\{\widehat{i}_1, \ldots, \widehat{i}_{k-1}, \widehat{n}\}$ determine a semi-k-fence D' of D_p (with pivot $n\widehat{n}$) such that

$$x(AD'_+) - x(AD'_-) = \alpha(k^2 - k) + 1 = k^2 - 2k + 3 - 1/k > k^2 - 2k + 2 \ .$$

Suppose now that D_p contains a semi-k-fence D' such that $x(AD'_+) - x(AD'_-) > k^2 - 2k + 2$. Let $s_1, \ldots, s_k$ be the superior nodes, and $l_1, \ldots, l_k$ be the inferior nodes and (s_k, l_k) the pivot of D'. Note that the arc $(n, \widehat{n}) \in AD'_+$, otherwise, noting that D' has no directed 2-path with weight 2α, we can conclude that

$$x(AD'_+) - x(AD'_-) \leq \alpha(k^2 - k) = k^2 - 2k + 2 - 1/k \ ,$$

a contradiction. Also $x(s_k, l_k) > 0$ and $x(l_j, s_i) > 0$ for $i \neq j$, otherwise we have

$$x(AD'_+) - x(AD'_-) \leq k^2 - 2k + 2 - 1/k^2 \ ,$$

a contradiction.

Now it is easy to conclude that $(s_k, l_k) = (n, \widehat{n})$, otherwise D' would have $k-1$ arcs in AD'_+, with one endnode l_k or s_k, with weight zero, a contradiction. Thus $\{s_1, \ldots, s_{k-1}\} \subseteq \{1, \ldots, n-$

$1\}$, $\{l_1, \ldots, l_{k-1}\} \subseteq \{\widehat{1}, \ldots, \widehat{n-1}\}$ and each pair of arcs (l_j, s_i), (l_i, s_j) for $1 \le i \ne j \le k-1$ in D' corresponds to an arc in G. Therefore G has a clique with at least $k-1$ nodes and the proof is complete. ∎

Since we have a partial characterization of $\mathcal{P}^n$ we can use this to design a linear programming based cutting plane algorithm to solve CPO. In such an algorithm it would be natural to consider first the triangle-inequalities, since they are facet-defining and can be tested in polynomial time. Thus, the following question arises naturally: can we solve the separation problem for the semi-k-fence inequalities knowing that the triangle-inequalities are satisfied?

The next theorem shows that the separation problem for the semi-k-fences is $\mathcal{NP}$-complete even in this case.

Theorem 11. The problem SFENCE is $\mathcal{NP}$-complete even if the weight function satisfies the triangle-inequalities.

Proof. Consider the same reduction as described in the proof of Theorem 12, except that we define another weight function, given below:

$$x(a) = \begin{cases} \beta & \text{if } a = (\widehat{i}, j) \text{ and } v_i v_j \in AG \ , \\ 1 & \text{if } a \in \{(\widehat{n}, i), (\widehat{i}, n) : i = 1, \ldots, n-1\} \ , \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = (k^2 - 4k + 4.1)/(k^2 - 3k + 2)$.

Note that $0.1 < \beta < 1$ and the weight function x satisfies the triangle-inequalities.

For any semi-k-fence $\widehat{D}$ of D_{2n} let us call *k-fence-weight of $\widehat{D}$*, and denote by $x(\widehat{D})$ the difference $x(A\widehat{D}_+) - x(\widehat{AD})$. That is,

$$x(\widehat{D}) := x(\widehat{AD}_+) - x(\widehat{AD}_-) \ .$$

As in the proof of Theorem 12, suppose that G has a clique with $k-1$ nodes $v_{i_1}, \cdots, v_{i_{k-1}}$. Then the node sets $\{i_1, \ldots, i_{k-1}, n\}$, $\{\widehat{i}_1, \ldots, \widehat{i}_{k-1}, \widehat{n}\}$, determine a semi-$k$-fence D' in D_{2n} (with pivot $n\widehat{n}$) such that

$$x(D') = \beta[(k-1)^2 - (k-1)] + 2(k-1) = k^2 - 2k + 2.1 > k^2 - 2k + 2 \ .$$

Suppose now that D_{2n} contains a semi-k-fence D' such that $x(D') > k^2 - 2k + 2$. Let $L = \{l_1, \ldots, l_k\}$ be the set of inferior nodes, $S = \{s_1, \ldots, s_k\}$ the set of superior nodes and (s_k, l_k) the pivot of D'.

Note that the subdigraph of D_{2n} spanned by the arcs with positive weights has a bipartition, say (V^-, V^+), such that all its arcs are oriented from V^- to V^+.

Considering the weights defined on the arcs of D_{2n}, it is easy to see that if $\widehat{D}$ is a semi-k-fence of D_{2n} with fence-weight greater than $k^2 - 2k + 2$ then $\widehat{D}$ is of maximum k-fence-weight, $\widehat{D}$ has precisely $2(k-1)$ positive arcs with weight 1, precisely k^2-3k+2 positive arcs with weight β and $x(\widehat{D}) = k^2-2k+2.1$.

Suppose by contradiction that $x(s_k,l_k) > 0$. Thus $s_k \in V^-$ and $l_k \in V^+$. Therefore $x(l_i,s_k) = x(l_k,s_i) = 0$ for $1 \le i \le k-1$. We have two cases:

Case 1: $s_k = \widehat{n}$

In this case D' can have at most k positive arcs with weight 1, since only one arc of AD_+ incident to s_k may have positive weight. As D' has k^2-3k+2 positive arcs not incident to the pivot, we have

$$x(D') \le \beta(k^2-3k+2)+k < k^2-2k+2 \ .$$

Case 2: $s_k \neq \widehat{n}$

If $l_k = n$, we can have at most k positive arcs with weight 1, since only one arc of AD'_+ incident to n may have positive weight. The conclusion follows as in Case 1.

Suppose now that $l_k \neq n$. Therefore $x(s_k,l_k) = \beta$. Observe that D' can have at most $2(k-2)$ positive arcs with weight 1 and this can occur if $n \in S$ and $\widehat{n} \in L$. Since D' has k^2-3k+2 positive arcs not incident to the pivot, we have

$$x(D') \le \beta(k^2-5k+6)+2(k-2)+\beta < k^2-2k+2 \ .$$

In both cases we have a contradiction, thus $x(s_k,l_k) = 0$. Since $k^2-2k+2 < x(D') \le k^2-2k+2.1$ and $\beta > 0.1$, it follows that D' is a semi-k-fence of maximum weight. Therefore $S \subseteq V^+$, $L \subseteq V^-$, $n \in S$, $\widehat{n} \in L$ and $x(l_i,s_j) > 0$ for all $1 \le i \neq j \le k$. We can conclude that each pair of arcs (l_j,s_i), (l_i,s_j) in D' ($l_i,l_j \in L\backslash\{\widehat{n}\}$ and $s_i,s_j \in S\backslash\{n\}$) corresponds an arc in G. Thus G has a clique with at least $k-1$ nodes and the proof is complete. ■

Remark 1. The weigth function considered in the last theorem does not satisfy the total-inequalities. It would be interesting to know the complexity status of the problem SFENCE when the weigth function satisfies both the triangle- and the total-inequalities.

Remark 2. The construction and the weigths defined by Müller [1996] for the *partial fences* can be used to show– for the *linear ordering polytope*– that the (general) separation problem for the simple k-fence inequalities is $\mathcal{NP}$-complete. One has to consider the complete digraph obtained from the completion of the *partial fence*, and give weights zero to the newly added arcs.

Acknowledgments. We thank Martin Grötschel for the suggestions which led to the semi-k-fence inequalities.

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