Adjacency of vertices of the complete pre-order polytope

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A complete pre-order (c.p.o.) on a finite set V is a relation on V that is transitive and total. We define the *complete pre-order polytope* $\mathcal{P}^n$ as the convex hull of the incidence vectors of all complete pre-orders on a set with n elements. We study the adjacency relation of the vertices of the complete pre-order polytope $\mathcal{P}^n$, $n \geq 3$. The main results are: (1) if A, B are c.p.o. with $|A \triangle B| = 2$, then x^A and x^B are adjacent in $\mathcal{P}^n$ if and only if the digraph induced by $A \triangle B$ is connected; (2) if $A \subseteq B$ are c.p.o., then x^A and x^B are adjacent in $\mathcal{P}^n$ if and only if the digraph induced by $B \setminus A$ is connected; (3) the diameter of the complete pre-order polytope is equal to 2.

Key words: complete pre-order, diameter, polytope, adjacency of vertices.

1 Introduction

A relation R on a set S is *total* if $(a, b) \in R$ or $(b, a) \in R$ for all a, $b \in S$, $a \neq b$; R is *anti-symmetric* if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all a, $b \in S$; and R is *transitive* if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all a, b, $c \in S$, $a \neq b \neq c \neq a$.

A *complete pre-order* is a relation that is transitive and total. A complete pre-order that is anti-symmetric is a *linear order*.

Given a finite set V_n with n elements, we represent each complete pre-order A on V_n by means of its incidence vector, that is, a vector $x^A \in \mathbb{R}^{n(n-1)}$, defined as $x^A_a = 1$ if $a \in A$ and $x^A_a = 0$ if $a \notin A$. Then we consider the polytope

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defined as the convex hull of the incidence vectors of all complete pre-orders on V_n. We call this polytope the *complete pre-order polytope* and denote it by $\mathcal{P}^n$. It is easy to see that the linear ordering polytope is a face of $\mathcal{P}^n$.

The interest on the study of the polytope $\mathcal{P}^n$ appeared in the investigation of the following problem, called the *complete pre-order problem*: given a finite set V_n with weights $w_a \in \mathbb{R}$ for all $a \in V_n \times V_n$, find a complete pre-order A on V_n such that $w(A) := \sum_{a \in A} w_a$ is as large as possible. This problem has applications in data analysis and is known to be $\mathcal{N}P$-hard [12]. In a former paper we have studied the facial structure of the polytope $\mathcal{P}^n$, and obtained various classes of facet-defining inequalities (see Gurgel and Wakabayashi [4]).

In this paper we study the adjacency structure of the vertices of $\mathcal{P}^n$. We do not characterize the adjacency of any two vertices of $\mathcal{P}^n$, but we present some results characterizing adjacency of particular vertices of $\mathcal{P}^n$ which allow us to determine the diameter of $\mathcal{P}^n$. We show that the diameter of $\mathcal{P}^n$ equals two, for all $n \geq 3$. This proof is based on the fact that there is a vertex of $\mathcal{P}^n$ which is adjacent to all other vertices. The results presented here are of theoretical nature, not known to be useful—as those about the facets of $\mathcal{P}^n$ — in the design of a cutting-plane based algorithm for the complete pre-order problem.

2 Definitions and notation

Most of the results to be presented in this paper are easier to be stated using terminology from graph theory. We assume the reader to be familiar with the basic concepts in graph theory, so we mention here only those necessary to establish our notation. Any concept not mentioned here can be found in [1,5].

A *directed graph* or *digraph* $D = (V, A)$ consists of a finite nonempty set V of *nodes* and a set A of *arcs* which are ordered pairs of distinct elements of V. An arc $\alpha = (u, v)$ will also be denoted by uv. The nodes u and v are called the *endnodes* of α. If $\alpha = (u, v)$ is an arc, $\overline{\alpha}$ denotes the *reverse arc* of α, i.e., $\overline{\alpha} = (v, u)$.

If H is a digraph (graph) sometimes we may denote its node set by $V(H)$ and its arc set by $A(H)$. The complete digraph on n nodes is denoted by $D_n = (V_n, A_n)$. If D is a digraph and $X \subseteq A(D)$, then $[X]$ denotes the subdigraph of D spanned by X, defined as the subdigraph of D whose arc set is X and node set consists of the endnodes of the arcs in X.

A path P in a digraph D will be denoted either by the sequence of its arcs or by the sequence of its nodes, i.e., $P = \langle w_1, w_2, \ldots, w_k \rangle$, where each w_i is in $A(D)$ or each w_i is in $V(D)$. A *directed path* is a path in which all arcs have the same direction. A digraph D is *connected* if there is a path with endnodes i and j, for all pairs i, j of distinct nodes of D; we call D *disconnected* if it is not connected. A digraph is *unilaterally connected* (respectively, *strongly connected*) if for any two distinct nodes i and j there is a directed path from i to j or (respectively, and) from j to i. A maximal strongly connected subdigraph of a digraph is called a *dicomponent*.

The short form *c.p.o.* stands for *complete pre-order*.

Note that whenever we have a c.p.o. on V_n, say T, we may consider that $[T]$ is a subdigraph of the complete digraph $D_n = (V_n, A_n)$. This will always be assumed, even if not explicitly stated. The dicomponents of $[T]$ are complete subdigraphs. For short, they will be called *dicliques*; if we contract each of them we obtain the so-called *condensed digraph*. The arcs of this digraph define a linear order (on the dicliques of $[T]$) that will be denoted by $LOT = \langle T_1, \ldots, T_k \rangle$, being $T_1, \ldots, T_k$ the dicliques of $[T]$. Note that LOT well-defines T.

If T is a c.p.o. on V_n, an arc $\alpha \in T$ is called *local* if its ends are in the same diclique of LOT; otherwise α is called a *binding* arc of T. Note that if $\alpha = (u, v)$ is a binding arc of T such that $u \in V(T_i)$ and $v \in V(T_j)$, then $i < j$.

Let S and T be two c.p.o. on V_n. We say that S is a *refinement* of T if $S \subseteq T$. To show the structure of the graph $[T \setminus S]$ when S and T are c.p.o. with $S \subseteq T$, we give an example below and indicate in Figure 1 the corresponding graphs $[S]$, $[T]$ and $[T \setminus S]$. In this figure, the arcs between two non-consecutive dicliques (implied by transitivity) are omitted.

Example. Let S and T be c.p.o. on $V_{10} = \{1, 2, \ldots, 10\}$ with

$$\begin{aligned} LOS &= \langle\ [\{1,2\}], [\{3,5\}], [\{4,6\}], [\{7\}], [\{8,9\}], [\{10\}]\ \rangle \quad \text{and} \\ LOT &= \langle\ [\{1,2\}], [\{3,4,5,6\}], [\{7\}], [\{8,9,10\}]\ \rangle\ . \end{aligned}$$

We denote by $A \triangle B$ the *symmetric difference* of two sets A and B, that is, $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Basic concepts in polyhedral theory can be found in Grünbaum [3] or Brønsted [2]. Definitions and more general results on adjacency of vertices on 0/1-polytopes can be found in Hausmann [6].

In the next lemma we mention some equivalent formulations of the adjacency concept to be used in this paper (see Pulleyblank [9]).

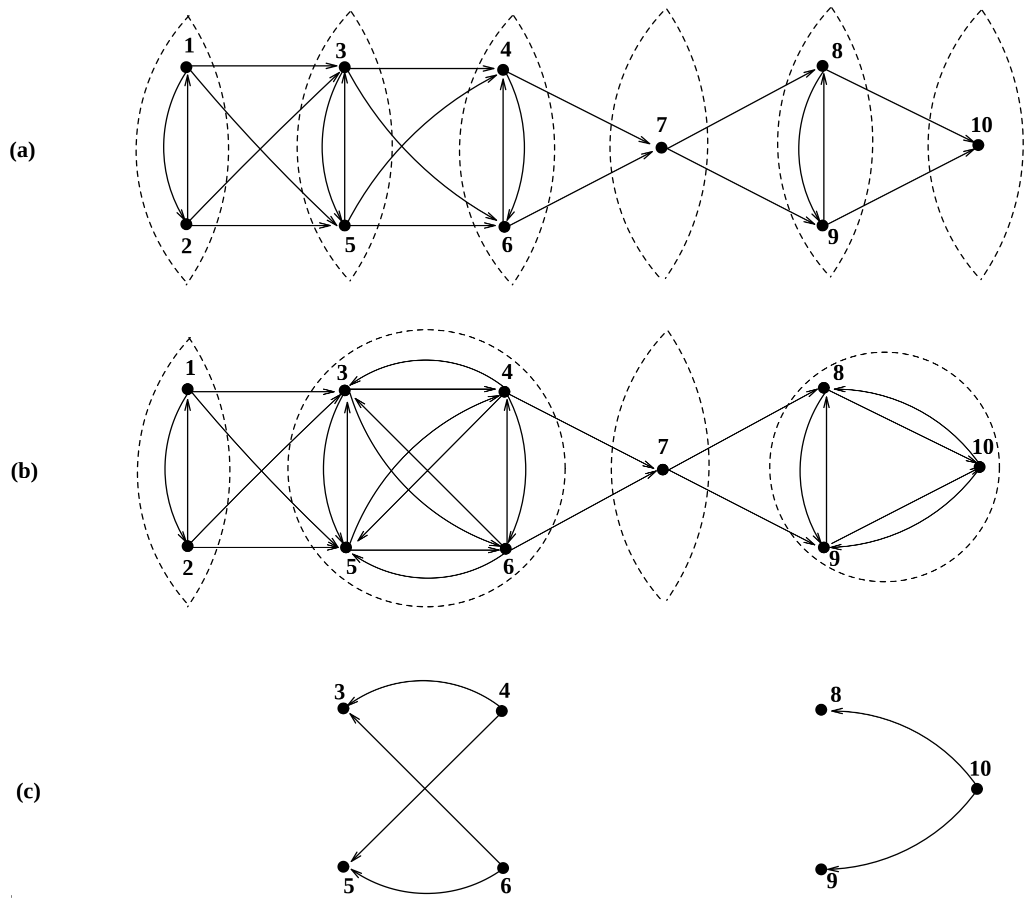

Figure 1. (a) The graph $[S]$. (b) The graph $[T]$. (c) The graph $[T \setminus S]$.

Lemma 1 *Let $P \subset R^n$ be a polytope and let x and y be two distinct vertices of P. Then the following statements are equivalent:*

(1) *x and y are adjacent (with respect to P);*
(2) *Any point on the line segment connecting x and y can be represented uniquely as a convex combination of vertices of P;*
(3) *The point $\frac{1}{2}x + \frac{1}{2}y$ can be represented uniquely as a convex combination of vertices of P;*
(4) *There exists a vector $c \in R^n$ such that x and y are the only vertices of P which maximize the function $c^T x$ over P.*

The *skeleton* of a polytope $\mathcal{P}$ is a graph that has a node for each vertex of $\mathcal{P}$, two nodes being adjacent if and only if the corresponding vertices are adjacent in $\mathcal{P}$. The *diameter* of $\mathcal{P}$ is defined as the diameter of the skeleton of $\mathcal{P}$, and is denoted by $\mathrm{diam}(\mathcal{P})$.

3 Adjacency of vertices of $\mathcal{P}^n$ and its diameter

In this section we first characterize adjacency of two vertices x^S and x^T of $\mathcal{P}^n$ when $|S \triangle T| = 2$. To this end, the following proposition will be used.

Proposition 2 *Let S and T be two c.p.o. on V_n such that $S \setminus T = \{a\}$ and $T \setminus S = \{b\}$; and let H be the digraph spanned by $\{a, b\}$. Then H is disconnected if and only if $S \cup T$ and $S \cap T$ are c.p.o. on V_n.*

PROOF. Suppose first that $S \cup T$ and $S \cap T$ are c.p.o. on V_n. Now assume by contradiction that H is connected. Since $(S \cap T) \cap \{a, b\} = \emptyset$ and $S \cap T$ is a c.p.o., it follows that $\{\overline{a}, \overline{b}\} \subseteq S \cap T$. Furthermore, since H is connected, $[\{\overline{a}, \overline{b}\}]$ is also connected. Let i and j be the endnodes of the path $\langle \overline{a}, \overline{b} \rangle$. Since T is a c.p.o., $\{\overline{a}, \overline{b}, b\} \subseteq T$ and $a \notin T$, it follows that $(i, j) \in T$ if and only if $(j, i) \notin T$. Let $\alpha \in \{(i, j), (j, i)\}$ and suppose, w.l.o.g., that $\alpha \notin T$. Since $\alpha \neq a$, we conclude that $\alpha \notin S$ and therefore $\alpha \notin S \cup T$. From $\{a, \overline{a}, b, \overline{b}\} \subseteq S \cup T$ and $\alpha \notin S \cup T$, it follows that $[S \cup T]$ is not transitive, which implies that $S \cup T$ is not a c.p.o. on V_n, a contradiction.

To prove the converse, suppose H is disconnected and let us prove that $S \cup T$ and $S \cap T$ are c.p.o. on V_n. Since $S \subseteq S \cup T$ and S is a c.p.o., it follows that $[S \cup T]$ is a total relation on V_n. Now, to show that $[S \cup T]$ is transitive, let (i, j) and (j, k) be two arcs in $S \cup T$. At least one of them is not contained in $\{a, b\}$. Thus both are contained in S or T and therefore the arc (i, k) is in $S \cup T$. Thus $S \cup T$ is a c.p.o. on V_n.

Clearly, $[S \cap T]$ is transitive, since S and T are c.p.o. on V_n. If $[S \cap T]$ were not total, then we would have $b = \overline{a}$. In this case, H would be connected, a contradiction.

Therefore, $S \cap T$ and $S \cup T$ are c.p.o. on V_n and the proof is complete. $\square$

We note that Proposition 2 remains true if we replace "H is disconnected" by "H is not unilaterally connected". The proof follows analogously. In fact, it can be proved directly that — under the assumptions of Proposition 2— "H is connected" implies "H is unilaterally connected".

Theorem 3 *Let S and T be two c.p.o. on V_n such that the subdigraph $H = [S \triangle T]$ has precisely two arcs. Then x^S and x^T are adjacent in $\mathcal{P}^n$ if and only if H is connected.*

PROOF. If $S \subseteq T$ or $T \subseteq S$ the result follows as a corollary of Theorem 8. Suppose therefore that $S \setminus T = \{a\}$ and $T \setminus S = \{b\}$ and consider the middle point y of the line segment which connects x^S and x^T in $\mathcal{P}^n$, that is, $y = \frac{1}{2}x^S + \frac{1}{2}x^T$.

Suppose also that $R_1, \ldots, R_k$ are distinct c.p.o. on V_n, such that:

$$y = \sum_{i=1}^{k} \lambda_i x^{R_i}, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad 0 < \lambda_i < 1.$$

In this case, if $\alpha \in S \cap T$, then

$$1 = \sum_{i=1}^{k} \lambda_i x_\alpha^{R_i} \ ,$$

and if $\alpha \in A_n \setminus (S \cup T)$, then

$$0 = \sum_{i=1}^{k} \lambda_i x_\alpha^{R_i} \ .$$

Since $\lambda_i > 0$ for $i = 1, \ldots, k$, it follows that

$$x_\alpha^{R_i} = \begin{cases} 1, & \text{if } \alpha \in S \cap T \\ 0, & \text{if } \alpha \in A_n \setminus (S \cup T) \end{cases} \ .$$

Therefore,

$$S \cap T \subseteq R_i \subseteq S \cup T \quad \text{for } i = 1, \ldots, k \ .$$

Considering the components related to the arcs a and b of the vertices x^{R_i}, x^S and x^T, it can be easily seen that the only alternative convex combination for $y = \frac{1}{2}x^S + \frac{1}{2}x^T$ is $y = \frac{1}{2}x^{S\cup T} + \frac{1}{2}x^{S\cap T}$. Thus, by Lemma 1, x^S and x^T are adjacent in $\mathcal{P}^n$ if and only if it does not hold that both $S \cup T$ and $S \cap T$ are c.p.o. on V_n. Hence, by Propositition 2, we conclude that x^S and x^T are adjacent in $\mathcal{P}^n$ if and only if H is connected. $\square$

In the sequel we characterize adjacency of vertices corresponding to complete pre-orders that are inclusionwise related, that is, vertices x^S and x^T such that S is a refinement of T. This result will play an important role to determine the diameter of $\mathcal{P}^n$.

Proposition 4 *Let S and T be two c.p.o. on V_n with $LOS = \langle S_1, \dots, S_p\rangle$ and $LOT = \langle T_1, \dots, T_q\rangle$. If $S \subseteq T$, then for every i, j such that $1 \le i < j \le p$, the following assertions hold:*

(a) *$S_i \subseteq T_\ell$, for some ℓ, $1 \le \ell \le q$.*
(b) *If $S_i, S_j \subseteq T_\ell$ $(1 \le \ell \le q)$, then $S_k \subseteq T_\ell$ for all k, $i \le k \le j$.*

PROOF. The proof of (a) is immediate since S is a refinement of T. To prove (b), suppose by contradiction that there are $S_i, S_j \subseteq T_\ell$ and $S_k \subseteq T_m$, $i < k < j$ and $m \ne \ell$. Let u, v and w be nodes in $V(S_i)$, $V(S_k)$ and $V(S_j)$, respectively. From $(u, v) \in S$, it follows that $(u, v) \in T$. Since $u \in V(T_\ell)$ and $v \in V(T_m)$, (u, v) is a binding arc of T; thus, $\ell < m$. On the other hand, since $(v, w) \in S$, we can conclude similarly that (v, w) is a binding arc of T, which implies $m < \ell$, a contradiction. $\square$

The following corollary is immediate.

Corollary 5 *Let S and T be as in Proposition 4, $S \subseteq T$. Then the nodes of each diclique of LOT are nodes of consecutive dicliques of LOS. More precisely, LOS can be written as:*

$$LOS = \langle S_1 = \underbrace{S_1^1, \dots, S_{t_1}^1}, \underbrace{S_1^2, \dots, S_{t_2}^2}, \dots, \underbrace{S_1^q, \dots, S_{t_q}^q} = S_p\rangle\ ,$$

where $\quad V(S_1^i) \cup V(S_2^i) \cup \dots \cup V(S_{t_i}^i) = V(T_i), \quad$ *for $i = 1, \dots, q$.*

Proposition 6 *Let S and T be c.p.o. on V_n and $H = [T \backslash S]$, with components $H_1, \dots, H_p$, $p \ge 2$. If $S \subseteq T$ and $LOT = \langle T_1, \dots, T_q\rangle$, then for every i, j, $1 \le i < j \le p$, the following assertions hold:*

(a) *$H_i \subseteq T_\ell$, for some ℓ, $1 \le \ell \le$.*
(b) *If $H_i \subseteq T_\ell$ and $H_j \subseteq T_m$ $(1 \le \ell, m \le q)$, then $\ell \ne m$.*

PROOF. The validity of (a) is immediate, since S is a refinement of T. To prove (b), suppose by contradiction that $H_i, H_j \subseteq T_\ell$, for some ℓ, $1 \le \ell \le q$. Since H is spanned by a set of arcs, H does not have isolated nodes. Let $\alpha = (u, v) \in A(H_i)$ and $w \in V(H_j)$. Since (u, w) and (w, v) are in $T \setminus A(H)$, they are in S. From the transitivity of S, it follows that $\alpha \in S$, a contradiction. $\square$

The next result shows that there exists a special relation between the dicliques of LOT and LOS, when $S \subseteq T$ and $H = [T \backslash S]$ is connected. The proof follows immediately from Corollary 5 and Proposition 6.

Corollary 7 *Let S and T be two distinct c.p.o. on V_n such that $S \subseteq T$ and let $H = [T \setminus S]$. Then H is connected if and only if there is precisely one diclique in LOT whose node set is the union of the node sets of two or more consecutive dicliques of LOS.*

Theorem 8 *Let S and T be two distinct c.p.o. on V_n such that $S \subseteq T$ and let $H = [T \setminus S]$. Then x^S is adjacent to x^T in $\mathcal{P}^n$ if and only if H is connected.*

PROOF. Suppose first that H is connected. From Proposition 6, it follows that $H \subseteq T_i$, for some diclique T_i of LOT. Suppose, w.l.o.g., that $H \subseteq T_1$. From Corollary 7, we can conclude that T_1 is the unique diclique of LOT whose node set is the union of the node sets of two or more dicliques of LOS. From Corollary 5 we can assume that

$$LOT = \langle T_1, \ldots, T_k \rangle \quad \text{and} \quad LOS = \langle S_1, \ldots, S_m, \ldots, S_{m+k-1} \rangle \ ,$$

where $\quad V(T_1) = \bigcup_{i=1}^{m} V(S_i),\ m \geq 2, \quad$ and $\quad V(T_j) = V(S_{m+j-1}),\ \ 2 \leq j \leq k.$

Let $\alpha = (w, z) \in A(H)$, with $w \in V(S_m)$ and $z \in V(S_1)$. Denote by t the number of arcs of $A(T_1) \setminus S$ and consider the function $p\ :\ A_n \rightarrow \mathbb{Z}$ defined as:

$$p(\beta) = \begin{cases} t-1, & \text{if } \beta = (w, z) \\ -1, & \text{if } \beta \in A(T_1) \setminus S, \quad \beta \neq (w, z) \\ t, & \text{if } \beta \in S \\ -t, & \text{if } \beta \in A_n \setminus T. \end{cases}$$

Then $\ p(T) = p(S) + p(A(T_1) \setminus S) = t|S|$, and therefore,

$$p(T) = p(S) = t|S|. \tag{1}$$

Let R be a c.p.o. on V_n whose incidence vector maximizes the function $\sum\limits_{\ell \in A_n} p_\ell x_\ell$. From (1), it follows that

$$\sum_{\ell \in A_n} p_\ell x_\ell^R \geq t|S|. \tag{2}$$

Since $p(\beta) = t$ for every arc $\beta \in S$ and, except for the arc (w, z), every other arc has negative weight, from (2) we can conclude that $S \subseteq R$ and if $p(\beta) = -t$, then $\beta \notin R$.

If $(w, z) \notin R$, we can conclude from (2) that $\beta \notin R$ for every arc $\beta \in A(T_1) \setminus S$. In this case, $R = S$. If $(w, z) \in R$, from the transitivity of R it is easy to see that $\beta \in R$ for every arc $\beta \in A(T_1)$. In this case, $R = T$. Therefore, x^T and x^S are the unique vertices of $\mathcal{P}^n$ that maximize the above function, which implies that they are adjacent.

To prove that the connectedness of H is a necessary condition, suppose on the contrary that H has components $H_1, \ldots, H_p$, $p \geq 2$, and consider the subsets of A_n defined by

$$Q_i = S \cup A(H_i), \quad i = 1, \ldots, p.$$

From Proposition 6, there exists $T_\ell \in LOT$ such that $H_1 \subseteq T_\ell$. Suppose, w.l.o.g., that $\ell = 1$. It is not difficult to see that Q_1 is a c.p.o. on V_n such that:

$$LOQ_1 = \langle T_1, S_t, S_{t+1}, \ldots, S_k \rangle, \quad \text{where}$$
$$LOS = \langle S_1, \ldots, S_t, \ldots, S_k \rangle, \quad \text{and} \quad V(T_1) = \bigcup_{j=1}^{t-1} V(S_j).$$

Analogously, it can be verified that each Q_i, $i = 2, \ldots, p$, is a c.p.o. on V_n. Thus, x^{Q_i} is a vertex of $\mathcal{P}^n$, for $i = 1, \ldots, p$.

Consider now an interior point, say y_λ, of the line segment joining x^T to x^S in $\mathcal{P}^n$, i.e.,

$$y_\lambda = \lambda x^S + (1 - \lambda) x^T, \quad \text{where } 0 < \lambda < 1 .$$

Note that, since $S \subseteq T$, then $y_\lambda = x^S + (1-\lambda)x^{T \setminus S}$. On the other hand, using previous results, we obtain

$$y_\lambda = (1 - p + p\lambda) x^S + \sum_{i=1}^{p} (1 - \lambda) x^{Q_i} .$$

Thus, if we take λ such that $\dfrac{p-1}{p} < \lambda < 1$, it follows that y_λ can be written in two different ways as a convex combination of vertices of $\mathcal{P}^n$. Therefore, x^S is not adjacent to x^T in $\mathcal{P}^n$, a contradiction. $\square$

Corollary 9 *Let T be a c.p.o. on V_n, $n \geq 3$, such that $T \neq A_n$. Then x^T is adjacent to x^{A_n} in $\mathcal{P}^n$.*

PROOF. By Theorem 8, it suffices to prove that $H = [A_n \setminus T]$ is connected. To this end, note that $V(H) = V_n$ and consider $LOT = \langle T_1, \ldots, T_k \rangle$, $u \in V(T_i)$ and $v \in V(T_j)$, such that $u \neq v$. If $i < j$ $(i > j)$ then $(v, u) \in A(H)$

$((u,v) \in A(H))$ and if $i = j$ there exists a path in H from u to v, passing through a node that does not belong to $V(T_i)$. $\square$

Note that in the two results seen for the adjacency of the special vertices x^S and x^T considered (Theorem 3 and Theorem 8), the connectedness of the subdigraph $[S \triangle T]$ is a necessary and sufficient condition.

In fact, one could suspect that this would be a necessary and sufficient condition for the adjacency of *any* two vertices x^S and x^T. The example given in the proof below, where $[S \triangle T]$ is connected and x^S is not adjacent to x^T, shows that the connectedness of $[S \triangle T]$ is not a sufficient condition.

We shall prove now the main result of this paper.

Theorem 10 $\mathrm{Diam}(\mathcal{P}^n) = 2$, *if* $n \geq 3$.

PROOF. From Corollary 9, we can conclude that $\mathrm{diam}(\mathcal{P}^n) \leq 2$. To conclude the theorem, it remains to show that there are c.p.o. S and T on V_n, such that x^S is not adjacent to x^T in $\mathcal{P}^n$. For that, let $V_n = \{1, 2, \ldots, n\}$ and take $S = \langle 1, 2, \ldots, n \rangle$ and $T = \langle n, n-1, \ldots, 1 \rangle$, linear orders on V_n. Note that $H = [S \triangle T]$ is connected. Now take i such that $1 \leq i < n$ and define the linear orders:

$$
\begin{aligned}
Q &= \langle 1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n \rangle \quad \text{and} \\
R &= \langle n, n-1, \ldots, i+2, i, i+1, i-1, \ldots, 1 \rangle.
\end{aligned}
$$

Then, it is immediate that if y is the middle point of the line segment joining x^S to x^T in $\mathcal{P}^n$, we have:

$$
y = \frac{1}{2}x^S + \frac{1}{2}x^T = \frac{1}{2}x^Q + \frac{1}{2}x^R \ .
$$

Thus, x^S is not adjacent to x^T in $\mathcal{P}^n$. $\square$

4 Concluding remarks

An interesting question that remains open concerns the characterization of the adjacency of any two vertices of $\mathcal{P}^n$. In 1978, Young [10] obtained a simple condition characterizing adjacency of any two vertices on the linear ordering polytope, in terms of the structure of the permutations. We were not able to obtain similar results extending the results due to Young. Note, however, that if S and T are different complete pre-orders on V_n with the same dicliques, then the condensed digraph of S (respectively, of T) defines a linear order on the dicliques of $[S]$ (respectively, of $[T]$). Thus, we can use Young's result to characterize adjacency of x^S and x^T in $\mathcal{P}^n$.

Considering also that we have proved that the diameter of $\mathcal{P}^n$ is two based on the fact that x^{A_n} is adjacent to every other vertex of $\mathcal{P}^n$, it would be interesting to know whether the diameter remains small if we delete the vertex x^{A_n} and consider the polytope defined as the convex hull of the remaining vertices.

There are other polytopes associated with $\mathcal{NP}$-hard problems that are known to have small diameter: the asymmetric TSP-polytope and the linear ordering polytope have both diameter 2 (Padberg and Rao [7]; Young [10]), and the clique partitioning polytope has diameter 3 (Wakabayashi [11]). For some of these polytopes there are polynomial time algorithms to check (non-)adjacency of two vertices; on the other hand, there are polytopes, like the symmetric TSP, for which checking non-adjacency of two vertices is an $\mathcal{NP}$-complete problem (Papadimitriou [8]). It should also be noted that the diameter of this last polytope has not been determined.

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