Clique Graphs and Edge-clique graphs

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Abstract

Not every edge-clique graph is a clique graph.

1 Introduction

Let G be a graph. The *clique graph of* G, $K(G)$, is the intersection graph of the family of all maximal cliques of G. The *edge-clique graph of* G, $K_e(G)$, is the one whose vertices are the edges of G, two vertices being adjacent in $K_e(G)$, when the corresponding edges of G belong to a same clique. A graph G is a *clique graph* (*edge-clique graph*) if there exist a graph H such that $K(H) = G$ ($K_e(H) = G$).

In a 1991 paper [4], Theorem 1, it is affirmed that *every edge-clique graph is a clique graph* (*). However, Prisner [1] noted that the proof of Theorem 1 is not correct. In this note we shall prove that (*) does not hold, i.e. we show that it is not always the case where an edge-clique graph is a clique graph.

In Section 2 we review some properties of both clique and edge-clique graphs and observe that every edge-clique graph whose largest clique has size at most three is a clique graph. In Section 3 we show an edge-clique graph that is not a clique graph and prove that it is a minimum counterexample to (*).

All graphs considered are finite, simple and undirected. The vertex and edge sets of a graph G are represented by $V(G)$ and $E(G)$, respectively. For $C \subseteq V(G)$, say that C is a *clique* when C induces a complete subgraph in G. A *maximal clique* is one not properly contained in any other. Let $\omega(G)$ denote

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the size of a largest clique of G. Let $\mathcal{C}$ be a family of cliques of G. Say that $\mathcal{C}$ is an *edge cover* (*K_4-cover*) if for all edge (complete subgraph of at most four vertices) e of G, there exists a member of $\mathcal{C}$ that contains all vertices of e. Say that $\mathcal{C}$ *satisfies the Helly Property* (*2-Helly Property*) if for every subfamily $\mathcal{C}' \subseteq \mathcal{C}$ such that the members of $\mathcal{C}'$ have pairwise one (two) element(s) in common, then all the members of $\mathcal{C}'$ have a (two) common element(s).

2 Clique graphs and edge-clique graphs

Clique graphs form one of the most interesting classes of intersection graphs [5]. In particular, they were characterized by Roberts and Spencer [6]. Whereas in [4,3] are presented characterizations of edge-clique graphs. Nevertheless, the recognition problems for both clique and edge-clique graphs remain open.

We shall use basically the two following fundamental results to the study of clique and edge-clique graphs, respectively.

Theorem 1 ([6]) *A graph is a clique graph if and only if it has an edge cover that satisfies the Helly Property.*

Theorem 2 ([2]) *There exists a one-to-one correspondence between maximal cliques (intersections of maximal cliques) of G and H, whenever $H = K_e(G)$. Moreover, if C is a maximal clique (intersection of maximal cliques) of G, then the corresponding clique of H has size $\binom{|C|}{2}$.*

Let G be a graph such that $\omega(G) \leq 3$. By Theorem 2, $\omega(K_e(G)) \leq 3$ and, moreover, there no exist in $K_e(G)$ two maximal cliques whose intersection has two elements. So the family of all maximal cliques of $K_e(G)$ is itself an edge cover that satisfies the Helly Property and by Theorem 1 we have:

Theorem 3 *If $\omega(G) \leq 3$ and G is an edge-clique graph, then G is a clique graph.*

By Theorem 2 does not exist an edge-clique graph G such that $\omega(G) = 4$ or 5. In the next section, we shall describe a graph whose edge-clique graph has largest clique of size 6 and is not a clique graph.

3 A counterexample

The existence and minimallity of such graph is based in the following.

Theorem 4 *If G has a K_4-cover that satisfies the 2-Helly Property, then $K_e(G)$ is a clique graph.*

PROOF. Direct translation of Theorem 1 to edge-clique graphs using the correspondence given by Theorem 2. ■

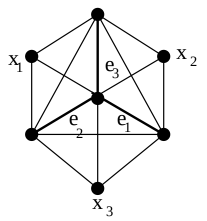
The graph P in Figure 1 shows that the converse of Theorem 4 does not hold.

Theorem 5 *Let G be a graph with $\omega(G) = 4$. Then there exists a K_4-cover of G satisfying the 2-Helly Property if and only if the graph P is not a subgraph of G.*

PROOF. Since $\omega(G) = 4$ every K_4-cover of G contains each K_4 of G as a member. As P contains three K_4's having pairwise two vertices in common that does not have two common vertices, if P is a subgraph of G, none K_4-cover of G could satisfy the 2-Helly Property.

To prove the converse assume that none K_4-cover of G satisfies the 2-Helly Property. Let $\mathcal{C}$ be the family of all maximal cliques of G. Observe that $\mathcal{C}$ is a K_4-cover of G. Let $\mathcal{C}' \subseteq \mathcal{C}$ be minimal with respect of not satisfying the 2-Helly Property. Let C_1, C_2 and C_3 be three different members of $\mathcal{C}'$ and let $\mathcal{C}'_i = \mathcal{C}' \setminus \{C_i\}$. Since each $\mathcal{C}'_i$ satisfies the 2-Helly Property, choose edge e_1 with both ends in $C_2 \cap C_3$ and at least one not in C_1. Analogously choose e_2 and e_3. So the ends of e_1, e_2 and e_3 are pairwise equal or adjacent and define at most four vertices since $\omega(G) = 4$. Let H be the subgraph of G generated by e_1, e_2 and e_3. We have $H \simeq K_3$, $H \simeq P_3$, or $H \simeq K_{1,3}$.

The two first cases contradict the choice of e_1, e_2 or e_3.



If $H \simeq K_{1,3}$. We have e_1, e_2, e_3 inducing a K_4 in G. Since C_1, C_2 e C_3 are maximal cliques, there exist three vertices $x_1 \in C_1$, $x_2 \in C_2$ and $x_3 \in C_3$ that are not ends of e_1, e_2, nor e_3. The subgraph induced by e_1, e_2, e_3, x_1, x_2 and x_3 contains P. ■

Since $\omega(G) = 4$, G does not contain a subgraph isomorphic to P is equivalent to G does not contain any induced subgraph isomorphic to one in Figure 1.

It is not difficult to prove that $K_e(P)$ and $K_e(P_1)$ are clique graphs. So the smallest candidate graph G with $\omega(G) = 4$ is P_2, and this in fact occur.

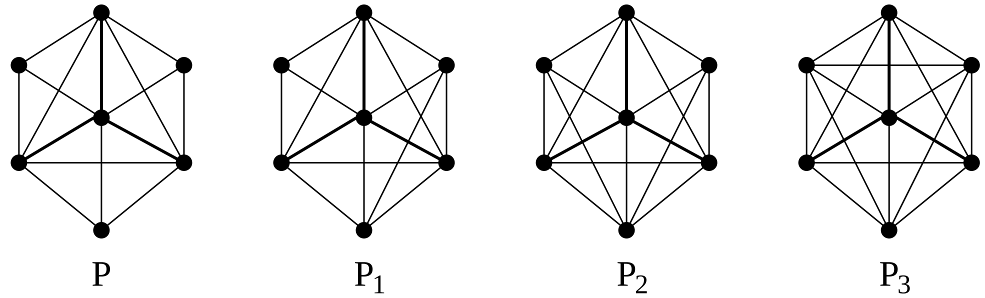


Fig. 1. Possibilities to the case $H \simeq K_{1,3}$.

Theorem 6 *$K_e(P_2)$ is not a clique graph.*

The graph $K_e(P_2)$ has 17 vertices and its family os maximal cliques is formed by exactly 6 maximal cliques of size 6.

The proof of Theorem 6 is by analyzing all the possibilities of an edge cover of $K_e(P_2)$ that satisfies the Helly Property. The search take advantage of the symmetry of the graph and of some forced configurations.

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