A Characterization of Edge Clique Graphs[1]

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Abstract

The *edge clique graph* of a graph G is one having as vertices the edges of G, two vertices being adjacent if the corresponding edges of G belong to a common clique.

We describe a characterization of edge clique graphs.

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1 Introduction

Edge clique graphs were introduced and first studied by Albertson and Collins in 1984 [1]. Some of the results concerning this class of graphs have been also described in [2, 3, 7, 8, 6] and they were also implicitly used by Kou, Stockmeyer and Wong, in 1978 [4]. In his book, Prisner [5] posed the problem of finding a good characterization for the class. So far, complete and simple characterizations of edge clique graphs have been only described for subclasses of chordal graphs [2]. In the present paper, we describe a simple general characterization for the class. Nevertheless, it leaves open the question of whether edge clique graphs can be recognized by a polynomial time algorithm.

All graphs considered are finite and simple. The vertex and edge sets of an undirected graph G are represented by $V(G)$ and $E(G)$, respectively. For $v \in V(G)$, denote by $N_G(v)$ the set of neighbours of v in G, while $N_G[v] = \{v\} \cup N_G(v)$. When convenient, drop the index G of the notation. If $u, v \in V(G)$ are neighbours, denote by uv the edge whose ends are u and v. A vertex adjacent to no other vertex is called *isolated*. For $S \subseteq V(G)$, say that S is a *clique* when S induces a complete subgraph in G. In special, if $N[v]$ is a clique, then v is a *simplicial* vertex. A *maximal clique* is one not properly contained in any other. In case that S induces a subgraph with no edges in G, then S is an *independent set*. On the other hand if $|S| = \binom{n}{2}$, for some $n = 0, 1, \ldots$, then S is a *triangular set* and $|S|$ a *triangular number*.

Let G be a graph. The *edge clique graph*, $K_e(G)$, of G is the one whose vertices are the edges of G, two vertices being adjacent in $K_e(G)$ when the corresponding edges of G belong to a same clique. A necessary condition for a graph H to be the edge clique graph of some graph G is that all its maximal cliques and intersections of maximal cliques ought to be triangular. That is,

Proposition 1 ([1]) *Let G, H be graphs such that $H = K_e(G)$. There exists a one-to-one correspondence between maximal cliques (intersections of maximal cliques) of G and H. Moreover, if C is a maximal clique (intersection of maximal cliques) of G then the corresponding clique of H has size $\binom{|C|}{2}$.*

An example of a graph (Figure 1) with triangular maximal cliques and intersections of maximal cliques that is not an edge clique graph has been described in [3].

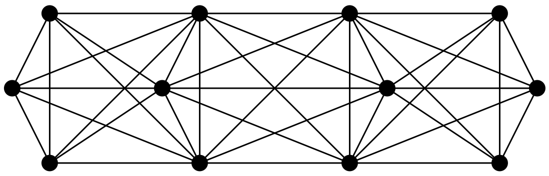
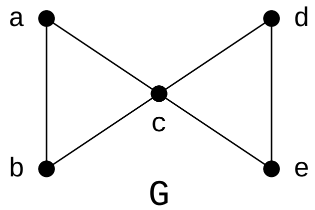


Figure 1: A graph that is not an edge clique graph.

2 Labellings and characterization

Let $G = (V, E)$ be a graph. A *k-labelling* of G is an assignment of a non-empty set $\ell(v) \subseteq \{1, \ldots, n\}$ to each vertex $v \in V(G)$, such that $|\ell(v)| = k$ and all label sets $\ell(v)$ are distinct.

Figure 2 shows a graph G and a 2-labelling of it.



$\ell(a) = \{1, 2\}$, $\ell(b) = \{2, 3\}$, $\ell(c) = \{1, 3\}$, $\ell(d) = \{1, 4\}$, $\ell(e) = \{2, 4\}$.

Figure 2: A graph and a 2-labelling.

For any subset $S \subseteq V(G)$, let $\ell(S)$ be the subset of integers i in $\{1, \ldots, n\}$, such that $i \in \ell(v)$, for some $v \in S$. That is, $\ell(S) = \cup\{\ell(v) : v \in S\}$.

A triangular set of vertices S is *strongly triangular*, with respect to a k-labelling ℓ of G, when $|S| = \binom{|\ell(S)|}{2}$.

In the example of Figure 2, let $A = \{a, c, d\}$, $B = \{a, b, c\}$, and $C = \{c, d, e\}$. Hence $\ell(A) = \ell(C) = \{1, 2, 3, 4\}$ and $\ell(B) = \{1, 2, 3\}$. All the sets A, B and C are triangular but only B is strongly triangular (with respect to ℓ). So C is a maximal clique of G that is not strongly triangular while $D = \{a, d, e\}$ is a strongly triangular set.

In this paper we only consider 1 and 2-labellings. The following simple properties will be useful.

Lemma 1 *If ℓ is a 1-labelling of a graph G then for every set $S \subseteq V(G)$, $|S| = |\ell(S)|$.*

Lemma 2 *If ℓ is a 2-labelling of a graph G and S is a strongly triangular set of G then for every $\{i, j\} \subseteq \ell(S)$, there exists a vertex $v \in S$ such that $\ell(v) = \{i, j\}$.*

The theorem below characterizes edge clique graphs in terms of a special 2-labelling.

Theorem 1 *A graph H is an edge clique graph if and only if it admits a 2-labelling such that every subset $S \subseteq V(H)$ satisfies*

(1) *If S is a maximal clique then S is strongly triangular;*

(2) *If S is strongly triangular then S is a clique.*

Proof: Let H be an edge clique graph. By hypothesis, there exists a graph G such that $H = K_e(G)$. Let $|V(G)| = n$ and consider an arbitrary 1-labelling g of G such that $g(V(G)) = \{1, \ldots, n\}$. Define a 2-labelling h of H, as follows. For each edge $uv \in E(G)$, the label of the vertex $uv \in V(H)$ is defined as $h(uv) = g(u) \cup g(v)$. Because g is an 1-labelling of G and $H = K_e(G)$ it follows that h is indeed a 2-labelling of H and, additionally, $h(V(H)) \subseteq \{1, \ldots, n\}$. Moreover, $h(V(H)) = \{1, \ldots, n\}$ if G has no isolated vertices. Let $S \subseteq V(H)$, and examine the two following cases:

Suppose S is a maximal clique of H. By Proposition 1, there exists a one-to-one correspondence between maximal cliques of H and G. Let S' be the maximal clique of G, corresponding to S. Consequently, $|S| = \binom{|S'|}{2}$ and, by the definition of h, $g(S') = h(S)$. On the other hand, by Lemma 1, $|S'| = |g(S')|$. Hence,

$$|S| = \binom{|S'|}{2} = \binom{|g(S')|}{2} = \binom{|h(S)|}{2},$$

meaning that S is strongly triangular, and (1) is satisfied.

Suppose S is strongly triangular (with respect to h). Then $|S| = \binom{|h(S)|}{2}$. By Lemma 2, each pair of distinct elements of $h(S)$ forms the label set of some vertex of S. Consider a subset of vertices S' of G such that $g(S') = h(S)$. Note that each label set $\{i, j\} \subseteq h(S)$ corresponds to an edge $uv \in E(G)$, such that $u, v \in S'$, $g(u) = \{i\}$ and $g(v) = \{j\}$. Therefore S' is a clique and consequently S is a clique and (2) is satisfied.

Conversely, suppose that H is a graph admitting a 2-labelling h, such that every subset $S \subseteq V(H)$ satisfies the conditions (1) and (2) of the theorem. We have to prove that H is an edge clique graph. Below, we describe how to construct a convenient graph G and show that $H = K_e(G)$, as required.

Assume, without loss of generality, that $h(V(H)) = \{1, 2, \ldots, n\}$. Define a graph G, together with a 1-labelling g of it, as follows. For each value $i \in h(V(H))$, G contains one vertex and its label in g is $\{i\}$. For each vertex of $V(H)$ having label $\{i, j\}$, G contains an edge incident to the pair of vertices, whose labels in g are $\{i\}$ and $\{j\}$. The description of G is completed. Below we show that $H = K_e(G)$.

Clearly, the construction of G assures that there is a one-to-one correspondence between edges of G and vertices of H.

Let e and f be two distinct edges of G, belonging to a common clique C. We need to prove that the corresponding vertices e and f in H are adjacent.

Denote by S the subset of edges of G having their both ends in C. Denote by S' the subset of vertices of H corresponding to the edges of S. Clearly, $|S'| = |S| = \binom{|C|}{2}$. In addition, $g(C) = h(S')$ and, by Lemma 1, $|g(C)| = |C|$. Therefore $|S'| = \binom{|h(S')|}{2}$, meaning that S' is strongly triangular. Applying (2), we conclude that S' is a clique of H. Since $e, f \in S'$, we conclude that e, f are adjacent vertices in H, as required.

Finally, consider two adjacent vertices e, f of H. It remains to prove that the corresponding edges e, f belong to a common clique in G. Denote by S' any maximal clique of H containing both e and f. Let S be the subset of edges of G, corresponding to $S' \subseteq V(H)$. Denote by $C \subseteq V(G)$ the set of ends of the edges of S. Then $h(S') = g(C)$ and $|S'| = |S|$. Since S' is a maximal clique, applying (1), we have $|S'| = \binom{|h(S')|}{2}$ and we conclude that $|S| = \binom{|g(C)|}{2}$. By Lemma 1, $|g(C)| = |C|$ and consequently $|S| = \binom{|C|}{2}$. That is, each pair of vertices $u, v \in C$, must be an edge of S. Hence C is a clique, meaning that $e, f \in E(G)$ belong to a common clique of G. This completes the proof of the theorem. ■

Corollary 1 *Let G, H be graphs such that $H = K_e(G)$. There exists a 2-labelling of H that assures a one-to-one correspondence between cliques of G and strongly triangular cliques of H.*

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