Ear decompositions of matching covered graphs*

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Abstract

Ear decompositions of matching covered graphs are important for understanding their structure. By exploiting the properties of the dependence relation introduced by Carvalho and Lucchesi in [2], we are able to provide simple proofs of several well-known theorems concerning ear decompositions. Our method actually provides proofs of generalizations of these theorems. For example, we show that every matching covered graph G different from K_2 and C_{2n} has at least Δ edge-disjoint removable ears, where Δ is the maximum degree of G. This shows that any matching covered graph G has at least $\Delta!$ different ear decompositions, and thus is a generalization of the fundamental theorem of Lovász and Plummer establishing the existence of ear decompositions. We also show that every brick G different from K_4 and $\overline{C}_6$ has $\Delta - 2$ edges, each of which is a removable edge in G, that is, an edge whose deletion from G results in a matching covered graph. This generalizes a well-known theorem of Lovász. We also give a simple proof of another theorem due to Lovász which says that every nonbipartite matching covered graph has a canonical ear decomposition, that is, one in which either the third graph in the sequence is an odd-subdivision of K_4 or the fourth graph in the sequence is an odd-subdivision of $\overline{C}_6$. Our method in fact shows that every nonbipartite matching covered graph has a canonical ear decomposition which is optimal, that is one which has as few double ears as possible. Most of these results appear in the Ph. D. thesis of the first author [1], written under the supervision of the second author.

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1 Introduction

There is an extensive literature on matching covered graphs. For a history of the subject we refer the reader to Lovász and Plummer [11], Lovász [12], or Murty [13].

Two types of decompositions of matching covered graphs, namely tight cut decompositions and ear decompositions, have played significant roles in the development of the subject. For example, both these decompositions are essential ingredients in Lovász's characterization of the matching lattice [12]. Here we use tight cut decompositions in our study of ear decompositions. We recall below some of the basic definitions and results related to matching covered graphs, tight cut and ear decompositions.

1.1 Matching Covered Graphs

We shall use $V(G)$ and $E(G)$, respectively, for the set of vertices and edges of a graph G. A **matching** of G is a set of edges of G no two of which share a common end. Matching M is **perfect** if every vertex of G is the end of (exactly) one edge of M. An edge of a graph G is **admissible** if it lies in some perfect matching of G. A **matching covered graph** is a nontrivial connected graph in which every edge is admissible. We denote by $\mathcal{M}(G)$ the set of perfect matchings of a graph G.

The following result characterizes bipartite graphs that admit a perfect matching:

Theorem 1.1 (Hall's Theorem [7]) *A graph G with bipartition (A, B) has a perfect matching if, and only if, $|A| = |B|$ and $|N(S)| \geq |S|$, for every subset S of A.*

Here, $N(S)$ stands for the **neighborhood** of S, that is, the set of vertices of G that are adjacent to some vertex of S. As a simple consequence of that result we then have the following characterization of matching covered bipartite graphs, which, according to Lovász and Plummer [11], is largely due to Hetyei [8], (see also [11, Theorem 4.1.1]):

Theorem 1.2 *Let G be a nontrivial graph with bipartition (A, B). The following assertions are equivalent:*

(i) *Graph G is matching covered,*

(ii) $|A| = |B|$ *and* $|N(S)| > |S|$, *for each nonnull proper subset S of A.*

(iii) *for each vertex v of A and each vertex w of B, graph $G - \{v, w\}$ has a perfect matching.*

Another fundamental result, due to Tutte [16], characterizes the existence of perfect matchings in general graphs:

Theorem 1.3 (Tutte's Theorem) *A graph G has a perfect matching if, and only if, $C_{\text{odd}}(G - X)| \leq |X|$, for every subset X of $V(G)$.*

Here, $C_{\text{odd}}(G - X)$ stands for the number of odd components of $G - X$. A connected component of $G - X$ is **odd** or **even**, depending on the parity of its number of vertices. A **barrier** of G is a set X of vertices of G such that $C_{\text{odd}}(G - X)| = |X|$. The empty set and the singletons are always barriers of matching covered graphs. We call these barriers **trivial**. As simple consequences of Tutte's Theorem we have the following important results:

Theorem 1.4 *Let G be a graph that has at least one perfect matching. An edge e of G is admissible if, and only if, no barrier of G contains both ends of e.*

Theorem 1.5 *Let G be a nontrivial graph that has at least one perfect matching. Graph G is matching covered if, and only if, for every nonnull barrier B of G, $G - B$ has no even components and no edge has both ends in B.*

Corollary 1.6 *Every matching covered graph is 2-connected.*

1.2 Tight Cuts

For every set X of vertices of a graph G, we denote by $\nabla(X)$ the edge cut associated with X, that is, the set of those edges of G that have one end in X, the other in $\overline{X}$. Set X is a **shore** of $\nabla(X)$. A cut is **trivial** if one of its shores is a singleton.

If X is a nonnull subset of $V(G)$, we shall denote the graph obtained by shrinking $\overline{X}$ to a single vertex $\overline{x}$ by $G\{X, \overline{x}\}$; if the name of the vertex $\overline{x}$ is irrelevant, we shall simply write $G\{X\}$; graph $G\{X\}$ is called a $\nabla(X)$-**contraction** of G. The proof of the following statement is straightforward.

Lemma 1.7 *Let G be a graph, X a nonnull, proper subset of $V(G)$. If $\nabla(X)$-contractions $G_1 := G\{X, \overline{x}\}$ and $G_2 := G\{\overline{X}, x\}$ of G are both matching covered then G is also matching covered.*

Let G be a connected graph with an even number of vertices (as is a matching covered graph). Since G is required to be connected, the shore of a cut is uniquely determined, up to complementation.

Graph G has an even number of vertices, therefore the number of vertices of both shores of any cut have the same parity. Cut $\nabla(X)$ is **odd** or **even**, depending on the parity of $|X|$. For every perfect matching M of G, the number of edges of M in $\nabla(X)$ has the same parity of $|X|$.

An odd cut C of G is called a **tight cut** if $|M \cap C| = 1$ for every perfect matching M of G. Every trivial cut is a tight cut of G.

Lemma 1.8 *Let G be a matching covered graph, C a tight cut of G, H a C-contraction of G. Graph H is connected and $\mathcal{M}(H) = \{M \cap E(H) : M \in \mathcal{M}(G)\}$. Consequently, H is matching covered and the tight cuts of H are the cuts of H that are tight cuts of G.*

1.2.1 Tight cut decompositions

The first type of decomposition of matching covered graphs, known as the tight cut decomposition, was introduced by Lovász in [12].

Let G be a matching covered graph, and let C be a nontrivial tight cut of G. Then, as already noted, the two C-contractions G_1 and G_2 of G are also matching covered. If either G_1 or G_2, say G_1, has a nontrivial tight cut D, then we can take D-contractions of G_1, in the same manner as above, and obtain smaller matching covered graphs than G_1. Thus, given any matching covered graph G, by repeatedly applying contractions on nontrivial tight cuts, we can obtain a list of graphs which do not have nontrivial tight cuts. This list is called a **tight cut decomposition** of G. Lovász proved the following remarkable result:

Theorem 1.9 (See [12]) *The results of any two applications of the tight cut decomposition procedure on a matching covered graph are the same list of graphs, except possibly for the multiplicities of edges.*

1.2.2 Tight cuts in bipartite graphs

The following assertion is easily proved by a simple counting argument.

Lemma 1.10 *Let G be a graph, $C := \nabla(X)$ denote an odd cut of G such that the subgraph $G[X]$ of G induced by X is bipartite. If C contraction $G\{X, \overline{x}\}$ is matching covered then it is also bipartite.*

¿From this result and Theorem 1.2 the following assertion is easily deduced:

Theorem 1.11 (See [11]) *Let G be a matching covered graph with bipartition (A, B). The following assertions are equivalent:*

(i) *graph G is free of nontrivial tight cuts,*

(ii) *for every subset X of A such that $0 < |X| < |A| - 1$, we have $|N(X)| > |X| + 1$,*

(iii) *removal of any two vertices of A and any two vertices of B yields a graph having a perfect matching.*

A bipartite matching covered graph free of nontrivial tight cuts is called a **brace**.

1.2.3 Tight cuts in nonbipartite graphs

There are two types of tight cuts, namely barrier cuts and 2-separation cuts, which are of special interest in this theory.

Barrier Cuts: If B is a nontrivial barrier and H is any nontrivial odd component of $G - B$, then $\nabla(V(H))$ is a nontrivial tight cut. Such a cut is called a **barrier cut** (see Figure 1(a)).

The second type of tight cuts is defined below.

2-Separation cuts: Let $\{u, v\}$ be a 2-separation that is not a barrier. Write G as the union of G_1 and G_2 in the usual manner. Then $\nabla(V(G_1) - u)$ and $\nabla(V(G_1) - v)$ are both tight cuts. Such cuts are referred to as **2-separation cuts** (see Figure 1(b)).

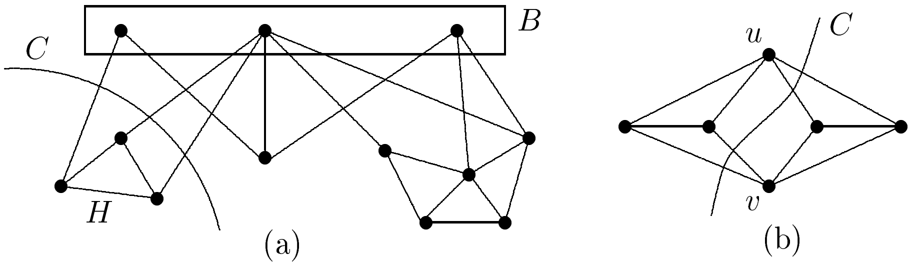

Figure 1: Barrier cuts and 2-separation cuts.

A matching covered graph may have a tight cut that is neither a barrier cut nor a 2-separation cut. However, the following fundamental theorem was proved by Edmonds, Lovász and Pulleyblank.

Theorem 1.12 (See [6]) *If a matching covered graph G has a nontrivial tight cut, then it either has a nontrivial barrier cut or it has a 2-separation cut.*

Until recently, the only known proof of the above Theorem was based on LP-duality. Recently Szigeti obtained a simple proof which does not use LP-duality [15].

A graph G with at least four vertices is **bicritical** if removal of any two vertices of G yields a graph having a perfect matching. From Tutte's Theorem it is easy to prove the following statement.

Lemma 1.13 *Let G be a graph with at least four vertices. Then G is bicritical if, and only if, G is matching covered and free of nontrivial barriers.*

Theorem 1.14 *A nonbipartite matching covered graph G is free of nontrivial tight cuts if, and only if, it is 3-connected and bicritical.*

A nonbipartite matching covered graph free of nontrivial tight cuts is called a **brick**.

1.3 Ear decompositions

A **single ear** of a graph G is a path of odd length whose internal vertices (if any) all have degree two in G. The following theorem provides a decomposition of bipartite matching covered graphs (see [11], Theorem 4.1.6.).

Theorem 1.15 *Given any bipartite matching covered graph G, there exists a sequence*

$$G_1 \subset G_2 \subset \ldots \subset G_r$$

of matching covered subgraphs of G, where (i) $G_1 = K_2$, $G_r = G$ and (ii) G_{i+1} is the union of G_i and a single ear of G_{i+1}, for $1 \leq i \leq r-1$.

The sequence $G_1, G_2, \ldots, G_r$ of subgraphs of G with the above properties is an ear decomposition of G in which each member of the sequence is a matching covered subgraph of G. Such decompositions do not exist for nonbipartite matching covered graphs. For example, K_4 has no ear decomposition as in Theorem (1.15). However, every matching covered graph has an ear decomposition with a slight relaxation of the above definition. For describing that decomposition, we require the notion of a double ear.

A **double ear** of a graph G is a pair of vertex-disjoint single ears of G. An **ear** of G is either a single ear or a double ear of G.

An **ear decomposition** of a matching covered graph G is a sequence

$$G_1 \subset G_2 \subset \ldots \subset G_r$$

of matching covered subgraphs of G where (i) $G_1 = K_2$, $G_r = G$, and (ii) for $1 \leq i \leq r-1$, G_{i+1} is the union of G_i and an ear (single or double) of G_{i+1}. The following fundamental theorem was established by Lovász and Plummer (see [11], Theorem 5.4.6).

Theorem 1.16 (The two-ear Theorem) *Every matching covered graph has an ear decomposition.*

There are various proofs of this theorem. The original proof due to Lovász and Plummer is somewhat involved. Later on, Little and Rendl [9] gave a simpler proof. Recently, another simple proof was given by Szigeti [14].

A matching covered subgraph H of a graph G is **nice** if every perfect matching of H extends to a perfect matching of G, or equivalently, if $G - V(H)$ has a perfect matching. It is easy to see that all the subgraphs in an ear decomposition of a matching covered graph must be nice matching covered subgraphs of the graph. In all the previous approaches, an ear decomposition of a matching covered graph G is established by showing how a nice matching covered proper subgraph G_i of G can be extended to a nice matching covered subgraph G_{i+1} by the addition of a single or double ear.

Our approach is to build an ear decomposition of a matching covered graph in the reverse order. To state our theorem precisely, we need to define the notion of a removable ear in a matching covered graph.

Let G be a matching covered graph. Let R be an ear (single or double) of G. Denote by $G - R$ the graph obtained from G by removing the edges and internal vertices of each single ear of R. We say that ear R is **removable** in G if $G - R$ is matching covered. (If R is a removable single ear of length one then the edge of R is a **removable edge**.)

In trying to establish the existence of ear decompositions with special properties, it is often convenient to find the subgraphs in the ear decomposition in the reverse order starting with $G_r = G$. Thus, after obtaining a subgraph G_i in the sequence which is different from K_2, we find a suitable removable ear (single or double) and obtain G_{i-1} from G_i by removing that ear from G_i. For example, to show that a matching covered graph G has an ear decomposition, it suffices to show that every matching covered graph different from K_2 has a removable ear. This is our approach. We in fact prove the following more general theorem, due to Carvalho and Lucchesi [2].

Theorem 1.17 *Any matching covered graph G different from K_2 and C_{2n} has Δ edge-disjoint removable ears (single or double).*

Clearly Theorem(1.16) is a special case of Theorem (1.17). In fact (1.17) shows that every matching covered graph G has at least $\Delta!$ different ear decompositions.

Suppose that

$$G_1 \subset G_2 \subset \ldots \subset G_r$$

is an ear decomposition of a matching covered graph G. If a member G_i of the sequence is obtained from its predecessor G_{i-1} by adding a single (double) ear, then we shall say that G_i is obtained from G_{i-1} by the addition of a single (double) ear. In any ear decomposition of a nonbipartite matching covered graph G, there must be at least one double ear addition. Furthermore, if G_i is the first member of an ear decomposition which is obtained by a double ear addition[1], then $G_1, \cdots, G_{i-1}$ are bipartite and $G_i, \cdots, G_r = G$ are nonbipartite.

Let $G_1 \subset G_2 \subset \ldots \subset G_r$ be an ear decomposition in which a double ear addition appears as soon as possible. By definition, G_1 is K_2, and G_2 is an even circuit. It is not difficult to check that if G_3 is nonbipartite, then it must in fact be an odd subdivision of K_4, and if G_3 is bipartite and G_4 is nonbipartite, then G_4 must be an odd subdivision of $\overline{C_6}$. We shall refer to an ear decomposition of a nonbipartite matching covered graph G as a **canonical ear decomposition** if either its third member G_3 is an odd subdivision of K_4 or its fourth member G_4 is an odd subdivision of $\overline{C_6}$. The following fundamental theorem was proved by Lovász in 1983 [10].

[1]We only consider ear decompositions which are fine. By this we mean that a double ear (P_1, P_2) is added to G_i to obtain G_{i+1} only if neither $G_i + P_1$, nor $G_i + P_2$ is matching covered.

Theorem 1.18 (The canonical ear decomposition Theorem) *Every nonbipartite matching covered graph G has a canonical ear decomposition.*

In [10], Lovász gave several applications of the above theorem to extremal problems in graph theory. Later on he used it to deduce the following theorem which was crucial for his work on the matching lattice.

Theorem 1.19 (The removable edge Theorem) *Every brick different from K_4 and $\overline{C}_6$ has a removable edge.*

Lovász's original proof of Theorem (1.18) was quite involved. A simpler proof was given by Carvalho and Lucchesi[3]. Our approach here shall be to prove Theorem (1.17) first and then to derive Theorem (1.18) from it. We also prove the following generalization of Theorem (1.19).

Theorem 1.20 *Every brick different from K_4 and $\overline{C}_6$ has at least $\Delta - 2$ removable edges.*

A proof of the above theorem was first given by Carvalho and Lucchesi in [2]. The proof we give here is along the same lines, but is somewhat simpler.

Any two ear decompositions of a bipartite matching covered graph have the same length. However, the length of an ear decomposition of a nonbipartite matching covered graph depends on the number of double ears used in the decomposition. An **optimal** ear decomposition of a matching covered graph G is one which uses as few double ears as possible. In Carvalho's Ph. D. thesis [1] (see also [5]), it is shown that the number of double ears in an optimal ear decomposition of a matching covered graph G is $b(G)+p(G)$ where $b(G)$ is the number of bricks of G, and $p(G)$ is the number of bricks of G whose underlying simple graphs are isomorphic to the Petersen graph, and have used this result to establish that the matching lattice of a matching covered graph has a basis consisting of incidence vectors of perfect matchings of the graph. In this connection, we proved the following generalization of Theorem (1.18).

Theorem 1.21 (Optimal canonical ear decomposition Theorem) *Every nonbipartite matching covered graph G has an optimal ear decomposition which is canonical.*

In the next section, we recall the definition of the dependence relation that we alluded to in the abstract, and state some of its salient properties. In section 3, we give a proof of Theorem (1.17). In section 4, we give a proof of Theorem (1.20). In section 5, we give proofs of Theorems (1.18) and (1.21).

2 A dependence relation

Let G be a matching covered graph, and let e and f be any two edges of G. Then e **depends** on f, or e **implies** f, if every perfect matching that contains e also contains f. (Equivalently, e depends on f if e is not admissible in $G - f$.)

We shall write $e \Rightarrow f$ to indicate that e depends on f. Clearly, $\Rightarrow$ is reflexive and transitive. It is convenient to visualize $\Rightarrow$ in terms of the digraph it defines on the set of edges of G (see Figure 2 for an illustration). A study of this relation, as shown by its many important uses in Carvalho's thesis [1], turns out to be very useful for understanding the structure of matching covered graphs.

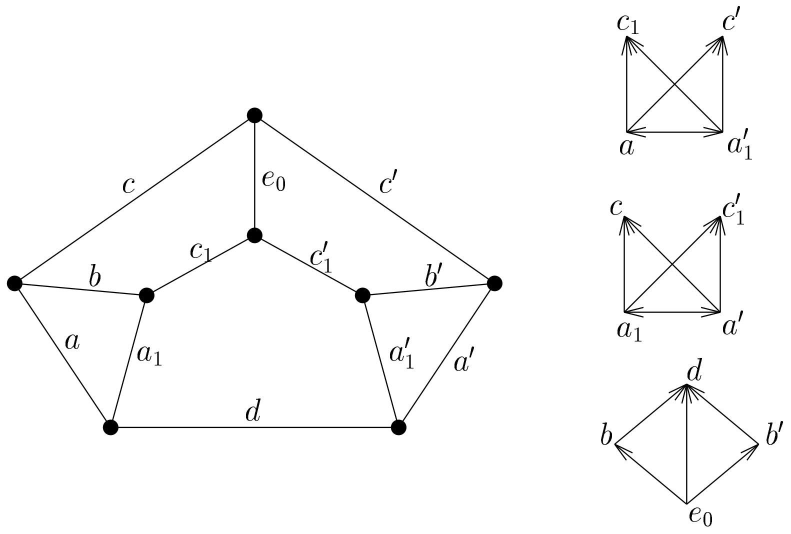

Figure 2: A matching covered graph G and the dependence relation on $E(G)$.

The following lemma can be proved easily using Theorem (1.4).

Lemma 2.1 *Let G be a matching covered graph, and let e and f be any two distinct edges of G. Then, the relation $e \Rightarrow f$ holds if, and only if, there exists a barrier B of $G - f$ such that (i) both ends of e are in B, and (ii) the ends of f lie in different components of $G - f - B$.*

We shall say that two edges e and f are **mutually dependent** if $e \Rightarrow f$ and $f \Rightarrow e$. In this case we shall write $e \Leftrightarrow f$. Clearly $\Leftrightarrow$ is an equivalence relation on $E(G)$. By identifying the vertices in the equivalence classes in the digraph representing dependence relation ($\Rightarrow$) on $E(G)$, we obtain the digraph $D(G)$ representing the dependence relation ($\Rightarrow$) on the set of equivalence classes. This digraph is clearly acyclic. The sources in this digraph are called *minimal classes*. Let e be an edge of G. Any source Q of $D(G)$

that contains an edge f of G that depends on e is said to be a **minimal class induced by** e. (We admit the possibility that $f = e$.) If Q is a minimal class induced by e that does not contain e then there is an edge from Q to the class containing e in the digraph $D(G)$, that is, Q depends on e. The following lemma is an immediate consequence of this definition.

Lemma 2.2 *If Q is a minimal class, then every edge not in Q is admissible in $G - Q$. (Thus, if $G - Q$ is connected, then $G - Q$ is matching covered.)*

The equivalence classes in a brick have some attractive properties:

Lemma 2.3 (See Lemma 3.4 in [12]) *Let G be a brick and let e and f be two distinct edges of G such that $e \Leftrightarrow f$. Then, $G - e - f$ is bipartite. Moreover, both ends of e lie in one part of the bipartition, and both ends of f lie in the other part of the bipartition.*

Proof: By Lemma (2.1), $G - f$ has a barrier B such that both ends of e are in B, and the two ends of f are in different components of $G - f - B$.

Suppose that there exists another edge e' which has both its ends in B. Let M be a perfect matching in G which contains the edge e'. Simple counting shows that $f \in M$, and $e \notin M$. But this contradicts the hypothesis that $e \Leftrightarrow f$. Therefore, e is the only edge with both ends in B.

Suppose now that $G - e - f$ is not bipartite. Then, some component H of $G - f - B$ is nontrivial. As G is a brick, $\nabla(V(H))$ is not a tight cut in G. Hence, there exists a perfect matching M of G such that $|M \cap \nabla(V(H))| \geq 3$. A simple counting argument shows $|M \cap \nabla(V(H))| = 3$, $f \in M$, and $e \notin M$. Again, this contradicts the hypothesis. Therefore $G - e - f$ is bipartite. ■

Lemma 2.4 *Let G be a brick and let Q be an equivalence class of G. Then, $|Q| \leq 2$. Moreover, if $|Q| = 2$, then $G - Q$ is bipartite.*

Proof: If possible, let e, f, and g, be any three edges in Q. Then, by Lemma 2.3, $E \setminus \{e, f\}$ and $E \setminus \{e, g\}$ are cuts of G. The symmetric difference of these two cuts is $\{f, g\}$. But the symmetric difference of any two cuts of a graph is a cut of the graph. Thus, $\{f, g\}$ is a cut of G. This is a contradiction because G is 3-connected. ■

3 Removable ears in matching covered graphs

In this section, we shall prove Theorem (1.17). We shall first consider the cases in which G is either a brick or is a brace on at least six vertices. We shall then be able to prove the theorem in general by induction using the tight cut decomposition procedure.

Lemma 3.1 *Let G be a brick. Then G has Δ edge-disjoint removable ears.*

Proof: Let v be a vertex of degree Δ of G, and let $e_1, \cdots, e_\Delta$ be the edges incident with v. Let $Q_1, \cdots, Q_\Delta$ be minimal equivalence classes induced by $e_1, \cdots, e_\Delta$, respectively. It follows from the definitions that, for each i, $1 \le i \le \Delta$, there exists a perfect matching M_i, such that M_i contains the edges in Q_i, and the edge e_i (because, Q_i, being induced by e_i, depends on e_i). It follows that the M_i are distinct, and hence, the Q_i are disjoint. But, by Lemma (2.4) $|Q_i| \le 2$ for $1 \le i \le \Delta$. Since G is 3-connected, $G - Q_i$ is connected for each i. By Lemma (2.2), $G - Q_i$ is matching covered for each i. ∎

Lemma 3.2 *Let H be a brace on six or more vertices. Then, every edge e of H is removable.*

Proof: Let H be a brace with bipartition (U, W), where $|U| = |W| \ge 3$. By Theorem (1.11), for any subset X of U (or of W), $0 < |X| < |U| - 1$, we have $|N(X)| \ge |X| + 2$.

Let e be any edge of H. Then, $|N_{H-e}(X)| \ge |X| + 1$, for each subset X of U (or of W), $0 < |X| < |U| - 1$. Suppose now that there exists a subset X of U (or of W), $|X| = |U| - 1$, such that $|N_{H-e}(X)| \le |X|$. As $H - e$ has a perfect matching, we must in fact have $|N_{H-e}(X)| = |X|$. Then, there exists a vertex $w \in W$ such that $|N_{H-e}(w)| = 1$. Consequently, $|N_H(w)| \le 2$. This is not possible because H is a brace on six or more vertices.

Therefore $|N_{H-e}(X)| \ge |X| + 1$ for every proper nonnull subset X of U. Hence $H - e$ is matching covered, by Theorem (1.2). ∎

Proof of Theorem (1.17): First consider the case in which G has two parallel edges, say e and f. Then, $G - f$ is matching covered and has fewer edges than G. If $G - f = K_2$, then $G = C_2$ and the assertion holds trivially in this case (although C_2 is not considered in the hypothesis of the Theorem). If $G - f = C_{2n}$ then $\Delta = 3$ and G has three edge-disjoint removable ears, namely, $\{e\}$, $\{f\}$ and $E(G) \setminus \{e, f\}$. We may thus assume that $G - f$ is neither K_2 nor C_{2n}. By induction, $G - f$ has at least $\Delta(G - f)$ edge-disjoint removable ears. Of these, at most one uses edge e. Therefore, G has at least $\Delta(G - f) - 1$ edge-disjoint removable ears, none of which uses any of e and f. But $\{e\}$ and $\{f\}$ are also removable ears of G. We conclude that G has at least Δ edge-disjoint removable ears.

We may thus assume that G is simple. There are only three matching covered simple graphs with at most four vertices: K_2, C_4 and K_4. By hypothesis, G cannot be any of the first two. If $G = K_4$ then the assertion holds. We may thus assume that G is a simple matching covered graph with at least six vertices.

If G is either a brick or a brace, then the validity of the theorem follows from Lemmas (3.1) and (3.2). To prove the theorem in general, assume inductively that any matching covered graph G', distinct from K_2 and C_{2n}, with $|E(G')| + |V(G')| < |E(G)| + |V(G)|$ has $\Delta(G')$ edge-disjoint removable ears. We shall first deal with a few simple cases.

Now consider the case in which G has two adjacent vertices, say v and w, of degree two. Let u be the neighbor of v different from w, and t be the neighbor of w different from

v. If u is the same vertex as t, then it would be a cut vertex of G, which is impossible. Therefore u and v are distinct, and $P = (u, v, w, t)$ is a path of length three which is internally disjoint from the rest of the graph. Obtain the graph G' from G by deleting v and w and joining u and t by a new edge e. Clearly, G' is matching covered. Moreover, G' has fewer vertices than G, and $\Delta(G') = \Delta(G)$. Graph G' cannot be K_2, else G would not be matching covered. Graph G' cannot be C_{2n}, else G would be C_{2n+2}. Therefore, by induction, G' has Δ edge-disjoint removable ears. Any removable ear of G' that does not include the edge e is also a removable ear of G. And, if Q' is a removable ear of G' which includes e, we can obtain a removable ear Q of G from Q' by replacing e by the path P. It follows that G has at least $\Delta(G)$ edge-disjoint removable ears. Thus we may assume that no two of its vertices of degree two are adjacent.

We divide the rest of the proof into two main cases depending on whether or not G is bicritical. If G is bicritical, and is not a brick, then G has a 2-separation cut. This is the first case we consider.

Case 1. G is bicritical and $\{v_1, v_2\}$ is a 2-separation of G: Firstly note that if there is an edge f joining v_1 and v_2 in G, then $G - f$ is also bicritical. In particular, f is removable in G. Moreover, if Q is a removable ear of $G - f$, then Q is also a removable ear of G. Thus, if there is an edge joining v_1 and v_2, we can delete it and apply induction to obtain the required result. So, we may assume that there is no such edge in G.

Express G, in the usual manner, as the union of two edge-disjoint graphs H_1, H_2 with $V(H_1) \cap V(H_2) = \{v_1, v_2\}$. Let $e = v_1v_2$ be a new edge. Then, $G_1 = H_1 + e$, and $G_2 = H_2 + e$ are both matching covered. Furthermore, $\Delta(G_1), \Delta(G_2) \geq 3$, and thus neither G_1 nor G_2 is K_2 or C_{2n}.

Let Δ_i be the maximum degree in G_i. Then, clearly $\Delta_1 + \Delta_2 \geq \Delta + 2$. By induction, G_i has at least Δ_i edge-disjoint removable ears, and at most one of them contains the edge e. Thus, G has at least $\Delta_1 + \Delta_2 - 2 \geq \Delta$ edge-disjoint removable ears.

To complete the proof, we must consider the case in which G is not bicritical. In this case, G has at least one nontrivial barrier. Since G is not a brace, it has nontrivial tight cuts which are barrier cuts. We shall apply induction hypothesis to graphs obtained by contracting the shores of suitably chosen barrier cuts.

Let $C = \nabla(S)$ be a tight cut of G. Recall that, by Lemma (1.8), C-contractions $G\{S\}$ and $G\{\overline{S}\}$ of G are both matching covered. The following lemma is then an easy consequence of Lemma (1.7).

Lemma 3.3 *Let G be a matching covered graph. Let $C = \nabla(S)$ be a nontrivial tight cut of G and let Q be a removable ear of $G\{\overline{S}\}$. If either Q is edge-disjoint from C, or if $E(Q) \cap C = \{e\}$ and e is removable in $G\{S\}$, then Q is also removable in G.*

With the aid of this lemma, we can now deal of the remaining case in which G has a nontrivial barrier. It is convenient to first consider the case in which G has a barrier of size two. Note that if G has a vertex u of degree two, with v and w as its neighbors, then $\{v, w\}$ is a barrier of size two of G. Furthermore, since G does not have adjacent vertices of degree two, neither v nor w has degree two. Thus, if G has a barrier of size two, then it has one in which neither vertex has degree two in G.

Case 2.1. **G has barriers of size two**: In this case, as noted above, there exists a barrier of size two in which neither vertex has degree two. Let $\{v_1, v_2\}$ be such a barrier, and let S_1 and S_2 be the vertex sets of the two components of $G \setminus \{v_1, v_2\}$.

Suppose one of S_1, S_2, say S_2 is a singleton. Then there are at least two edges from each of v_1 and v_2 to S_1. So all edges in $\nabla(S_1)$ are multiple edges in $G\{\overline{S_1}\}$, and hence are removable in $G\{\overline{S_1}\}$. Furthermore, $G\{S_1\}$ is different from K_2 and C_{2n} and has a vertex whose degree in $G\{S_1\}$ is at least $\Delta(G)$. By induction hypothesis, $G\{S_1\}$ has at least Δ removable ears. By Lemma (3.3), they are also removable in G.

Suppose now that neither S_1 nor S_2 is a singleton. In general, for $i, j = 1, 2$, if there is more than one edge from v_i to S_j, then all these edges would be multiple edges in $G\{\overline{S_j}\}$, and hence would be removable edges of $G\{\overline{S_j}\}$. Thus, since both v_1 and v_2 have degree at least three, the number of edges in the cut $\nabla(S_1)$ which are not removable in $G\{\overline{S_1}\}$, plus the number of edges of $\nabla(S_2)$ which are not removable in $G\{\overline{S_2}\}$ is at most two. By considering the cuts $\nabla(S_1)$, $\nabla(S_2)$, using the induction hypothesis, and applying Lemma (3.3) twice, we can deduce that G has at least Δ removable ears.

Case 2.2. **G is not bicritical, and the size of a smallest nontrivial barrier in G is at least three.**: Let B be a minimal nontrivial barrier of G. Let S_1, S_2, ..., S_b, where $b = |B|$, be the vertex sets of the odd components of $G - B$. The minimality of B implies that the bipartite graph H obtained by shrinking S_1, S_2, ..., S_b to single vertices is a brace. Furthermore, since $|B| \geq 3$, H is a brace on six or more vertices. Therefore, by Theorem (3.2), every edge of H is removable.

Consider the graphs $G_i = G\{S_i, \overline{s_i}\}$, for $1 \leq i \leq b$. The degree of $\overline{s_i}$ in G_i is at least three, therefore G_i is neither K_2 nor C_{2n}.

Let v be a vertex of degree Δ in G. Without loss of generality, v is either in B or in S_1. First suppose that v is in B, and let Δ_1, Δ_2,..., Δ_b be the numbers of edges from v to S_1, S_2, ..., S_b, respectively. Then, clearly, $\Delta(G_i) \geq \Delta_i$, for each i. By induction, for $1 \leq i \leq b$, G_i has at least Δ_i edge-disjoint removable ears. Since each edge of $\nabla(S_i)$ is removable in H, it follows by Lemma (3.3) that, for $1 \leq i \leq b$, G has at least Δ_i edge-disjoint removable ears contained in $E(G_i)$. Thus, G has at least $\sum_{i=1}^{b} \Delta_i = \Delta$ edge-disjoint removable ears.

Now consider the case in which v is in S_1. Then, $\Delta(G_1) \geq \Delta(G)$. By induction, G_1 has at least $\Delta(G)$ edge-disjoint removable ears. Since each edge of $\nabla(S_1)$ is removable in H, it follows that all removable ears of G_1 are also removable in G. ■

Remark: The above theorem implies that every matching covered graph has at least $\Delta!$ ear decompositions. This bound is best possible because the graph on two vertices and Δ parallel edges joining the two vertices has exactly $\Delta!$ ear decompositions.

4 Removable edges in bricks

The following lemma establishes a property of equivalent edges in bipartite matching covered graphs. It is used in the proof of Theorem (1.20).

Lemma 4.1 *Let H be a bipartite matching covered graph, and let e and f be two distinct equivalent edges of H. Then $\{e, f\}$ is a cut of H.*

Proof: By Lemma (2.1), there is a barrier in $H - f$ which contains both ends of edge e. Choose a maximal barrier B in $H - f$ which contains both ends of e.

Let K denote any component of $H - B - f$. Then K is odd, because H is matching covered. Therefore cut $\nabla(V(K))$ is tight. Component K, a subgraph of a bipartite graph, is bipartite. Let (A_K, B_K) denote a bipartition of K. Adjust notation so that $|A_K| > |B_K|$. By Lemma (1.10), $G\{V(K)\}$ is bipartite, $|A_K| = |B_K| + 1$ and $\nabla(V(K)) = \nabla(A_K) \setminus \nabla(B_K)$. Therefore, $B \cup B_K$ is a barrier of G that contains both ends of e. By the maximality of B, we conclude that B_K is null. Thus, A_K is a singleton, whence K is trivial. This conclusion holds for each component K of $H - B - f$.

If possible, let h be an edge of H which also has both its ends in B. Then, by simple counting, we can see that any perfect matching through h contains f, but not e. This contradicts the hypothesis that e and f are equivalent. We conclude that $E(H) \setminus \{e, f\}$ is a cut of H. But $E(H)$ is also a cut of H. The symmetric difference of these two cuts is also a cut. That is, $\{e, f\}$ is a cut of H. ■

Theorem 4.2 *Let G be a brick and let v be a vertex of G. Let $f_1, f_2, \cdots, f_{|\nabla(v)|}$ be the edges of $\nabla(v)$. For each i such that $1 \leq i \leq |\nabla(v)|$, let Q_i be a minimal class induced by f_i. If (at least) three of those classes are doubletons then either G is K_4 or G is $\overline{C_6}$, up to multiple edges joining vertices of both triangles of $\overline{C_6}$.*

Proof: Each Q_i is either a singleton (a removable edge) or a doubleton. Assume that there are three distinct doubletons, say, $Q_1 = \{e_1, \overline{e}_1\}$, $Q_2 = \{e_2, \overline{e}_2\}$, and $Q_3 = \{e_3, \overline{e}_3\}$.

We know that for $1 \leq i \leq 3$, $G - Q_i$ is a bipartite matching covered graph. Let us write $H = G - Q_1$. Every perfect matching of H is a perfect matching of G, therefore the edges of Q_2 are equivalent in H. By Lemma (4.1), Q_2 is a cut of H. Likewise, Q_3 is a cut of H. Graph H, a matching covered graph, is 2-connected. Thus, $H - Q_2$ has precisely two connected components, whence $H - (Q_2 \cup Q_3)$ has at least three and at most four connected components. That is, $G - (Q_1 \cup Q_2 \cup Q_3)$ has at least three and at most four connected components.

Let r denote the number of connected components of $G-(Q_1 \cup Q_2 \cup Q_3)$. Let $H_1, \cdots, H_r$ denote the connected components of $G-(Q_1 \cup Q_2 \cup Q_3)$. For each i, $1 \leq i \leq r$, let C_i denote cut $\nabla(V(H_i))$ of G. Clearly, $C_i \subseteq Q_1 \cup Q_2 \cup Q_3$.

For $1 \leq j \leq 3$, let M_j denote any perfect matching of G that contains (both) edges of Q_j. Recall that classes Q_1, Q_2 and Q_3 depend on distinct edges that are incident with vertex v. Therefore M_j does not contain any edges in any Q_k, for $k \neq j$.

If cut C_i is even, then $|M_j \cap C_i|$ is even, therefore C_i must contain either none or both edges of Q_j. Moreover, graph G, a brick, is 3-connected, therefore C_i must contain at least three edges. We conclude that if C_i is even then it has either four or six edges, and it must contain either none or both edges of any of the three classes Q_1, Q_2 and Q_3.

If cut C_i is odd then it must contain an odd number of edges of M_j, therefore just one edge of Q_j. It follows that if C_i is odd then C_i has precisely three edges, one in each of the three classes Q_1, Q_2 and Q_3. In that case, C_i is tight in G. But graph G, a brick, is free of nontrivial tight cuts, therefore H_i is trivial.

Case 1. Graph $G-(Q_1 \cup Q_2 \cup Q_3)$ has four connected components.: Since G is 3-connected, we must have that $|C_i| \geq 3$, for $1 \leq i \leq 4$. But $\sum |C_i| = 2\sum Q_j = 12$. Therefore, each C_i consists precisely of three edges. By the analysis above, each C_i is odd, whence each H_i is trivial. That is, G consists of four vertices and Q_1, Q_2 and Q_3 are the three perfect matchings of G. We conclude that G is K_4.

Case 2. Graph $G-(Q_1 \cup Q_2 \cup Q_3)$ has three connected components.: Since graph G is 3-connected, we must have that $|C_i| \geq 3$, for $1 \leq i \leq 3$. But $\sum |C_i| = 2\sum Q_j = 12$. Therefore, $|C_i| \geq 4$, for some i such that $1 \leq i \leq 3$. Adjust notation, so that $|C_1| \geq 4$. Then C_1 is even and consists of either four or six edges. If it had six edges, then the other two cuts, C_2 and C_3, would have three edges each and therefore H_2 and H_3 would both be trivial. In that case, $G-(Q_1 \cup Q_2)$ would be connected, a contradiction. Therefore, C_1 has precisely four edges. We conclude that each C_i $(1 \leq i \leq 3)$ has precisely four edges. Therefore, we may adjust notation so that $C_1 = Q_1 \cup Q_2$, $C_2 = Q_2 \cup Q_3$ and $C_3 = Q_1 \cup Q_3$.

Recall that $G-Q_1$ is bipartite. Let (A, B) denote a bipartition of $G-Q_1$. Each H_i is a subgraph of H, therefore also bipartite. For each i, $1 \leq i \leq 3$, let $A_i = A \cap V(H_i)$, $B_i = B \cap V(H_i)$. Then (A_i, B_i) is a bipartition of H_i. Note also that perfect matching M_3 of G, which contains both edges of Q_3, has no edge in C_1, whence $M_3 \cap E(H_1)$ is a perfect matching of H_1. Therefore, $|A_1| = |B_1|$. Likewise, $|A_2| = |B_2|$ and $|A_3| = |B_3|$. Thus, in summary, we have:

- An edge of Q_1, say e_1, joins a vertex u_1 in A_1 with a vertex v_1 in A_3, and the other edge $\overline{e}_1$ of Q_1 joins a vertex $\overline{u}_1$ in B_1 with a vertex $\overline{v}_1$ in B_3,

- an edge of Q_2, say e_2, joins a vertex u_2 in A_1 with a vertex v_2 in B_2, and the other edge $\overline{e}_2$ of Q_2 joins a vertex $\overline{u}_2$ in B_1 with a vertex $\overline{v}_2$ in A_2, and

- an edge of Q_3, say e_3, joins a vertex u_3 in A_2 with a vertex v_3 in B_3, and the other edge $\overline{e}_3$ of Q_3 joins a vertex $\overline{u}_3$ in B_2 with a vertex $\overline{v}_3$ in A_3.

Figure 3 shows all the incidences described above.

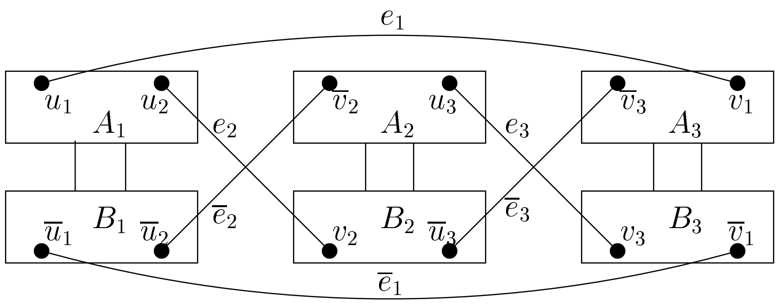

Figure 3:

Now, we shall show that all A_i and B_i are singletons, and thereby deduce that G is $\overline{C}_6$, up to multiple edges in some of the H_i. Towards this end, we shall first show that $u_1 = u_2$. Suppose that $u_1 \neq u_2$. Since G is bicritical, $G - \{u_1, u_2\}$ has a perfect matching N. This perfect matching necessarily contains $\overline{e}_1$ and $\overline{e}_2$. Thus, $\overline{u}_1 \neq \overline{u}_2$.

Let N' denote $N \cap E(H_1)$. Let M_1' denote $M_1 \cap E(H_3)$. Let M_2' denote $M_2 \cap E(H_2)$. Then, $M = N' \cup \{e_1, \overline{e_1}, e_2, \overline{e_2}\} \cup M_1' \cup M_2'$ is a perfect matching of G which includes both Q_1 and Q_2. This is impossible.

Thus, $u_1 = u_2$. It now follows that B_1 is a barrier of G. But since G is a brick, $|B_1| = 1$. Similar arguments show that, in fact, all A_i and B_i are singletons. Up to multiple edges in some of the subgraphs H_i, it follows that G is $\overline{C}_6$. ■

Remark: The above theorem is best possible. For example, the brick in Figure 2 has exactly one removable edge.

Proof of Theorem (1.20): Let v be a vertex of degree Δ in G. As in the statement of Theorem (4.2), let $Q_1, Q_2, \cdots, Q_\Delta$ be minimal classes in G, each depending on an edge incident with v. Each Q_i is either a singleton (a removable edge) or a doubleton. Thus, in order to prove that there are at least $(\Delta - 2)$ singletons, it suffices to prove that there are at most two doubletons. Assume to the contrary that there are three distinct doubletons.

By Theorem (4.2), either G is K_4 or G is $\overline{C_6}$, up to multiple edges. If G is K_4 we are done. Assume thus that G is $\overline{C_6}$, up to multiple edges. Graph $\overline{C_6}$ is cubic, therefore no multiple edges are incident with v, else at most two classes would be doubletons. But v is a vertex of maximum degree in G, whence G is cubic. Therefore G is free of multiple edges. We conclude that G is $\overline{C_6}$. ■

5 Canonical ear decompositions

In this section we shall present a proof of Theorem (1.21).

A graph G is **near-bipartite** if it is matching covered, nonbipartite, and it has a removable double ear D such that the (matching covered) graph $G - D$ is bipartite. In any ear decomposition of a nonbipartite matching covered graph, the first nonbipartite member of the sequence is a near-bipartite graph. Thus, by studying ear decompositions of near-bipartite matching covered graphs, we are able to deduce theorems concerning ear decompositions of nonbipartite matching covered graphs.

Theorem 5.1 *Let G be a matching covered graph free of vertices of degree two. Let e and f denote two edges of G that constitute a removable double ear of G. Let H denote (matching covered) graph $G - \{e, f\}$. If H is bipartite and G is neither K_4 nor $\overline{C_6}$ then G has a removable edge that is also removable in H.*

Proof: By induction on $|E(G)|$.

Assume that H is bipartite. Since G is not bipartite, one of e and f has both ends in one of A and B. Adjust notation so that edge e has both ends in A. Since H is matching covered, $|A| = |B|$. Since G is matching covered, edge f has both ends in B.

Lemma 5.2 *Graph H has no removable edges if, and only if, graph G is K_4.*

Proof: If graph G is K_4 then graph H is C_4, a graph without removable edges.

Assume that graph G is not K_4. Each vertex of G which is not incident with any of e and f has the same degree in G and in H. We conclude that at most four vertices of H have degree two in H. Graph H is not C_{2n}, otherwise it would be C_4, whence G would be K_4, a contradiction. Therefore, $\Delta(H) \geq 3$.

By Theorem (1.17), graph H has at least three edge-disjoint removable (single) ears. Since at most four vertices of H have degree two, at least one of those three removable ears is a removable edge. We conclude that graph H has removable edges. ■

By hypothesis, graph G is not K_4. By Theorem (5.2), graph H has at least one removable edge, h. If that edge is also removable in G, we are done. We may thus assume that edge h is not removable in G.

The next result characterizes removable edges of H that are not removable in G.

Lemma 5.3 *Let h be a removable edge of H. Edge h is not removable in G if, and only if, there exists a partition (A_1, A_2) of A and a partition (B_1, B_2) of B such that $|A_1| = |B_1| + 1$, $|B_2| = |A_2| + 1$, edge e has both ends in A_1, edge f has both ends in B_2 and edge h is the only edge of G having one end in B_1, the other in A_2 (Figure 4).*

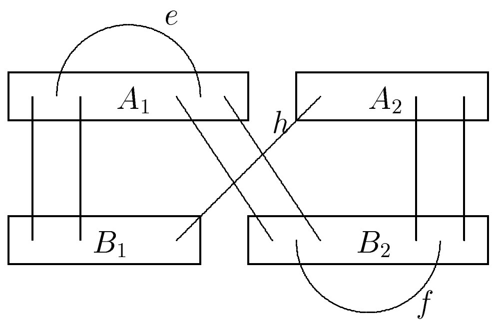

Figure 4: Edge h is removable in H but not in G.

Proof: Assume that the asserted partitions exist. Then A_1 is a barrier of $G - h$ that contains both ends of e, whence e is not admissible in $G - h$. Therefore, $G - h$ is not matching covered. That is, edge h is not removable in G.

Conversely, assume that edge h is not removable in G. Edge h is removable in H, therefore both edges e and f depend on h in G. Let A_1 denote a maximal barrier of $G - h$ that contains both ends of edge e. Since e is admissible in G, edge h has its ends in distinct components of $G - h - A_1$.

For each component K of $G - h - A_1$, let C_K denote cut $\nabla_G(V(K))$. Let G_K denote C_K-contraction $G\{V(K), \overline{v_K}\}$ of G.

We shall now show that G_K is matching covered, for each component K of $G - h - A_1$. Certainly, G_K is connected. Let k be any edge of G_K. Let us show that there exists a perfect matching M_k of G that contains edge k. Certainly, $k \neq e$; if $k \notin \{f, h\}$ then $k \in E(H - h)$, let M_k be a perfect matching of $H - h$ that contains edge h; if $k \in \{f, h\}$, let M_k be a perfect matching of G that contains edges e, f and h. Thus, M_k either includes $\{e, h\}$ or is disjoint with $\{e, h\}$. A simple counting argument then shows that $|M_k \cap C_K| = 1$ in both cases. Therefore, $M_k \cap E(G_K)$ is a perfect matching of G_K. Moreover, it contains edge k. We conclude that G_K is matching covered, for each component K of $G - h - A_1$.

We now observe that every bipartite component of $G - h - A_1$ is trivial. For this, let K denote any bipartite component of $G - h - A_1$. Let (X_K, Y_K) denote its bipartition, where $|X_K| > |Y_K|$. We have seen above that graph G_K is matching covered, whence G_K is bipartite, with bipartition $(X_K, Y_K \cup \{\overline{v_K}\})$, by Lemma (1.10). Set $A_1 \cup Y_K$ is then a barrier of $G - h$ that contains both ends of edge e. By the maximality of A_1, set Y_K is null, whence X_K is a singleton. Indeed, every bipartite component of $G - h$ is trivial.

If edge f does not lie in some component K of $G - h - A_1$, then K is a subgraph of H, in turn a bipartite graph: in that case, K is bipartite, whence trivial. We conclude that at most one component of $G - h - A_1$ is nontrivial.

On the other hand, at least one component of $G - h - A_1$ is nontrivial, otherwise edges e and h would be mutually dependent in G, which would contradict the hypothesis that H is matching covered. We conclude that $G - h - A_1$ has precisely one nontrivial component, say L.

The set of vertices of the trivial components of $G - h - A_1$ is a subset of $N_{G-e}(A_1)$, whence a subset of B: denote it B_1. Then $|A_1| = |B_1| + 1$. Let $A_2 = A \setminus A_1$, let $B_2 = B \setminus B_1$. Then $|B_2| = |A_2| + 1$. Note that $L - f$, a subgraph of H, is bipartite. Moreover, $V(L) = A_2 \cup B_2$. We conclude that (A_2, B_2) is a bipartition of $L - f$. Moreover, edge f has both ends in B_2, because component L must contain edge f and edge f has both ends in B.

Finally, we have seen that G_L is matching covered. Therefore edge h lies in G_L, it has one end in A_2, the other end is $\overline{v_L}$. Edge h is the only edge that has one end in B_1 and one end in $V(L)$. Therefore, it is the only edge that joins a vertex of B_1 to a vertex of A_2. ■

Let X denote set $A_1 \cup B_1$. Let C denote cut $\nabla(X)$. Let G_1 be the C-contraction $G\{X, \overline{x}\}$, let G_2 denote C-contraction $G\{\overline{X}, x\}$. Let H_1 and H_2 denote bipartite graphs $G_1 - \{e, h\}$ and $G_2 - \{f, h\}$, respectively.

Cut $C - h$ is tight in $H - h$, therefore each of H_1 and H_2 is matching covered. Let M_e denote any perfect matching of G that contains edge e. Then it contains edges f and h. Moreover, edge h is the only edge of M_e in C. We conclude that graphs G_1 and G_2 are both near-bipartite, with removable double ears e and h, and f and h, respectively.

Observe that for any perfect matching M_h of H that contains edge h, precisely two more edges of C lie in M_h, in addition to h. Thus, $|C| \geq 3$, whence each of G_1 and G_2 is free of vertices of degree two.

By induction hypothesis, either G_1 is one of K_4 and $\overline{C_6}$ or it has a removable edge that is also removable in H_1. Likewise, either G_2 is one of K_4 and $\overline{C_6}$ or it has a removable edge that is also removable in H_2.

Lemma 5.4 *Let r be an edge of $H - h$. If each of $G_1 - r$ and $G_2 - r$ is matching covered then r is removable in G. If each of $H_1 - r$ and $H_2 - r$ is matching covered then r is removable in H.*

Proof: Assume that each of $G_1 - r$ and $G_2 - r$ is matching covered. These two graphs are the $(C - r)$-contractions of $G - r$. Therefore, graph $G - r$ is matching covered, by Lemma (1.7).

Assume that each of $H_1 - r$ and $H_2 - r$ is matching covered. Observe that those two graphs are the $(C - h - r)$-contractions of graph $H - h - r$. Therefore, $H - h - r$ is matching covered, by Lemma (1.7).

Graph $H - h - r$ is also a subgraph of bipartite graph $H - r$. Therefore, by Theorem (1.2), removal of the ends of h in $H - h - r$ yields a graph with a perfect matching. Thus, edge h is admissible in $H - r$. We conclude that graph $H - r$ is matching covered. ■

So far, we have established notation and proved properties related to an edge h that is removable in H but not in G. We shall now choose a particular edge h: among the removable edges h of H that are not removable in G, choose one such that the block A_1 of partition (A_1, A_2) of A, as stated in Lemma (5.3), is minimal.

Lemma 5.5 *Let r be an edge of H_1. If r is removable in H_1 then it is removable in G_1.*

Proof: Assume, to the contrary, that edge r is removable in H_1 but not in G_1.

By Lemma (5.3), with G_1 playing the role of G and H_1 that of H, there exists a partition (A'_1, A''_1) of A_1 and a partition (B'_1, B''_1) of $B_1 \cup \{\overline{x}\}$ such that $|A'_1| = |B'_1| + 1$, $|B''_1| = |A''_1| + 1$, edge e has both ends in A'_1, edge h has both ends in B''_1 and edge r is the only edge of G_1 having one end in B'_1, the other in A''_1.

Let $A'_2 = A''_1 \cup A_2$, let $B'_2 = (B''_1 - \overline{x}) \cup B_2$. Then, (A'_1, A'_2) and (B'_1, B'_2) are partitions of A and B, respectively, such that $|A'_1| = |B'_1| + 1$, $|B'_2| = |A'_2| + 1$, edge e has both ends in A'_1, edge f has both ends in B'_2 and edge r is the only edge of G that has one end in B'_1, the other in A'_2. This contradicts the choice of h, because A'_1 is a proper subset of A_1.

We conclude that if r is removable in H_1 then the minimality of A_1 implies that it is also removable in G_1. ■

Case 1: graph G_2 is one of K_4 or $\overline{C_6}$. In that case, cut C consists of precisely three edges, one of which is h. This implies that the contraction vertex $\overline{x}$ has degree two in H_1. Therefore, no edge of $C - h$ is removable in H_1.

If graph H_1 has a removable edge, r, it does not lie in C. By Lemmas (5.4) and (5.5), edge r is removable in both H and G, and the assertion holds.

If graph H_1 has no removable edges, then, by Lemma (5.2), graph H_1 is K_4. If G_2 is K_4 then G is $\overline{C_6}$, a contradiction to the hypothesis. If G_2 is $\overline{C_6}$, then G is graph R_8, depicted in Figure 2. To see this, observe that if, in R_8, one takes as X the set of vertices of one of its triangles, then one of the $\nabla(X)$-contractions of R_8 is K_4, the other is $\overline{C_6}$. Moreover, "splicing" of $\overline{C_6}$ and K_4 to form a cubic graph is unique, up to isomorphism. So G is R_8, and R_8 is matching covered. Graph R_8 is near-bipartite, there is, up to isomorphism, just one choice for the pair of special edges: $\{a, a'_1\}$ (Figure 2). The only removable edge of R_8 is edge e_0 indicated in Figure 2, and that edge is also removable in the underlying bipartite subgraph of R_8.

Graph R_8 is the third smallest graph of a significant class that contains K_4 and $\overline{C_6}$, obtained by sequencial splicings of K_4, as introduced by Carvalho in his Ph. D. thesis [1]; this series will be presented in another paper, written by the first two authors [4].

We conclude that the assertion holds if G_2 is one of K_4 or $\overline{C_6}$.

Case 2: graph G_2 is not one of K_4 and $\overline{C_6}$. By induction hypothesis, H_2 has an edge, r, that is removable in both G_2 and H_2.

If edge r does not lie in C, then, by Lemma (5.4), r is an edge of H that is removable in each of H and G. The assertion then holds. We may thus assume that edge r lies in $C - h$.

If edge r is removable in H_1, then, by Lemmas (5.4) and (5.5), r is an edge of H that is removable in each of H and G. Again, the assertion holds. We may thus assume that edge r lies in $C - h$, is removable in each of H_2 and G_2 but is not removable in H_1.

Since edge r lies in $C - h$ and is removable in H_2, cut C has at least four edges. This implies that contraction vertex $\overline{x}$, one of the ends of h, has degree at least three in H_1. We conclude that graph H_1 has at most three vertices of degree two. By Theorem (1.17), graph H_1 has at least $|C| - 1$ edge-disjoint removable (single) ears. Since at most three vertices of H_1 have degree two, at least $|C| - 2$ of those ears are removable edges.

If graph H_1 has at least $|C| - 1$ removable edges, and since r, an edge of $C - h$, is not removable in H_1, we necessarily must have a removable edge of H_1 that does not lie in C. If graph H_1 has a removable edge that does not lie in C, then, by Lemmas (5.4) and (5.5), that edge lies in H and is removable in each of G and H. The assertion holds in this case.

We may thus assume that graph H_1 has precisely $|C| - 2$ removable edges, which are the edges of $C \setminus \{r, h\}$. Then graph H_1 also has a removable single ear R_1 that is edge-disjoint from those $|C| - 2$ removable edges. Since H_1 has at most three vertices of degree two, that removable ear is a path of length three. Let us denote that path by (v_0, v_1, v_2, v_3), where the v_i $(0 \leq i \leq 3)$ are the vertices of the path (Figure 5).

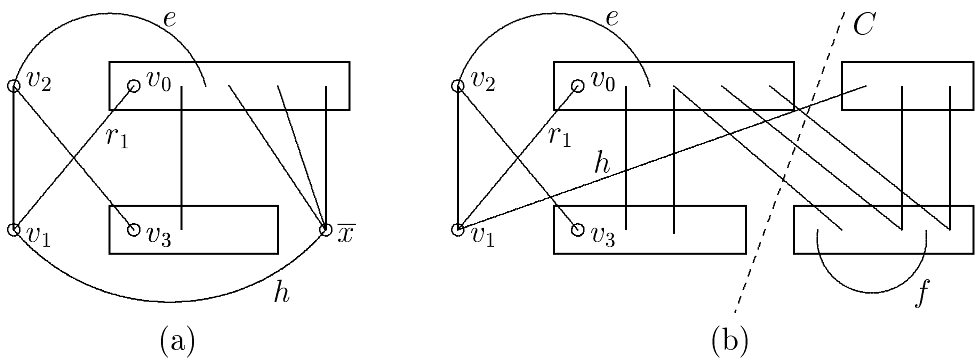

Figure 5: Single ear R_1.

Adjust notation, considering the reverse path if necessary, so that vertex v_0 lies in A_1. Let r_1 denote the edge in the ear that joins vertices v_0 and v_1. We assert that r_1 is removable in each of H and G.

Vertex v_1 is a vertex of degree two in H_1 and does not lie in A_1. Since the end $\overline{x}$ of h in H_1 has degree at least three in H_1, it follows that v_1 is the end of edge h in B_1. Thus, edges v_0v_1 and v_1v_2 do not lie in C. Therefore, the the only edge of R_1 that may possibly lie in C is edge v_2v_3, and, in that case, vertex v_3 is $\overline{x}$. Moreover, the $|C|-1$ removable ears of H_1 are edge-disjoint, therefore if v_2v_3 lies in C then it is edge r.

Thus, the $(C-h-v_2v_3)$-contractions of $H-h-R_1$ are the graphs H_1-R_1 and $H_2-v_2v_3$, and they are both matching covered. Therefore, graph $H-h-R_1$ is matching covered, by Lemma (1.7). Let R the path of length three in H obtained from R_1 by replacing edge r_1 by edge h. Then R joins a vertex of A to a vertex of B in H. Moreover, graph $H-r_1-R = H-h-R_1$. Therefore, graph $H-r_1-R$ is matching covered and bipartite. By Theorem (1.2), graph $H-r_1$ is matching covered.

Finally, let M_e denote any perfect matching of G that contains edge e: then M_e contains also edges f and h. But edge h is adjacent to edge r_1. Therefore, M_e is a perfect matching of $G-r_1$ that contains edges e and f. We conclude that edge r_1 is removable also in graph G. ■

Theorem 5.6 *Every near-bipartite graph G has a canonical ear decomposition that uses just one double ear.*

Proof: By induction on $|E(G)|$. Let D denote a removable double ear of G such that graph $G-D$ is matching covered and bipartite. Let H denote graph $G-D$. Let D_1 and D_2 denote the two single ears of D.

Consider first the case in which G is free of vertices of degree two. If G is one of K_4 and $\overline{C_6}$, then the assertion holds trivially. Assume thus that G is neither. By Theorem (5.1), there exists in H an edge, r, that is removable in both G and H. That is, each of $G-r$ and $H-r$ is matching covered. But graph $H-r$, a bipartite graph, is graph $G-r-D$, therefore graph $G-r$ is near-bipartite. By induction hypothesis, it has a canonical ear decomposition that uses just one double ear. Add at the end of that decomposition graph G, which is the result of the addition of single ear r to graph $G-r$. We obtain a canonical ear decomposition of G that uses just one double ear.

We may thus assume that graph G has vertices of degree two. Let u denote a vertex of degree two in G. If possible, choose u to lie in one of the single ears D_1 and D_2 of D.

The two edges incident with vertex u are not multiple edges, otherwise, since G is 2-connected, graph G is C_2, a graph which is not near-bipartite. Let v and w denote the two vertices of G that are adjacent to vertex u. Let $S = \{u, v, w\}$, let $C = \nabla(S)$. Let G_1 and G_2 denote the C-contractions $G\{\overline{S}, s\}$ and $G\{S, \overline{s}\}$, respectively.

Set $\{v, w\}$ is a barrier of G, therefore cut C is tight in G, whence each of G_1 and G_2 is matching covered.

Lemma 5.7 *Graph G_1 is near-bipartite.*

Proof: Consider first the case in which vertex u lies in a single ear of D. Adjust notation so that u is an internal vertex of D_1. Let D_1' denote the path obtained from D_1 by replacing (v, u, w), or its reverse, by vertex s. Let D' denote the double ear of G_1 obtained from D by replacing D_1 by D_1'.

By hypothesis, graph G is not bipartite. But graph G_2 is bipartite. Thus, G_1 is not bipartite. Moreover, $H = G - D = G_1 - D'$. We conclude that D' is a removable double ear of G_1 such that $G_1 - D'$ is bipartite. Therefore graph G_1 is near-bipartite in this case.

Consider last the case in which vertex u does not lie in any ear of D. By the choice of vertex u, each single ear of D contains just one edge. Let e_1 denote the edge of D_1, e_2 the edge of D_2.

Both edges lie in G_1, because neither is incident with vertex u. Moreover, cut C is tight in G and edges e_1 and e_2 are mutually dependent in G, thus at most one of them lies in C.

Let H_1 denote graph $G_1 - e_1 - e_2$. Graph H_1 is a $(C - e_1 - e_2)$-contraction of graph H, and cut $C - e_1 - e_2$ is tight in H. Therefore, graph H_1 is matching covered.

Let (A, B) denote a bipartition of graph H. Then edge e_1 has both ends in one of A and B, edge e_2 has both ends in the other. Vertices v and w are both adjacent to vertex u in H, therefore they lie both in one of A and B, say A. Therefore, graph H_1 is bipartite, with bipartition (A_1, B_1), where $A_1 = A - v - w + s$ and $B_1 = B - u$. Moreover, one of e_1 and e_2 has both ends in A_1, the other has both ends in B_1. It follows that graph G_1 is near-bipartite, with e_1 and e_2 constituting a removable double ear of G_1 such that $G_1 - e_1 - e_2$ is bipartite. ■

By induction hypothesis, let $\mathcal{P}$ be a canonical ear decomposition of G_1 with just one double ear, where

$$\mathcal{P} = (G_1, G_2 = G_1 + Q_1, \cdots, G_r = G_{r-1} + Q_{r-1}).$$

We shall see how an ear decomposition $\mathcal{P}'$ of G can be obtained from $\mathcal{P}$ by modifying one of the ears in it.

Let (C_v, C_w) denote the partition of C where C_v is the set of edges of C that are incident in G with vertex v, and C_w is the set of edges of C that are incident in G with vertex w. Graph G is 2-connected, therefore each of C_v and C_w is nonnull.

If P is a path in G_1 which does not contain s, then there is a path $\tilde{P}$ in G with $E(\tilde{P}) = E(P)$. Let us refer to $\tilde{P}$ as *the path that corresponds to* P. All modifications of ears, except one, consist of merely replacing paths in the ears in $\mathcal{P}$ by the paths in G that correspond to them. In the exceptional case, a path in an ear of $\mathcal{P}$ is modified so as to include the edges vu and uw, which are the two edges of G not in G_1. A precise description of this modification is given below.

Let G_{i+1} be the first graph in $\mathcal{P}$ that contains edges in both C_v and C_w. In that case, the ear Q_i either is, or contains, a path P which contains vertex s.

Adjust notation, so that s is not the terminus of P. Let e_1 denote the edge that follows s in P. Then $e_1 \in C$. Adjust notation, so that $e_1 \in C_w$. Then, G_i is the first graph in $\mathcal{P}$ that contains edges of C_w. Therefore replacement of s in P by (v, u, w) yields a path in G with the same ends as P.

For $1 \leq i \leq (r-1)$, let Q'_i be the path or the pair of paths obtained by modifying Q_i according to the above specified rules. Consider the sequence

$$\mathcal{P}' = (G'_1 = G_1, G'_2 = G'_1 + Q'_1, \cdots, G'_r = G'_{r-1} + Q'_{r-1})$$

of subgraphs of G. Then, it is easy to verify that $\mathcal{P}'$ is a canonical ear decomposition of G with just one double ear. ■

Proof of Theorem 1.21: Let G be any graph which has an ear decomposition

$$\mathcal{P} = (G_1, G_2, \cdots, G_r = G)$$

which uses d double ears. Then, we wish to prove that G has a canonical ear decomposition $\mathcal{P}^*$ which also uses exactly d double ears. In order to prove this, it suffices to show that the first nonbipartite graph in $\mathcal{P}$ has a canonical ear decomposition which uses exactly one double ear. Let G_i be the first nonbipartite graph in $\mathcal{P}$. Then, G_{i-1} is bipartite and G_i is obtained from G_{i-1} by adding a double ear (P_1, P_2), where P_1 is an odd path with ends in one part of the bipartition of G_{i-1}, and P_2 is an odd path with ends in the other part of the bipartition of G_{i-1}. Thus G_i is near-bipartite. To prove the theorem, it clearly suffices to show that G_i has a canonical ear decomposition which uses exactly one double ear. But this is valid by Theorem (5.6). ■

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