Uniqueness of the decompositon of the rank function of a 2-polimatroid*

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Abstract

In [5], Murty and Simon asked the following question: when the rank function of a polimatroid can be decomposed as the sum of rank funcitons of matroids? Another natural question to ask is the following: when this decomposition is unique? In this paper, we shall adress the second question.

1 Introduction

We shall denote the multiset $\{X_1, X_2, \ldots, X_m\}$ of matroids $X_1, X_2, \ldots, X_m$ by $\{X\}_m$. We say that a multiset of matroids $\{M\}_m$ is *admissible*, when $M_1, M_2, \ldots, M_m$ are matroids defined over the same ground set, which is denoted by $E(\{M\}_m)$, and

$$r_{M_1}(\{e\}) + r_{M_2}(\{e\}) + \cdots + r_{M_m}(\{e\}) \leq 2,$$

for every $e \in E(\{M\}_m)$. That is, the sum of the rank function of all the matroids belonging to $\{M\}_m$ is equal to the rank function of a 2-polimatroid P. We say that $\{M\}_m$ *defines* P or that $\{M\}_m$ is a *decomposition* of P. When this decompositon exists, P is said to be *matroidal*. We say that $(\{M\}_m, \{N\}_n)$ is a *pair*, when $\{M\}_m$ and $\{N\}_n$ are admissible multisets of matroids, $E(\{M\}_m) = E(\{N\}_n)$ and

$$r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_m}(X) = r_{N_1}(X) + r_{N_2}(X) + \cdots + r_{N_n}(X),$$

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for every $X \subseteq E(\{M\}_m)$. That is, $\{M\}_m$ and $\{N\}_n$ defines the same 2-polimatroid.

A pair $(\{M\}_m, \{N\}_n)$ is said to be *connected*, when, for every partition $\{X, Y\}$ of $E(\{M\}_m)$ such that $\min\{|X|, |Y|\} \geq 1$,

$$r_{M_i}(X) + r_{M_i}(Y) - r(M_i) \geq 1,$$

for some $i \in \{1, 2, \ldots, k\}$. That is, the matroids belonging to $\{M\}_m$ do not have a common 1-separation. Thus, $\{M\}_m$ defines a connected 2-polimatroid.

When M is a matroid and N a connected component of M, we denote by $\overline{N}$ the matroid such that $E(\overline{N}) = E(M)$, $\overline{N}|E(N) = N$ and $r(\overline{N}|E(M) \setminus E(N)) = 0$. We define

$$\mathcal{A}(M) = \{\overline{N} : N \text{ is a connected component of } M \text{ and } r(N) \neq 0\}.$$

Observe that

$$r_M(X) = \sum_{H \in \mathcal{A}(M)} r_H(X),$$

for every $X \subseteq E$. For a multiset $\{M\}_m$, consider the following multiset:

$$\mathcal{A}(\{M\}_m) = \bigcup_{i=1}^{m} \mathcal{A}(M_i).$$

Observe that $(\{M\}_m, \mathcal{A}(\{M\}_m))$ is a pair. Two admissible multisets of matroids $\{M\}_m$ and $\{N\}_n$ are said to be *equivalent*, when

$$\mathcal{A}(\{M\}_m) = \mathcal{A}(\{N\}_n).$$

We say that a pair $(\{M\}_m, \{N\}_n)$ is *trivial*, when $\{M\}_m$ is equivalent to $\{N\}_n$. When $(\{M\}_m, \{N\}_n)$ is a trivial pair, we can get $\{N\}_n$ from $\{M\}_m$ by moving around connected components and ignoring some loops. Thus, we shall not consider $\{M\}_m$ and $\{N\}_n$ different decompositons for the 2-polimatroid defined by $\{M\}_m$.

The *size* of an admissible multiset of matroids $\{M\}_m$ is defined as

$$size(\{M\}_m) = |\mathcal{A}(\{M\}_m)|.$$

We define the *size* of a 2-polimatroid P as

$$size(P) = \max\{size(\{M\}_m) : \{M\}_m \text{ decomposes } P\}.$$

In this paragraph, we shall discuss an ideia originated on Seymour's paper [8]. For a matroid M, we define

$$\mathcal{C}_{\geq 2}(M) = \{C \in \mathcal{C}(M) : |C| \geq 2\},$$

that is, $\mathcal{C}_{\geq 2}(M)$ is the set of circuits of M which are not loops. For an admissible multiset of matroids $\{M\}_m$, we define the following multiset:

$$\mathcal{C}_{\geq 2}(\{M\}_m) = \bigcup_{i=1}^{m} \mathcal{C}_{\geq 2}(M_i).$$

We say that a pair $(\{M\}_m, \{N\}_n)$ is *preserving* when

$$\mathcal{C}_{\geq 2}(\{M\}_m) = \mathcal{C}_{\geq 2}(\{N\}_n).$$

In section three, we shall study the non-preserving connected pairs and conclude that they are very few. Essentially, there is just one of them, since all the others can be obtained from it making 2-sums of appropriated matroids, as described in the next theorem. To avoid a more cumbersome statement for it, we need to extend the 2-sum operation for matroids having more than one element. When N and M are matroids such that $E(N) \cap E(M) = \{e\}$ and $E(N) \setminus E(M) = \{f\}$, the 2-sum of N with M is the matroid obtained from M by renaming e by f.

Theorem 1. Let H_1, H_2, K_1, K_2, K_3 be matroids defined over the same ground set $\{a, b, c\}$ such that

$$H_1 \simeq U_{2,3}, H_2 \simeq U_{1,3}, K_i \simeq U_{1,2} \oplus U_{0,1}, \text{ for every } i \in \{1, 2, 3\},$$

and a, b, c are loops of K_1, K_2, K_3, respectively. When $\{a, b, c\} \cap E(\{M\}_m) = \emptyset$, $(\{M\}_m, \{N\}_n)$ is a non-preserving connected pair if and only if there are connected matroids P_a, P_b, P_c such that $x \in P_x$, $|E(P_x)| \geq 2$, for every $x \in \{a, b, c\}$, $\{P_a \setminus \{a\}, P_b \setminus \{b\}, P_c \setminus \{c\}\}$ is a partition of $E(\{M\}_m)$, $\{M\}_m$ or $\{N\}_n$ is equivalent to

$$\{H_1 \oplus_2 N_a \oplus_2 N_b \oplus_2 N_c, H_2 \oplus_2 N_a \oplus_2 N_b \oplus_2 N_c\}$$

and the other admissible multiset of matroids in this pair is equivalent to

$$\{(L_a \oplus K_1 \backslash a) \oplus_2 N_b \oplus_2 N_c, (L_b \oplus K_2 \backslash b) \oplus_2 N_a \oplus_2 N_c, (L_c \oplus K_3 \backslash c) \oplus_2 N_a \oplus_2 N_b\},$$

where L_x is a matroid over $P_x \setminus \{x\}$ such that $r(L_x) = 0$, for every $x \in \{a, b, c\}$.

As the non-preserving pairs are rare and well described, the rest of the paper shall be dedicated to prove results about preserving pairs. In section four, we shall prove the following theorem:

Theorem 2. If $(\{M\}_m, \{N\}_n)$ is a non-trivial preserving connected pair, then there is a pair $(\{M'\}_2, \{N'\}_2)$ such that $\{M\}_m$ is equivalent to $\{M'\}_2$ and $\{N\}_n$ is equivalent to $\{N'\}_2$.

Thus, when a connected 2-polimatroid has non-equivalent decompositons, we can reorganize the matroids on these decompositons moving around connected components and ignoring loops, in such way that at the end we get only two matroids in each decompositon, unless these decompositions forms a non-preserving pair.

Now one example to show that theorem 2 is quite surprising: for each vertex v of the complete graph K_n, let M_v be a matroid such that $E(M_v) = E(K_n), M_v|st(v)$ is a connected component of M and $r_{M_v}(E(K_n) \setminus st(v)) = 0$. When $n \geq 2$, then $\{M_v : v \in V(K_n)\}$ is an admissible multiset of matroids which is not equivalent to one having less that n matroids.

A vector $\mathbf{M} = (M_1, M_2, M_3, M_4)$ with four matroids defined on the same ground set E is called a *quad* when these matroids satisfies:

$$r_{M_1}(X) + r_{M_2}(X) = r_{M_3}(X) + r_{M_4}(X),$$

for every $X \subseteq E$. $\mathbf{M}$ is said to be *non-trivial*, when $\{M_1, M_2\} \neq \{M_3, M_4\}$. So, theorem 1 and theorem 2 reducess the problem of uniqueness of decomposition of a connected 2-polimatroid to the study of non-trivial quads.

In [3], Lemos and Mota shall present an algorithm to decide when a vector with two matroids (M_1, M_2) can beginning a non-trivial quad. Moreover, this algorithm gives all the possibilities for the others two matroids that completes that quad. Thus, the algorithm, theorem 1 and theorem 2 solves the uniqueness problem of decomposition of the rank function of a 2-polimatroid as the sum of rank function of matroids.

We shall have the following corollaries from these theorems, which guaranties uniqueness in many cases:

Corollary 1. If P is a connected matroidal 2-polimatroid such that $size(P) \geq 4$, then the decompositon of P is unique.

Corollary 2. Suppose that $P = (E, \rho)$ is a connected matroidal 2-polimatroid such that $\rho(E)$ is odd. If the decomposition of P is not unique, then P has only two non-equivalent decompositions which forms a non-preserving pair.

Corollary 3. Suppose that $P = (E, \rho)$ is a connected matroidal 2-polimatroid. If $\rho(e) = 1$, for some $e \in E$, then the decomposition of P is unique.

We say that a 2-polimatroid is *binary* when it decomposes as an admissible multiset of binary matroids.

Corollary 4. If P is a connected binary 2-polimatroid, then it has at most two non-equivalent decompositions.

Now, we give an example to show that a connected 2-polimatroid can have arbitrary large number of non-equivalent decompositions: let $\mathcal{H}$ be a family of subsets of a set E such that $|H| = r$ and $|H \cup H'| \geq r + 2$, for every $H, H' \in \mathcal{H}$

such that $H \neq H'$. For $\mathcal{Z} \subseteq \mathcal{H}$, let $M(\mathcal{Z})$ be the matroid having E as ground set and every subset of E with r elements not belonging to $\mathcal{Z}$ as a basis. When $\{\mathcal{X}, \mathcal{Y}\} \neq \{\mathcal{H}, \emptyset\}$ is a partition of $\mathcal{H}$, then $(\{M(\mathcal{H}), M(\emptyset)\}, \{M(\mathcal{X}), M(\mathcal{Y})\})$ is a non-trivial pair.

2 Tutte's geometry

In this section, we shall present some results from Tutte's geometry. We shall use Tutte's geometry, since it is natural to work with it when one is dealing with circuits.

We say that L is a *Tutte-line* of a matroid M, when L is the uniun of circuits of M and $r^*(M|L) = 2$. P is said to be a *Tutte-plane* of M, when P is the uniun of circuits of M and $r^*(M|P) = 3$. Every Tutte-line has a partition $\{L_1, L_2, \dots, L_k\}$, which is called *canonical*, such that C is a circuit of M contained in L if and only if $C = L \setminus L_i$, for some $i \in \{1, 2, \dots, k\}$. We say that a Tutte-line L or a Tutte-plane P is *connected*, when $M|L$ or $M|P$ is connected, respectively. When a line L is connected, its canonical partition has at least three sets. Now, we shall present some results of Tutte's geometry.

(2.1) (4.26 of [9]) Suppose that P is a connected Tutte-plane of a matroid M. If C is a circuit of M contained in P, then there are at least two connected Tutte-lines of M contained in P which contains C. ∎

(2.2) (4.171 of [9]) Suppose that P is a Tutte-plane of a matroid M. If L and L' are different Tutte-lines of M contained in P, then $L \cap L'$ contains exactly one circuit of M. ∎

(2.3) (4.35 of [9]) Suppose that M is a connected matroid. If C and D are different circuits of M, then there is a sequence

$$C = C_0, C_1, C_2, \dots, C_n = D$$

of circuits of M such that $C_{i-1} \cup C_i$ is a connected Tutte-line of M, for every $i \in \{1, 2, \dots, n\}$. ∎

Now, we shall present a result of Lehman. Before the introduction of its statement, we remember that the set of all circuits of a matroid M which contains an element e is denoted by $\mathcal{C}_e(M)$.

(2.4) (4.3.2 of [6]) Suppose that M and N are connected matroids defined on the same ground set. If $\mathcal{C}_e(M) = \mathcal{C}_e(N)$, then $N = M$. ∎

3 The non-preserving case

Throughout this section, we shall suppose that $(\{M\}_m, \{N\}_n)$ is a pair. When X and Y are disjoint subsets of $E(\{M\}_m)$, then

$\{M\backslash X/Y\}_m$ denotes the multiset $\{M_1\backslash X/Y, M_2\backslash X/Y, \ldots, M_m\backslash X/Y\}$.

We shall define a *minor* of $(\{M\}_m, \{N\}_n)$ as

$$(\{M\}_m, \{N\}_n)\backslash X/Y = (\{M\backslash X/Y\}_m, \{N\backslash X/Y\}_n).$$

We denote $(\{M\}_m, \{N\}_n)\backslash(E(\{M\}_m) \setminus X)$ and $\{M\backslash(E(\{M\}_m) \setminus X)\}_m$ by $(\{M\}_m, \{N\}_n)|X$ and $\{M|X\}_m$, respectively. We shall use the next result along this paper without refering to it.

(3.1) $(\{M\}_m, \{N\}_n)\backslash X/Y$ is a pair.

Proof: When $Z \subseteq E(\{M\}_m) \setminus (X \cup Y)$, we have that

$$\begin{aligned}\sum_{i=1}^{m} r_{M_i\backslash X/Y}(Z) &= \sum_{i=1}^{m}[r_{M_i}(Z \cup Y) - r_{M_i}(Y)] \\ &= \sum_{i=1}^{m} r_{M_i}(Z \cup Y) - \sum_{i=1}^{m} r_{M_i}(Y) \\ &= \sum_{i=1}^{n} r_{N_i}(Z \cup Y) - \sum_{i=1}^{n} r_{N_i}(Y) \\ &= \sum_{i=1}^{n}[r_{N_i}(Z \cup Y) - r_{N_i}(Y)] = \sum_{i=1}^{n} r_{N_i\backslash X/Y}(Z) \quad \blacksquare\end{aligned}$$

(3.2) If $M_i|X$ is loopless, for some $i \in \{1, 2, \ldots, m\}$, then $\{M|X\}_k$ is equivalent to $\{M_i|X, K\}$, for some matroid K.

Proof: As $r_{M_i}(\{e\}) = 1$, for every $e \in X$, it follows that

$$r_{M_1}(\{e\}) + \cdots + r_{M_{i-1}}(\{e\}) + r_{M_{i+1}}(\{e\}) + \cdots + r_{M_m}(\{e\}) \leq 1.$$

Thus, there is a matroid K such that every connected component N of M_j satisfying $r(N) \neq 0$, for $j \neq i$, is also a connected component of K. $\blacksquare$

Observe that every chain of elements of $\mathcal{C}_{\geq 2}(\{M\}_m)$ has size one or two, otherwise there are three elements C_1, C_2 and C_3 of $\mathcal{C}_{\geq 2}(\{M\}_m)$ such that $C_1 \subset C_2 \subset C_3$. Thus,

$$r_{M_1}(\{e\}) + r_{M_2}(\{e\}) + \cdots + r_{M_m}(\{e\}) \geq 3,$$

for every element $e \in C_1$, since e cannot be a loop in M_i, when $C_j \in \mathcal{C}(M_i)$. Moreover, an element of $\mathcal{C}_{\geq 2}(\{M\}_m)$ can have multiplicity at most two. When this happens, it does not contains or it is not contained in another element of $\mathcal{C}_{\geq 2}(\{M\}_m)$. We denote by $\mathcal{C}_{min}(\{M\}_m)$ the multiset of minimal elements of $\mathcal{C}_{\geq 2}(\{M\}_m)$. Note that an element has multiplicity two in $\mathcal{C}_{min}(\{M\}_m)$ if and only if is has multiplicity two in $\mathcal{C}_{\geq 2}(\{M\}_m)$.

For a matroid K, we shall define

$$\delta(K) = r(K) - \sum_{e \in E(K)} r_K(\{e\}).$$

The following observation is fundamental in the proof of the next lemma:

(3.3) For a matroid K, $\delta(K) \leq 0$. Moreover, $\delta(K) = 0$ if and only if $\mathcal{C}_{\geq 2}(K) = \emptyset$ and $\delta(K) = -1$ if and only if $|\mathcal{C}_{\geq 2}(K)| = 1$. ■

(3.4) $\mathcal{C}_{min}(\{M\}_m) = \mathcal{C}_{min}(\{N\}_n)$.

Proof: By symmetry, we need only to prove that $\mathcal{C}_{min}(\{M\}_m) \subseteq \mathcal{C}_{min}(\{N\}_n)$. Let C be an element of $\mathcal{C}_{min}(\{M\}_m)$. For $X \subseteq C$, observe that

$$\begin{aligned}
\sum_{i=1}^{m} \delta(M_i|X) &= \sum_{i=1}^{m} \left(r_{M_i}(X) - \sum_{x \in X} r_{M_i}(\{x\}) \right) \\
&= \sum_{i=1}^{m} r_{M_i}(X) - \sum_{i=1}^{m} \sum_{x \in X} r_{M_i}(\{x\}) \\
&= \sum_{i=1}^{m} r_{M_i}(X) - \sum_{x \in X} \sum_{i=1}^{m} r_{M_i}(\{x\}) \\
&= \sum_{i=1}^{n} r_{N_i}(X) - \sum_{x \in X} \sum_{i=1}^{n} r_{N_i}(\{x\}) \\
&= \sum_{i=1}^{n} \left(r_{N_i}(X) - \sum_{x \in X} r_{N_i}(\{x\}) \right) = \sum_{i=1}^{n} \delta(N_i|X)
\end{aligned}$$

Now, we shall prove that:

Step 1: $\mathcal{C}_{\geq 2}(\{N|X\}_n) = \emptyset$, for every $X \subset C$.

When $X \subset C$, $\mathcal{C}_{\geq 2}(M_i|X) = \emptyset$, for every $i \in \{1, 2, \ldots, m\}$, since $C \in \mathcal{C}_{min}(\{M\}_m)$. By **(3.3)**, we have that

$$0 = \sum_{i=1}^{m} \delta(M_i|X) = \sum_{i=1}^{n} \delta(N_i|X).$$

Thus, by **(3.3)**, $\delta_{N_i}(X) = 0$ and $\mathcal{C}_{\geq 2}(N_i|X) = \emptyset$, for every $i \in \{1, 2, \ldots, n\}$.

So, for every $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, $\mathcal{C}_{\geq 2}(M_i|C) \in \{\emptyset, \{C\}\}$ and $\mathcal{C}_{\geq 2}(N_j|C) \in \{\emptyset, \{C\}\}$. Thus, the result follows when we stabilish:

Step 2: $|\mathcal{C}_{\geq 2}(\{M|C\}_m)| = |\mathcal{C}_{\geq 2}(\{N|C\}_n)|$.

From **(3.3)** and step 1, we have that

$$-|\mathcal{C}_{\geq 2}(\{M|C\}_m)| = \sum_{i=1}^{m} \delta(M_i|X) = \sum_{i=1}^{n} \delta(N_i|X) = -|\mathcal{C}_{\geq 2}(\{N|C\}_n)|$$

and the result follows. ∎

We define the following multiset

$$\mathcal{C}_{uni}(\{M\}_m) = \mathcal{C}_{\geq 2}(\{M\}_m) \setminus \mathcal{C}_{\geq 2}(\{N\}_n).$$

We define $\mathcal{C}_{uni}(\{N\}_n)$ similarly. In **(3.7)**, we shall describe the nice behaviour of the elements of $\mathcal{C}_{uni}(\{M\}_m)$. To do this, we need to prove some preliminary lemmas.

(3.5) If $C \in \mathcal{C}_{uni}(\{M\}_m)$ and $C' \in \mathcal{C}_{uni}(\{N\}_n)$, then $C \not\subseteq C'$ and $C' \not\subseteq C$.

Proof: We shall argue by contradiction and we assume that the result fails. Without loss of generality, we may suppose that $C \subseteq C'$. By definition of these multisets, we have that $C \subset C'$. By **(3.2)**, there are matroids K and K' such that $\{M|C\}_m$ and $\{N|C\}_n$ are equivalent to $\{M_i|C, K\}$ and $\{N_j|C, K'\}$, respectively, where C is a circuit of M_i and C' is a circuit of N_j. If $C'' \in \mathcal{C}_{\geq 2}(\{M\}_m)$ is contained in C and $C''' \in \mathcal{C}_{\geq 2}(\{N\}_n)$ is contained in C', then $C'' \in \mathcal{C}_{min}(\{M\}_m)$ and $C''' \in \mathcal{C}_{min}(\{N\}_n)$. So, by **(3.4)**, C'' and C''' belongs to $\mathcal{C}_{\geq 2}(\{M\}_m) \cap \mathcal{C}_{\geq 2}(\{N\}_n)$. Thus, K and K' have the same circuits and $K = K'$. We arrive at a contradiction, since $M_i|C \neq M_j|C$, and the result follows. ∎

(3.6) For $i \in \{1, 2, \ldots, m\}$, if L is a Tutte-line of M_i having canonical partition $\{L_1, L_2, L_3\}$ such that $C_j = L \setminus L_j$ is a circuit of N_j, for every $j \in \{1, 2, 3\}$, then $L \in \mathcal{C}_{uni}(\{M\}_m)$.

Proof: By **(3.2)**, there is a matroid K such that $E(K) = L$ and $\{M|L\}_m$ is equivalent to $\{M_i|L, K\}$. For $j \in \{1, 2, 3\}$, observe that $r(N_j|L_j) = 0$ because $r_{N_k}(\{e\}) = 1$, for every $e \in L_j$ and $k \in \{1, 2, 3\} \setminus \{j\}$. Thus, $\mathcal{C}_{\geq 2}(N_i|L) = \{C_i\}$. Hence $\mathcal{C}_{\geq 2}(\{N|L\}_n) = \{C_1, C_2, C_3\}$. Observe that

$$r_K(L) = \left(\sum_{j=1}^{3} r_{N_j}(L)\right) - r_{M_i}(L) = \left(\sum_{j=1}^{3} (|L \setminus L_j| - 1)\right) - (|L| - 2) = |L| - 1.$$

Hence $\mathcal{C}_{\geq 2}(\{N|L\}_n) \triangle \mathcal{C}_{\geq 2}(\{M_i|L, K\}) = \{C\}$, for some circuit C of K. So, $C \in \mathcal{C}_{uni}(\{M\}_m)$. When $L = C$, then the result follows. We may suppose that there is $e \in L \setminus C$. In this case, we arrive at a contradiction, since $r_H(L \setminus \{e\}) = r_H(L)$, for every $H \in \{K, N_1, N_2, N_3\}$ and $r_{M_i}(L) = r_{M_i}(L \setminus \{e\}) + 1$. ∎

(3.7) If $C \in \mathcal{C}_{uni}(\{M\}_m)$, then C is a Tutte-line of M_i containing exactly three circuits of M_i, for some $i \in \{1, 2, \ldots, m\}$, and these circuits are circuits of different N_j.

Proof: We do not loose generality when we suppose that C is a circuit of M_1. By **(3.2)**, $\{M|C\}_m$ is equivalent to $\{M_1|C, K\}$. By **(3.4)** and **(3.5)**, we have that

$$\mathcal{C}_{\geq 2}(\{M_1|C, K\}) \triangle \mathcal{C}_{\geq 2}(\{N|C\}_n) = \{C\}. \tag{\textbf{1}}$$

Thus,

$$\mathcal{C}_{\geq 2}(K) = \mathcal{C}_{\geq 2}(\{N\}_n). \tag{\textbf{2}}$$

Observe that K has a connected component H such that $r(H) > 0$ and H is not a connected component of N_i, for every $i \in \{1, 2, \ldots, n\}$, otherwise $\mathcal{A}(\{K', K\}) = \mathcal{A}(\{N|C\}_n)$, where K' is a matroid over C such that $r(K') = |C|$.

In this paragraph, we shall prove that H has circuits C' and C'' such that

$$\{C', C''\} \not\subseteq \mathcal{C}(N_i), \text{ for every } i \in \{1, 2, \ldots, n\}. \tag{\textbf{3}}$$

If **(3)** does not occur, then $\mathcal{C}(H) \subseteq \mathcal{C}(N_i)$, for some $i \in \{1, 2, \ldots, n\}$, since every element of $\mathcal{C}_{\geq 2}(\{N\}_n)$ has multiplicity one, by **(2)**. So, N_i has a connected component H' such that $\mathcal{C}(H) \subseteq \mathcal{C}(H')$. By **(2)**, $\mathcal{C}(H') \subseteq \mathcal{C}(K)$. Hence $\mathcal{C}(H') = \mathcal{C}(H)$ and $H = H'$. We arrive at a contradiction and **(3)** follows. Now, we shall prove that:

Step 1: There are circuits C_1 and C_2 of H such that $C_1 \cup C_2$ is a connected Tutte-line of H, C_1 is a circuit of N_i and C_2 is a circuit of N_j, for $j \neq i$, say $i = 1$ and $j = 2$.

By **(3)**, there are circuits C' and C'' of H such that, for every $i \in \{1, 2, \ldots, n\}$, $\{C, C''\} \not\subseteq \mathcal{C}(N_i)$. So, by **(2.3)**, there is a sequence of circuits of H

$$C' = D_0, D_1, D_2, \ldots, D_k = C''$$

such that $D_{i-1} \cup D_i$ is a connected Tutte-line of H, for every $i \in \{1, 2, \ldots, k\}$. Thus, for some $i \in \{1, 2, \ldots, k\}$, $\{D_{i-1}, D_i\} \not\subseteq \mathcal{C}(N_j)$, for every $j \in \{1, 2, \ldots, n\}$. Take $C_1 = D_{i-1}$ and $C_2 = D_i$.

Let $\{L_1, L_2, \ldots, L_k\}$ be the canonical partition of $L = C_1 \cup C_2$. We may suppose that $C_1 = L \setminus L_1$ and $C_2 = L \setminus L_2$. As L is connected, it follows that $k \geq 3$. By **(1)**, we have that

$$\mathcal{C}_{\geq 2}(\{M|L\}_m) \setminus \{C\} = \mathcal{C}_{\geq 2}(\{N|L\}_n) = \mathcal{C}(H|L) = \{L \setminus L_1, L \setminus L_2, \ldots, L \setminus L_k\}.$$

Step 2: For every $i \in \{1,2,\ldots,n\}$, $|\mathcal{C}(H|L) \cap \mathcal{C}(N_i|L)| \leq 1$.

Suppose that step 2 is not true. So, there is an $i \in \{1,2,\ldots,n\}$ such that $|\mathcal{C}(H|L) \cap \mathcal{C}(N_i|L)| \geq 2$, say $i=2$. Let C_3 be a circuit of $N_2|L$ different from C_2. Observe that $(C_2 \cup C_3) \setminus e$, for $e \in L_1$, does not contains a circuit of $N_2|L$, otherwise this circuit would be C_1. We arrive at a contradiction and the result follows.

Without loss of generality, we may suppose that $C_i = L \setminus L_i$ is a circuit of N_i, for $i \in \{1,2,\ldots,k\}$. As an element e of $E(\{M\}_m)$ cannot be a loop in at most two N_i's, it follows that $k=3$. By **(3.6)**, $L \in \mathcal{C}_{uni}(\{M\}_m)$. Thus, $L = C$. ∎

(3.8) Suppose that $m=2$, L is a connected Tutte-line of M_2 and $r([M_1|L]^*) \leq 1$. If $\mathcal{C}_{uni}(\{M|L\}_m) = \emptyset$, then L is a Tutte-line of N_i having the same canonical partition as in M_2, for some $i \in \{1,2,\ldots,n\}$.

Proof: By **(3.7)**, $\mathcal{C}_{\geq 2}(\{N|L\}_n) \subseteq \mathcal{C}_{\geq 2}(\{M|L\}_m)$, since $m=2$. Hence, by hypothesis,

$$\mathcal{C}_{\geq 2}(\{N|L\}_n) = \mathcal{C}_{\geq 2}(\{M|L\}_m).$$

Suppose that $\{L_1, L_2, \ldots, L_k\}$ is the canonical partition of L in M_2. Consider the set $X = \{i : \mathcal{C}(N_i) \cap \mathcal{C}(M_2|L) \neq \emptyset\}$.

In this paragraph, we shall prove that $|X| \leq 2$. Suppose that $|X| \geq 3$. If $k \geq 4$, then, for $l \in \{1,2,3\}$, there are i_l and j_l such that $|\{i_1,i_2,i_3\}| = |\{j_1,j_2,j_3\}| = 3$ and $C_{j_l} \in \mathcal{C}(N_{i_l}) \cap \mathcal{C}(M_2|L)$. Hence

$$\sum_{i \in X} r_{N_i}(\{e\}) \geq 3,$$

for every $e \in C_{j_1} \cap C_{j_2} \cap C_{j_3}$. We arrive at a contradiction. Thus, $k=3$ and we arrive at a contradiction by **(3.6)**. So, $|X| \leq 2$.

If $|X| = 1$, then the result follows. Thus, we may suppose that $|X| = 2$. We can order the sets on the canonical partition of L in M_2 and the matroids in $\{N\}_n$ such that $C_1, \ldots, C_t$ are circuits of N_1 and $C_{t+1}, \ldots, C_k$ are circuits of N_2. Without loss of generality, we may suppose that $t \geq 2$. Hence L is a Tutte-line of N_1 having canonical partition $\{L_1, \ldots, L_t, L \setminus (L_1 \cup \cdots \cup L_t)\}$. If $t = k-1$, then $C' = C_k$, where C' is the circuit of $M_1|L$, and the result follows. So, we may suppose that $t \leq k-2$. Observe that $C' = L_1 \cup \cdots \cup L_t$. Similarly, we have that $C' = L_{t+1} \cup \cdots \cup L_k$ and we arrive at a contradiction. ∎

If K is a matroid and $Z \subseteq E(K)$, a *Z-arc* of K is a minimal non-empty subset $A \subseteq E(K) \setminus Z$ such that there is a circuit C of K with $C \setminus Z = A$ and $C \cap Z \neq \emptyset$. A *fundamental for* A is a circuit C such that $C \setminus Z = A$, where A is a Z-arc. We denote by $arc_K(Z)$ the set of all Z-arcs of K. When $A \in arc_K(Z)$ and $P \subseteq Z$, we say that $A \to P$ in K if there is a Z-fundamental for A included in $A \cup P$. For its negation, we write $A \not\to P$.

(3.9) Suppose that $L \in \mathcal{C}_{uni}(\{M\}_m)$. If L is a circuit of M_1, L is a Tutte-line of M_2 and A is a L-arc of M_2 which does not contain properly a L-arc of M_1, then $\mathcal{C}_{uni}(\{M\}_m) \cap \mathcal{C}(M_1|(L \cup A)) \neq \{L\}$.

Proof: Suppose that $\mathcal{C}_{uni}(\{M\}_m) \cap \mathcal{C}(M_1|(L \cup A)) = \{L\}$. Without loss of generality, we may suppose that $E(\{M\}_m) = L \cup A$. Choose $a \in A$. By **(3.2)**, there is a matroid K such that $\{M/(A \setminus \{a\})\}_m$ is equivalent to $\{M_2/(A \setminus \{a\}), K\}$. Observe that $L \cup \{a\}$ is a connected Tutte-plane of $M_2/(A \setminus \{a\})$, since A is a L-arc of M_2. As A does not contain properly a L-arc of M_1, it follows that L is a circuit of K and $r(K^*) \leq 2$. Moreover, a is a loop of K or a coloop of K or $L \cup \{a\}$ is a Tutte-line of K.

By **(3.7)**, we may suppose that $\mathcal{C}(M_2|L) = \{C_1, C_2, C_3\}$ and that C_i is a circuit of N_i, for every $i \in \{1,2,3\}$. By **(2.1)**, it follows that there is a connected Tutte-line L'_i of $M_2/(A \setminus \{a\})$ such that $C_i \subset L'_i \subset L \cup \{a\}$ and $L'_i \neq L$. Hence $|\mathcal{C}(M_1/(A \setminus \{a\})|L'_i)| \leq 1$, since $L'_i \subset L \cup \{a\}$. Thus, applying **(3.8)** to the pair $(\{M\}_m, \{N\}_n)/(A \setminus \{a\})|L'_i$, we have that L'_i is a connected line of $N_i/(A \setminus \{a\})$. We arrive at a contradiction because $a \in L'_i$, for every $i \in \{1,2,3\}$, and hence $r_{N_1}(\{a\}) + r_{N_2}(\{a\}) + r_{N_3}(\{a\}) = 3$. ■

(3.10) Suppose that L is a circuit of M_1, L is a Tutte-line of M_2 having canonical partition $\{L_1, L_2, L_3\}$ and $C_i = L \setminus L_i$ is a circuit of N_i, for every $i \in \{1,2,3\}$. Then, $arc_{M_1}(L) = arc_{M_2}(L)$. Moreover, there is a partition $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ of $arc_{M_1}(L)$ such that

(i) $arc_{N_i}(L) = arc_{M_1}(L) \setminus \mathcal{A}_i$, for every $i \in \{1,2,3\}$;

(ii) for every $i \in \{1,2,3\}$ and $\{i,j,k\} = \{1,2,3\}$, if $A \in \mathcal{A}_i$, then $A \to L_i$ in K, for every $K \in \{M_1, M_2, N_j, N_k\}$.

Proof: Observe that L_i is a set of loops of N_i, for every $i \in \{1,2,3\}$, and that $L \in \mathcal{C}_{uni}(\{M\}_m)$. When $A \in arc_{M_i}(L)$, for some $i \in \{1,2\}$, then $M_i|(L \cup A)$ is connected. By **(3.2)**, $\{M|(L \cup A)\}_m$ is equivalent to $\{M_i|(L \cup A), K_A\}$, for some matroid K_A defined on $L \cup A$.

Step 1: $\mathcal{C}_{\geq 2}(\{N|(L \cup A)\}_n) \subseteq \mathcal{C}_{\geq 2}(\{M|(L \cup A)\}_m)$, when $A \in arc_{M_1}(L) \cup arc_{M_2}(L)$.

As A is a L-arc of M_i, for some $i \in \{1,2\}$, by **(3.7)**, it follows that

$$\mathcal{C}_{\geq 2}(\{N|(L \cup A)\}_n) \subseteq \mathcal{C}_{\geq 2}(\{M_i|(L \cup A), K_A\}),$$

since $|\{M_i|(L \cup A), K_A\}| = 2$ and the step follows.

Observe that $arc_{M_1}(L) = arc_{M_2}(L)$ follows from the next step:

Step 2: If A is a minimal set belonging to $arc_{M_1}(L) \cup arc_{M_2}(L)$, then A belongs to $arc_{M_1}(L) \cap arc_{M_2}(L)$.

Suppose that this step fails. Choose a minimal set A of $arc_{M_1}(L) \cup arc_{M_2}(L)$ which does not belong to $arc_{M_1}(L) \cap arc_{M_2}(L)$. We have two cases to deal with: (a) $A \in arc_{M_1}(L) \setminus arc_{M_2}(L)$; or (b) $A \in arc_{M_2}(L) \setminus arc_{M_1}(L)$. In both cases, for $i \in \{1, 2\}$, $\{a\}$ is a L-arc of $M_i|(L \cup A)/(A \setminus \{a\})$ if and only if A is a L-arc in M_i. Now, we shall prove that

$$\mathcal{C}_{\geq 2}(\{M|(L \cup A)/(A \setminus \{a\})\}_m) \triangle \mathcal{C}_{\geq 2}(\{N|(L \cup A)/(A \setminus \{a\})\}_n) = \{L\}. \quad \mathbf{(1)}$$

By setp 1, we need only to prove that $\mathcal{C}_{uni}(\{M|(L \cup A)/(A \setminus \{a\})\}_m) = \{L\}$. In both cases, a is a loop or coloop of $K_A/(A \setminus \{a\})$. By **(3.7)**, if $C \in \mathcal{C}_{uni}(\{M|(L \cup A)/(A \setminus \{a\})\}_m)$, then $a \notin C$. So, $C = L$ and we have **(1)**.

Suppose that (a) occurs. Let $L = D_0, D_1, D_2, \ldots, D_s$ be the circuits of $M_1|(L \cup A)/(A \setminus \{a\})$ contained in $L \cup \{a\}$. As $D_i \cup D_j$ contains L, when $i \neq j$, and L_i is a set of loops of N_i, it follows that each $N_i|(L \cup A)/(A \setminus \{a\})$ has at most one D_j as a circuit. Thus, by **(1)**, we may suppose that $s = 2$ and D_i is a circuit of $N_i|(L \cup A)/(A \setminus \{a\})$. Hence, by **(1)**, $C_i \cap D_i = \emptyset$, otherwise $C_i \cup D_i$ contains another circuit of $M_1|(L \cup A)/(A \setminus \{a\})$. So, $(D_1 \cup D_2) \cap L_3 = \emptyset$ and we arrive at a contradiction. When (ii) happens, we arrive at a contradiction by **(3.9)** and **(1)**. Thus, step 2 follows.

Step 3: If $A \in arc_{M_1}(L)$, then, there is an $i \in \{1, 2, 3\}$, such that $A \in arc_K(L)$ and $A \to L_i$ in K, for every $K \in \{M_1, M_2, N_j, N_k\}$, where $\{i, j, k\} = \{1, 2, 3\}$.

By step 2, for every $A \in arc_{M_1}(L)$, $P = A \cup L$ is a Tutte-line of M_1 and P is a Tutte-plane of M_2. By **(3.9)**, there is $L' \in \mathcal{C}_{uni}(\{M\}_m) \cap \mathcal{C}(M_1|P)$ such that $L \neq L'$. By **(3.7)**, L' is a Tutte-line of P. Thus, the circuit of M_2 contained in $L \cap L'$, which exists by **(2.2)**, must be equal to C_i, for some $i \in \{1, 2, 3\}$, say $i = 1$.

First, we shall determine the structure of the Tutte-plane P of M_2 using the Tutte-lines L and L' of M_2. By **(3.7)**, we may suppose that the other circuits contained in L' are C_2' and C_3', with C_i' being a circuit of N_i, for $i \in \{2, 3\}$. By **(2.1)**, there is a connected Tutte-line L_i' contained in P such that $C_i' \subset L_i'$ and $L_i' \neq L'$. The circuit of M_2 contained in $L_i' \cap L$ is C_2 or C_3. Observe that L_i' does not contains a circuit of N_1, since A is a set of loops of N_1 and every circuit of $M_2|L_i'$ different of C_j contains A. By **(3.7)**, $\mathcal{C}_{uni}(\{M\}_m) \cap \mathcal{C}(M_1|L_i') = \emptyset$. Hence, by **(3.8)** applied to the pair $(\{M_1|P, M_2|P\}, \{N|P\}_n)$, L_i' is a Tutte-line of N_2 or N_3 having the same canonical partition. Thus, $L_i' = C_i \cup C_i'$, for $i \in \{2, 3\}$.

Let C be the circuit of M_2 contained in $L_2' \cap L_3'$, which exists by **(2.2)**. So, C is a circuit of both N_2 and N_3. Hence C is a circuit of both M_1 and M_2. Moreover, $C \cap (L_2 \cup L_3) = \emptyset$. Hence $A \to L_1$ in H, for $H \in \{M_1, M_2, N_2, N_3\}$

because C is a circuit of all these matroids and $A \subset C \subseteq A \cup L_1$. We have stablished step 3.

So, we get the partition $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ of $arc_{M_1}(L)$ setting:

$$\mathcal{A}_i = \{A \in arc_{M_1}(L) : A \to L_i\},$$

for $i \in \{1, 2, 3\}$, and the result follows, from step 1, step 2 and step 3. $\blacksquare$

For the proof of theorem 1, we need the following result of Seymour:

(3.11) (3.8 of [7]) Let K be a matroid, let $Z \subseteq E(K)$, and let $\{P_1, P_2\}$ be a partition of Z. Then either there is a Z-arc A such that $A \not\to P_1, A \not\to P_2$, or there is a partition $\{X_1, X_2\}$ of $E(K)$ such that $X_i \cap Z = P_i (i = 1, 2)$ and

$$r(X_1) + r(X_2) - r(K) = r(P_1) + r(P_2) - r(Z). \quad \blacksquare$$

We shall apply Seymour's result only in the case that $Z \neq \emptyset$. Looking in the proof of this result, one concludes that X_i is choosen such that

$$X_i = \bigcup \{P_i \cup A : A \in arc_Z(K) \text{ and } A \to P_i \text{ in } K\}.$$

So, when we refer to this result, we are supposing that the partitions are taken as described above.

Proof of theorem 1: As $(\{M\}_m, \{N\}_n)$ is non-preserving, it follows that $\mathcal{C}_{uni}(\{M\}_m) \neq \emptyset$ or $\mathcal{C}_{uni}(\{N\}_n) \neq \emptyset$, say $\mathcal{C}_{uni}(\{M\}_m) \neq \emptyset$. Chose $C \in \mathcal{C}_{uni}(\{M\}_m)$. Without loss of generality, we may suppose that C is a circuit of M_1. By **(3.7)**, C is a line of M_i, for some $i \geq 2$, say $i = 2$. Moreover, we may suppose that the canonical partition of C in M_2 is $\{L_1, L_2, L_3\}$ and $C_i = L \setminus L_i$ is a circuit of N_i, for $i \in \{1, 2, 3\}$, say. Now, we shall define

$$X_i = \bigcup \{A \cup L_i : A \in arc_{M_1}(C) \text{ and } A \to L_i \text{ in } M_1\}.$$

By **(3.10)**, we have that $\{X_1, X_2, X_3\}$ is a partition of $E(\{M\}_m)$, otherwise $\{E(\{M\}_m) \setminus (X_1 \cup X_2 \cup X_3), X_1 \cup X_2 \cup X_3\}$ is a 1-separation of M_i, for every $i \in \{1, 2, \ldots, m\}$. By **(3.11)**, when $|X_i| \geq 1$, we have that $\{X_i, E(\{M\}_m)\}$ is an exact 2-separation of M_1, M_2 and N_j, for $j \neq i$. The result follows, since the matroids that are decomposed using these 2-separations and has X_i in its ground set are equal, by **(3.8)**(ii) and **(2.4)**. $\blacksquare$

4 The preserving case

Throughout this section, we shall suppose that $(\{M\}_m, \{N\}_n)$ is a non-trivial preserving pair. We shall denote $E(\{M\}_m)$ by E. Let ρ be the rank function of the 2-polimatroid P defined by this pair.

(4.1) If $\rho(e) = 1$, for some $e \in E$, then $\mathcal{A}(\{M\}_m) \cap \mathcal{A}(\{N\}_n) \neq \emptyset$.

Proof: Let M_i and N_j be matroids such that $r_{M_i}(\{e\}) = r_{N_j}(\{e\}) = 1$. Let M' and N' be the connected component of M_i and N_j that contains e, respectively. As e is a loop of M_k and N_l, for every $k \neq i$ and $l \neq j$, it follows that M' and N' have the same circuits containing the element e, since $(\{M\}_m, \{N\}_n)$ is preserving. Thus, $M' = N'$, by **(2.4)**. ∎

(4.2) If P is connected, then $\mathcal{A}(\{M\}_m) \cap \mathcal{A}(\{N\}_n) = \emptyset$.

Proof: Suppose that $\mathcal{A} = \mathcal{A}(\{M\}_m) \cap \mathcal{A}(\{N\}_n)$. Observe that $(\mathcal{A}(\{M\}_m) \setminus \mathcal{A}, \mathcal{A}(\{N\}_n) \setminus \mathcal{A})$ is a non-trivial preserving pair which defines a 2-polimatroid P' having rank function ρ'. By **(4.1)**, $\rho'(\{e\}) \neq 1$, for every $e \in E$. So, $X = \{e \in E : \rho'(\{e\}) = 0\}$ and $Y = \{e \in E : \rho'(\{e\}) = 2\}$ is a partition of E. As X is a set of loops, for every matroid in $(\mathcal{A}(\{M\}_m) \cup \mathcal{A}(\{N\}_n)) \setminus \mathcal{A}$, and Y a set of loops, for every matroid in $\mathcal{A}$, it follows that $\{X, Y\}$ is a 1-separation for the 2-polimatroid P, when $X \neq \emptyset$ and $Y \neq \emptyset$. So, $X = \emptyset$ or $Y = \emptyset$. Thus, $X = \emptyset$, otherwise $(\{M\}_m, \{N\}_n)$ is trivial. ∎

For the proof of theorem 2, we need the following result of Lemos:

(4.3) (2.12 of [2]) If (M_1, M_2, M_3, M_4) is a non-trivial connected quad, then there is at most one $i \in \{1, 2, 3, 4\}$ such that M_i is not a connected matroid. Moreover, when M_i is not connected, it has exactly two connected components. ∎

Proof of theorem 2: For a matroid H, we denote by $loop(H)$ the set of its loops. Now, we shall consider a maximal member S of

$$\mathcal{S} = \{E \setminus loop(H) : H \in \mathcal{A}(\{M\}_m) \cup \mathcal{A}(\{N\}_n)\}.$$

Without loss of generality, we may suppose that $M_1|S$ is connected. By **(3.2)**, there is a matroid K such that $\{M|S\}_m$ is equivalent to $\{M_1|S, K\}$.

In this paragraph, we shall prove that $\{N|S\}_n$ is also equivalent to an admissible multiset containing two matroids. Suppose that this is not the case, and let $\{K\}_p$ be an admissible multiset of matroids equivalent to $\{N|S\}_n$ having p minimum. Thus, $p \geq 3$. For each $i \in \{1, 2, \ldots, p\}$, set $L_i = E \setminus loop(K_i)$. So, for $\{i, j\} \subset \{1, 2, \ldots, p\}$, $L_i \cap L_j \neq \emptyset$, otherwise we could replace K_i and K_j by $K_i|L_i \oplus K_j|L_j \oplus K_i|(loop(K_i) \cap loop(K_j))$ in the admissible multiset $\{K\}_p$ and we arrive at a contradiction by the choice of p. Choose an element a_{ij} belonging to $L_i \cap L_j$. Observe that $a_{ij} \in loop(K_l)$, for $l \notin \{i, j\}$. Let L be a connected Tutte-line of $M_1|S$ containing $\{e_{12}, e_{13}, e_{23}\}$. So, $|\mathcal{C}(N_i) \cap \mathcal{C}(M_1|L)| \leq 1$ and L contains only three circuits of M_1. We arrive at a contradiction, by **(3.6)**.

Let $(\{M'\}_2, \{N'\}_2)$ be a pair such that $\{M|S\}_m$ and $\{N|S\}_n$ are equivalent to $\{M'\}_2$ and $\{N'\}_2$, respectively. So, (M'_1, M'_2, N'_1, N'_2) is a quad. It must be non-trivial, otherwise $M'_1 = N'_1$, say, and by the choice of S, M'_1 and N'_1 are connected components of some matroid in $\{M\}_m$ and $\{N\}_n$, recpectively. But this is contrary to **(4.2)**. By **(4.3)**, M_1' and M_2' are connected or N_1' and N_2' are connected, say M_1' and M_2' are connected. Thus, by the choice of S, for $i \in \{1, 2\}$, M_i' is a connected component of M_i. So, S is a separator for both M_1 and M_2. Hence $\{S, E \setminus S\}$ is a 1-separation for P, unless $S = E$. The result follows. $\blacksquare$

5 The proof of the colloraries

Throughout this section, we shall suppose that $(\{M\}_m, \{N\}_n)$ is a non-trivial pair which defienes a connected 2-polimatroid P. We shall denote $E(\{M\}_m)$ by E. Let ρ be the rank function of P. First, we shall state a result of Lemos [?]:

(5.1) (theorem 2 of [1]) If (M_1, M_2, M_3, M_4) is a non-trivial connected quad, then
$$r(M_1) = r(M_2) = r(M_3) = r(M_4). \quad \blacksquare$$

(5.2) If $(\{M\}_m, \{N\}_n)$ is non-preserving, then it is the unique pair defining P.

Proof: By theorem 1, we may suppose that $m = 2$ and $n = 3$. Let $(\{M'\}_m$, $\{N'\}_n)$ be another non-trivial pair that defines P. When $(\{M'\}_{m'}, \{N'\}_{n'})$ is non-preseving, we may suppose that $m' = 2$ and $n' = 3$. By theorem 1, $\{N'\}_{n'}$ is equivalent to $\{N\}_n$. So, (M_1, M_2, M_1', M_2') is a non-trivial connected quad. We arrive at a contaridiction, by **(5.1)**, because $r(M_1) \neq r(M_2)$. Thus, $(\{M'\}_{m'}, \{N'\}_{n'})$ is preserving. By theorem 2, we may suppose that $m' = n' = 2$. In this case, we also arrive at the same contradiction, by **(5.1)**, since (M_1, M_2, M_1', M_2') is a non-trivial connected quad. $\blacksquare$

All the corollaries are imediate consequences of **(5.2)** and theorem 1 in the case that $(\{M\}_m, \{N\}_n)$ is non-preseving. Thus, we may suppose that every pair of admissible multisets which decomposes P is preserving. By theorem 2, we do not loose generality, if we suppose that $m = n = 2$. So, (M_1, M_2, N_1, N_2) is a non-trivial quad.

(5.3) (2.2 of [2]) If (M_1, M_2, M_3, M_4) is a non-trivial connected quad, then M_i is loopless, for every $i \in \{1, 2, 3, 4\}$. $\blacksquare$

Corollary 1 follows **(4.3)**. Corollary 2 from **(5.1)**. Corollary 3 from **(5.3)**. Corollary 4 from **(4.2)** of Lemos [2]. We shall not state the last result of Lemos here, since it envolves some tecnical definitions. The interested reader should consult the original paper. Moreover, the binary connected 2-polimatroid with two non-equivalent decompositions are completely described by Lemos result.

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