The reconstruction of a matroid from its connectivity function*

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Abstract

In this paper, we shall present an algorithm to decide when a connected matroid M is reconstructible from its connectivity function. When M is not reconstructible, this algorithm gives all the matroids with the same connectivity fucntion as M.

1 Introduction

The matroid theory terminology used in this paper will follow Oxley [4]. The *connectivity function* of a matroid M is defined as

$$\xi_M(X,Y) = r_M(X) + r_M(Y) - r(M) + 1,$$

where $\{X, Y\}$ is a partition of $E(M)$. This function is invariant under duality, since it may be written as

$$\xi_M(X,Y) = r_M(X) + r_{M^*}(X) - |X| + 1.$$

In particular, a matroid and its dual have the same connectivity function. When $M = M_1 \oplus M_2$, then

$$\xi_M(X,Y) + 1 = \xi_{M_1}(X \cap E(M_1), Y \cap E(M_1)) + \xi_{M_2}(X \cap E(M_2), Y \cap E(M_2)).$$

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Thus, $M_1 \oplus M_2$ and $M_1 \oplus M_2^*$ have the same connectivity function.

In this paper, we shall adress the problem of determining all matroids with the same connectivity function. By the last identity, we may restrict our attention to connected matroids. In this case, Cunningham conjectured that, up to duality, a connected matroid is reconstructed from its connectivity functon, that is:

Cunningham conjecture. If M and N are connected matroids over the same ground set having the same connectivity function, then $M = N$ or $M = N^*$.

In [5], Seymour proved the Cunningham conjecture for the class of binary matroids. In the same paper, Seymour gave a counter-example for it in general. In [1], Lemos proved that this conjecture is true provided that $r(M) \neq r(M^*)$.

Note that

$$r_M(X) + r_{M^*}(X) = r_N(X) + r_{N^*}(X),$$

for every $X \subseteq E$, when M and N are matroids over the same ground set E having the same connectivity function. In general, we say that $\mathbf{M} = (M_1, M_2, M_3, M_4)$ is a *quad* when M_1, M_2, M_3 and M_4 are matroids over the same ground set, which is denoted by $E(\mathbf{M})$, such that

$$r_{M_1}(X) + r_{M_2}(X) = r_{M_3}(X) + r_{M_4}(X),$$

for every $X \subseteq E(\mathbf{M})$. Moreover, when $M_1 = M_2^*$ and $M_3 = M_4^*$, $\mathbf{M}$ is said to be a *s-quad*.

We are setting a different notation from Lemos [1] to get a more compact description of the results. Whem $\mathbf{X}$ denotes a quad, X_i denotes the i-th coordenate of this quad, for $i \in \{1, 2, 3, 4\}$.

For a quad $\mathbf{M}$ and disjoint subsets X and Y of $E(\mathbf{M})$, we define a *minor* of $\mathbf{M}$ as

$$\mathbf{M}\backslash X/Y = (M_1\backslash X/Y, M_2\backslash X/Y, M_3\backslash X/Y, M_4\backslash X/Y)$$

and the *dual* $\mathbf{M}^*$ of $\mathbf{M}$ as $(M_1^*, M_2^*, M_3^*, M_4^*)$. We shall use $\mathbf{M}|X$ to denote $\mathbf{M}\backslash(E(\mathbf{M}) \setminus X)$. In [1], Lemos have proved that

(1.1) (1.1, 1.2 and 1.3 of [1]) $\mathbf{M}\backslash X/Y$ and $\mathbf{M}^*$ are quads. ■

When $\mathbf{M}$ and $\mathbf{N}$ are quads such that $E(\mathbf{M}) \cap E(\mathbf{N}) = \emptyset$, we denote by $\mathbf{M} \oplus \mathbf{N}$ the quad

$$(M_1 \oplus N_1, M_2 \oplus N_2, M_3 \oplus N_3, M_4 \oplus N_4).$$

Case $N_1 = N_2 = N_3 = N_4 = N$, $\mathbf{M} \oplus \mathbf{N}$ is also denoted by $\mathbf{M} \oplus N$. For a matroid N such that $E(N) \cap E(\mathbf{M}) = \{e\}$, $\mathbf{M} \oplus_2 N$ denotes

$$(M_1 \oplus_2 N, M_2 \oplus_2 N, M_3 \oplus_2 N, M_4 \oplus_2 N),$$

which was proved to be a quad by Lemos in [1].

For a partitition $\{X, Y\}$ of $E(\mathbf{M})$, we define

$$\xi_{\mathbf{M}}(X, Y) = \xi_{M_1}(X, Y) + \xi_{M_2}(X, Y).$$

When $\xi_{\mathbf{M}}(X, Y) = 2$, then $\mathbf{M} = (\mathbf{M}|X) \oplus (\mathbf{M}|Y)$. So, we can restrict our attention to quads satisfying $\xi_{\mathbf{M}}(X, Y) \geq 3$, for every partition $\{X, Y\}$ of $E(\mathbf{M})$ such that $\min\{|X|, |Y|\} \geq 1$. Such quads are called *connected*. $\mathbf{M}$ is said to be *non-trivial*, when $\{M_1, M_2\} \neq \{M_3, M_4\}$. In [1], Lemos proved the following result:

(1.2) Suppose that $\mathbf{M}$ is a non-trivial connected quad. Then,

(i) (2.6 of [1]) $\xi_{\mathbf{M}}(X, Y) \geq 4$, for every partititon $\{X, Y\}$ of $\mathbf{M}$ such that $\min\{|X|, |Y|\} \geq 1$.

(ii) (2.7 of [1]) When $\xi_{M_1}(X, Y) = \xi_{M_2}(X, Y) = \xi_{M_3}(X, Y) = \xi_{M_4}(X, Y) = 2$, for a partition $\{X, Y\}$ of $E(\mathbf{M})$ such that $\min\{|X|, |Y|\} \geq 2$, then there is a matroid N such that $E(N) = X \cup \{e\}$ or $E(N) = Y \cup \{e\}$, for an element $e \notin E(\mathbf{M})$, and a quad $\mathbf{N}$ such that $E(\mathbf{N}) = (E(\mathbf{M}) \setminus E(N)) \cup \{e\}$ satisfying $\mathbf{M} = \mathbf{N} \oplus_2 N$. ■

In [1], Lemos proved the next result which characterizes a quad:

(1.3) (1.4 and 1.5 of [1]) Suppose that M_i is a matroid having E as its ground set, for every $i \in \{1, 2, 3, 4\}$. (M_1, M_2, M_3, M_4) is a quad if and only if

$$\mathcal{C}(M_1) \cup \mathcal{C}(M_2) = \mathcal{C}(M_3) \cup \mathcal{C}(M_4) \text{ and } \mathcal{C}(M_1) \cap \mathcal{C}(M_2) = \mathcal{C}(M_3) \cap \mathcal{C}(M_4). \quad ■$$

This result is essential in Seymour's proof that a connected binary matroid is reconstructible from its connectivity function, up to duality, in the class of binary matroids. We shall state Seymour's original result, whose proof can be adapted to prove **(1.3)**:

Theorem 1. Suppose that M and N are matroids such that $E(M) = E(N)$. M and N have the same connectivity function if and only if

$$\mathcal{C}(M) \cup \mathcal{C}^*(M) = \mathcal{C}(N) \cup \mathcal{C}^*(N) \text{ and } \mathcal{C}(M) \cap \mathcal{C}^*(M) = \mathcal{C}(N) \cap \mathcal{C}^*(N). \quad ■$$

So, from Seymour's result, a matroid M is reconstructible from its connectivity function, when there is no subset $\mathcal{A}$ of $\mathcal{C}(M) \triangle \mathcal{C}^*(M)$ such that

(F1) $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq \mathcal{C}(M) \triangle \mathcal{C}^*(M)$;

(F2) $\mathcal{C} = \mathcal{C}(M) \triangle \mathcal{A}$ and $\mathcal{C}^* = \mathcal{C}^*(M) \triangle \mathcal{A}$ are, respectively, the family of circuits and the family of cocircuits of a matroid.

A subset $\mathcal{A}$ of $\mathcal{C}(M) \triangle \mathcal{C}^*(M)$ is called *feasible* for M when it satisfies **(F1)** and **(F2)**. Thus, we need to answer the following question: when does M have a feasible set?

Clearly, a feasible set $\mathcal{A}$ for M must satisfies:

(A1) $|\{X, Y\} \cap \mathcal{A}| \neq 1$, for every $X, Y \in \mathcal{C}(M) \triangle \mathcal{C}^*(M)$ such that $X \subset Y$.

(A2) $|\{X, Y\} \cap \mathcal{A}| \neq 1$, for every $X, Y \in \mathcal{C}(M) \triangle \mathcal{C}^*(M)$ such that $|X \cap Y| = 1$.

A subset $\mathcal{A}$ of $\mathcal{C}(M) \triangle \mathcal{C}^*(M)$ is called *admissible* for M when it satisfies **(F1)**, **(A1)** and **(A2)**. The next result determines which admissible sets are feasible.

Theorem 2. Suppose that $\mathcal{A}$ is an admissible set for a connected matroid M such that $r(M) = r(M^*)$. $\mathcal{A}$ is feasible if and only if $|\{X, Y\} \cap \mathcal{A}| \in \{0, 2\}$, for every

$$X, Y \in (\mathcal{C}(M) \cap \mathcal{I}(M^*)) \cup (\mathcal{C}(M^*) \cap \mathcal{I}(M))$$

such that $X \cap Y = \emptyset$.

Observe that one can easily determine all the feasible sets for M using theorem 2, provided that the circuits and cocircuits of M are know.

2 Almost preserving the rank function

In this section, we shall improve the next result of Lemos [1]:

(2.1) (theorem 2 of [1]) If $\mathbf{M}$ is a non-trival connected quad, then

$$r(M_1) = r(M_2) = r(M_3) = r(M_4). \quad \blacksquare$$

By **(2.1)**, the condition of the rank of M in theorem 2 is not restricitive, since, for a connected matroid M, feasible sets exist only when $r(M) = r(M^*)$.

(2.2) Suppose that $\mathbf{M}$ is a non-trivial connected quad. If $\{M_1|A, M_2|A\} \neq \{M_3|A, M_4|A\}$, then

$$r_{M_1}(A) = r_{M_2}(A) = r_{M_3}(A) = r_{M_4}(A)$$

and there is a subset B of A such that

(i) $\mathbf{M}|A = \mathbf{M}|B \oplus \mathbf{M}|(A \setminus B)$;

(ii) $\mathbf{M}|B$ is non-trivial and connected;

(iii) $M_1|(A \setminus B) = M_2|(A \setminus B) = M_3|(A \setminus B) = M_4|(A \setminus B)$.

Proof: We shall prove this result by induction on $|E(\mathbf{M})|$. The result is vacuously true for $|E(\mathbf{M})| = 1$. When $E(\mathbf{M}) = A$, the first part of the result follows from **(2.1)** and for the second, we can take $B = A$. So, we may suppose that $E(\mathbf{M}) \setminus A \neq \emptyset$. Let e be an element of $E(\mathbf{M}) \setminus A$.

When $\mathbf{M} \backslash e$ is connected, then the result follows by induction, since

$$\{M_1 \backslash e, M_2 \backslash e\} \neq \{M_3 \backslash e, M_4 \backslash e\}$$

because $\{M_1|A, M_2|A\} \neq \{M_3|A, M_4|A\}$.

When $\mathbf{M} \backslash e$ is not connected, then, by **(1.2)**, there is a matroid N and a quad $\mathbf{N}$ such that $E(N) = X \cup \{f\}$, for some $X \subset E(\mathbf{M})$, $E(\mathbf{N}) = (E(\mathbf{M}) \setminus X) \cup \{f\}$ and $\mathbf{M} = \mathbf{N} \oplus_2 N$, where $f \notin E(\mathbf{M})$ and $\{e, f\}$ is contained in a series class of N. As $\mathbf{M}$ is connected, it follows that $\mathbf{N}$ is a connected quad. Observe that $\mathbf{N}$ is a non-trivial quad, since

$$M_i|A = M_i|(A \cap X) \oplus M_i|(A \setminus X) = N|(A \cap X) \oplus N_i|(A \setminus X).$$

By induction, the first part of the result holds for $\mathbf{N}$ and hence

$$r_{N_1}(A \setminus X) = r_{N_2}(A \setminus X) = r_{N_3}(A \setminus X) = r_{N_4}(A \setminus X).$$

As $r_{M_i}(A) = r_N(A \cap X) + r_{N_i}(A \setminus X)$, the first part of the result follows for $\mathbf{M}$. By induction, the second part of the result also holds for $\mathbf{N}$, and there is a subset B of $A \setminus X$ such that

(i) $\mathbf{N}|(A \setminus X) = \mathbf{N}|B \oplus \mathbf{N}|(A \setminus (X \cup B))$;

(ii) $\mathbf{N}|B$ is non-trivial and connected;

(iii) $N_1|(A \setminus (X \cup B)) = N_2|(A \setminus (X \cup B)) = N_3|(A \setminus (X \cup B)) = N_4|(A \setminus (X \cup B))$.

The second part of the result also follows for $\mathbf{M}$ because $\mathbf{N}|B = \mathbf{M}|B, \mathbf{N}|(A \setminus (B \cup X)) = \mathbf{M}|(A \setminus (B \cup X))$ and $\mathbf{N}|(A \setminus (B \cup X)) \oplus \mathbf{M}|(A \cap X) = \mathbf{M}|(A \setminus B)$. ■

(2.3) If $\mathbf{M}$ is a non-trivial connected quad, then

$$\{r_{M_1}(A), r_{M_2}(A)\} = \{r_{M_3}(A), r_{M_4}(A)\},$$

for every $A \subseteq E(\mathbf{M})$.

Proof: The result follows when $\{M_1|A, M_2|A\} = \{M_3|A, M_4|A\}$. So, we may suppose that this is not the case. By **(2.2)**, it follows that $r_{M_1}(A) = r_{M_2}(A) = r_{M_3}(A) = r_{M_4}(A)$. ■

Observe that the previous result says that the rank function is locally preserved in a non-trivial connected quad $\mathbf{M}$, that is,

$$\{r_{M_1}(A), r_{M_2}(A)\} = \{r_{M_3}(A), r_{M_4}(A)\}, \text{ for every } A \subseteq E(\mathbf{M}).$$

Cunningham conjecture may be extended to this context as: for a non-trivial connected quad $\mathbf{M}$, is it true that

$$\{r_{M_1}, r_{M_2}\} = \{r_{M_3}, r_{M_4}\}?$$

So, **(2.3)** is the closest positive answer that we can get in general for Cunningham conjecture.

In view of **(2.3)**, when $\mathbf{M}$ is a non-trivial connected quad, then the rank functions of M_3 and M_4 are obtained mixing the rank functions of M_1 and M_2, that is, there is a family $\mathcal{R}$ of subsets of $E(\mathbf{M})$ such that

$$(r_{M_3}(X), r_{M_4}(X)) = \begin{cases} (r_{M_1}(X), r_{M_2}(X)) & \text{when } X \in \mathcal{R} \\ (r_{M_2}(X), r_{M_1}(X)) & \text{when } X \notin \mathcal{R} \end{cases}$$

In the next section, we shall define an operation of mix between the rank functions of two matroids, that is, we prove that for some $\mathcal{R}$ this identity gives rise to a pair of rank functions of matroids. This operation of mix will be enough to characterize the quads, as we shall prove in section four.

3 The mix of two matroids

Suppose that M_1 and M_2 are matroids on the same ground set E such that

$$\xi_{M_1}(X, Y) + \xi_{M_2}(X, Y) \geq 3,$$

for every partition $\{X, Y\}$ of E satisfying $\min\{|X|, |Y|\} \geq 1$.

We define

$$\mathcal{X} = \{X \subseteq E : r_{M_1}(X) \neq r_{M_2}(X)\}.$$

Let G_{M_1,M_2} be a simple graph whose vertex set is $\mathcal{X}$ and, for $X, Y \in \mathcal{X}$, XY is an edge of this graph if and only if

(i) $X \subset Y$; or

(ii) $X \cap Y \notin \mathcal{X}, X \cup Y \notin \mathcal{X}$ and $|X \triangle Y| = 2$.

Let $\{\mathcal{A}, \mathcal{B}\}$ be a partition of $\mathcal{X}$ such that $AB \notin E(G_{M_1,M_2})$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We define two functions:

$$r_3(X) = \begin{cases} r_{M_1}(X) & \text{when } X \notin \mathcal{A} \\ r_{M_2}(X) & \text{when } X \in \mathcal{A} \end{cases} \quad \text{and } r_4(X) = \begin{cases} r_{M_2}(X) & \text{when } X \notin \mathcal{A} \\ r_{M_1}(X) & \text{when } X \in \mathcal{A} \end{cases}$$

Observe that these functions are also given by

$$r_3(X) = \begin{cases} r_{M_1}(X) & \text{when } X \in \mathcal{B} \\ r_{M_2}(X) & \text{when } X \notin \mathcal{B} \end{cases} \quad \text{and } r_4(X) = \begin{cases} r_{M_2}(X) & \text{when } X \in \mathcal{B} \\ r_{M_1}(X) & \text{when } X \notin \mathcal{B} \end{cases}$$

Note that the definition of mix is algorithmic, but it depends on the rank functions of the matroids involved. In section five, we shall look to the mix in terms of circuits and hyperplanes, getting a more efficient algorithm for its computation.

(3.1) r_3 and r_4 are rank functions of matroids, which are denoted by M_3 and M_4, respectively.

Proof: By symmetry, it is enough to prove that r_3 is the rank function of a matroid. To demonstrate this, we need to verify the following (1.4.14 of [4]):

(1) $r_3(\emptyset) = 0$;

(2) $r_3(X) \leq r_3(Y)$, when $X \subseteq Y$;

(3) $r_3(X \cup \{e\}) \leq r_3(X) + 1$;

(4) $r_3(X \cup \{e, f\}) = r_3(X)$, when $r_3(X \cup \{e\}) = r_3(X \cup \{f\}) = r_3(X)$.

First, observe that the next affirmation follows from the definition of r_3:

(3.2) When $X \in \mathcal{X}$ and $r_3(X) = r_{M_i}(X)$, for $i \in \{1, 2\}$, then $r_3(Y) = r_{M_i}(Y)$ for every Y such that $X \subseteq Y$ or $Y \subseteq X$.

Observe that (1) is trivial. Now, we shall prove (2). Suppose that (2) is not true, that is, there are sets X and Y such that $X \subseteq Y$ and $r_3(X) > r_3(Y)$. By definition, $r_3(X) = r_{M_i}(X)$, for some $i \in \{1, 2\}$. Thus, $r_3(Y) = r_{M_{3-i}}(Y)$. Hence

$$r_{M_{3-i}}(X) \leq r_{M_{3-i}}(Y) = r_3(Y) < r_3(X) = r_{M_i}(X) \leq r_{M_i}(Y).$$

So, X and Y belong to $\mathcal{X}$ and we arrive at a contradiction by **(3.2)**.

Now, we shall stablish (3). When (3) fails, by (2), there is a set X and an element e such that $r_3(X \cup \{e\}) > r_3(X) + 1$. By definition, $r_3(X) = r_{M_i}(X)$, for some $i \in \{1, 2\}$. Thus, $r_3(X \cup \{e\}) = r_{M_{3-i}}(X \cup \{e\})$, $r_3(X) \neq r_{M_{3-i}}(X)$ and $r_3(X \cup \{e\}) \neq r_{M_i}(X \cup \{e\})$, otherwise $r_3(X \cup \{e\}) - r_3(X) \in \{0, 1\}$. Thus, $X, X \cup \{e\} \in \mathcal{X}$. We arrive at a contradiction, by **(3.2)**.

In this paragraph, we shall prove (4). Suppose that (4) is not true, that is, there is a set X and elements e and f such that $r_3(X \cup \{e, f\}) \neq r_3(X)$ and $r_3(X \cup \{e\}) = r_3(X \cup \{f\}) = r_3(X)$. By **(3.2)**, $X \notin \mathcal{X}$ and $X \cup \{e, f\} \notin \mathcal{X}$. So,

$$r_3(X) = r_{M_1}(X) = r_{M_2}(X) \text{ and}$$

$$r_3(X \cup \{e, f\}) = r_{M_1}(X \cup \{e, f\}) = r_{M_2}(X \cup \{e, f\}).$$

Thus, $|cl_{M_i}(X) \cap \{e, f\}| = 1$, for $i \in \{1, 2\}$, and $r_3(X \cup \{e, f\}) = r_3(X) + 1$. Without loss of generality, we may suppose that

$$r_3(X) = r_3(X \cup \{e\}) = r_{M_1}(X \cup \{e\}) = r_{M_2}(X \cup \{e\}) - 1 \text{ and}$$

$$r_3(X) = r_3(X \cup \{f\}) = r_{M_2}(X \cup \{f\}) = r_{M_1}(X \cup \{f\}) - 1.$$

We arrive at a contradiction, since $X \cup \{e\}$ and $X \cup \{f\}$ belong to $\mathcal{X}$ and X and $X \cup \{e, f\}$ does not belong to $\mathcal{X}$. By definition, $(X \cup \{e\})(X \cup \{f\})$ is an edge of G_{M_1,M_2}. ■

We say that $\{M_3, M_4\}$ is a *mix* of $\{M_1, M_2\}$ — a pair of matroids $\{M_1, M_2\}$ can have more than one mix. Observe that $\{M_3, M_4\} = \{M_1, M_2\}$, when $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$. This is always the case when $E \in \mathcal{X}$, since EX is an edge of G_{M_1,M_2}, for every $X \in \mathcal{X} \setminus \{E\}$. By our construction, we have that

(3.3) If $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$, then (M_1, M_2, M_3, M_4) is a connected quad. ■

In the next section, we shall prove that all the quads can be obtained as in the previous proposition.

(3.4) Suppose that $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$, $M_1 = M_2^*$ and $r(M_1) = r(M_2)$. Then, $X \in \mathcal{X}$ if and only if $E \setminus X \in \mathcal{X}$.

Proof: As M_1 and M_2 have the same connectivity function, it follows that

$$r_{M_1}(X) + r_{M_1}(E \setminus X) - r(M_1) = r_{M_2}(X) + r_{M_2}(E \setminus X) - r(M_2).$$

Thus,

$$r_{M_1}(X) - r_{M_2}(X) = r_{M_2}(E \setminus X) - r_{M_1}(E \setminus X)$$

and the result follows. ■

(3.5) Suppose that $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$, $M_1 = M_2^*$ and $r(M_1) = r(M_2)$. Then, $M_3 = M_4^*$ if and only if $|\{A, E \setminus A\} \cap \mathcal{A}| \in \{0, 2\}$, for every $A \subseteq E$.

Proof: This result follows from the equivalence between the next affirmations:

(i) $M_3 = M_4^*$;

(ii) $r_{M_4}(X) = |X| + r_{M_3}(E \setminus X) - r(M_3)$, for every $X \subseteq E$;

(iii) for every $X \subseteq E$, there is an $i \in \{1, 2\}$ such that $r_{M_3}(X) = r_{M_i}(X)$ and $r_{M_3}(E \setminus X) = r_{M_i}(E \setminus X)$;

(iv) $|\{X, E \setminus X\} \cap \mathcal{A}| \in \{0, 2\}$, for every $X \subseteq E$.

The equivalences between (i) and (ii) is immediate and between (iii) and (iv) follows from **(3.4)**.

In this paragraph, we shall prove that (ii) and (iii) are equivalent. For $X \subseteq E$, we have that

$$|X| + r_{M_3}(E \setminus X) - r(M_3) = |X| + r_{M_i}(E \setminus X) - r(M_i) = r_{M_{3-i}}(X),$$

for some $i \in \{1,2\}$, where the last identity occurs because $M_1 = M_2^*$. Thus, by **(3.4)**, $r_{M_3}(X) = r_{M_i}(X)$ if and only if $r_{M_4}(X) = r_{M_{3-i}}(X)$. Hence, we have the equivalence between (ii) and (iii). ■

We say that $\{M_3, M_4\}$ is a *s-mix* of $\{M_1, M_2\}$, when $M_1 = M_2^*, r(M_1) = r(M_2), \mathcal{A} \neq \emptyset, \mathcal{A} \neq \mathcal{X}$ and $A \in \mathcal{A}$ if and only if $E \setminus A \in \mathcal{A}$, for every $A \subseteq \mathcal{X}$. So, by **(3.5)**,

(3.6) If $\{M_3, M_4\}$ is a s-mix of $\{M_1, M_2\}$, then (M_1, M_2, M_3, M_4) is a connected non-trivial s-quad. ■

4 A characterization of connected quads

In this section, we shall present a characterization of the non-trivial connected quads, using the construction of the mix which was defined in the previous section.

(4.1) Suppose that M_i is a matroid having E as its ground set, for every $i \in \{1,2,3,4\}$. (M_1, M_2, M_3, M_4) is a connected quad if and only if $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$.

Proof: By **(3.3)**, the "if" part of this result follows. So, we need to prove only the "only if " part. Suppose that (M_1, M_2, M_3, M_4) is a connected quad. Observe that this quad must be non-trivial, otherwise $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$.

To conclude that $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$, we just need to prove that:

(4.2) If $X, Y \in \mathcal{X}, XY \in E(G_{M_1,M_2})$ and $r_{M_1}(X) = r_{M_j}(X)$, for $j \in \{3,4\}$, then $r_{M_1}(Y) = r_{M_j}(Y)$.

By **(2.2)**, **(4.2)** is true when $X \subseteq Y$ or $Y \subseteq X$. So, we may suppose that $X \setminus Y = \{e\}, Y \setminus X = \{f\}, X \cap Y \notin \mathcal{X}$ and $X \cup Y \notin \mathcal{X}$. When **(4.2)** does not hold, we must have that $r_{M_1}(Y) = r_{M_{7-j}}(Y)$. As $X \cap Y$ and $X \cup Y$ do not belong to $\mathcal{X}$, it follows that

$$r_{M_1}(X \cap Y) = r_{M_2}(X \cap Y) = r \text{ and } r_{M_1}(X \cup Y) = r_{M_2}(X \cup Y) = s,$$

for some integers r and s. Thus,

$$\{r_{M_1}(X), r_{M_2}(X)\} = \{r_{M_1}(Y), r_{M_2}(Y)\} = \{r, r+1\},$$

since $|X \setminus Y| = |Y \setminus X| = 1$ and $X, Y \in \mathcal{X}$. So, $s = r + 1$. Without loss of generality, we may suppose that

$$r_{M_1}(X) = r_{M_2}(Y) = r \text{ and } r_{M_1}(Y) = r_{M_2}(X) = r + 1,$$

otherwise, there is an $i \in \{1,2\}$ such that

$$r_{M_i}(X \cap Y) = r_{M_i}(X) = r_{M_i}(Y) = r_{M_i}(X \cup Y) - 1,$$

which is contrary to the fact that M_i is a matroid.

So, there is an $i \in \{3,4\}$ such that

$$r_{M_i}(X \cap Y) = r_{M_i}(X) = r_{M_i}(Y) = r_{M_i}(X \cup Y) - 1,$$

since $r_{M_1}(X) = r_{M_j}(X)$ and $r_{M_1}(Y) = r_{M_{7-j}}(Y)$. Again, we arrive at a contradiction. ■

As a consequence of **(2.1)**, **(3.5)**, **(3.6)** and **(4.1)**, we have that:

(4.3) Suppose that M_i is a matroid having E as its ground set, for every $i \in \{1,2,3,4\}$. (M_1, M_2, M_3, M_4) is a connected non-trivial s-quad if and only if $\{M_3, M_4\}$ is a s-mix of $\{M_1, M_2\}$. ■

By the previous result, a connected matroid M is reconstructible from its connectivity function if and only if $\{M, M^*\}$ does not have a s-mix.

The explicit construction of all quads looks unlikely, since there are a lot of constructions which produce them, even in the case of s-quads (1-sum of s-quads, 2-sum of a s-quad with a identity self-dual matroid, the truncated lift of a s-quad — replace every entry of the quad by its truncated lift). Moreover, the construction made by Seymour [5] can be extended for a large class of paving matroids, giving non-trivial examples of s-quads, where we can apply all these operations to get more exotic examples.

5 Another way to compute the mix

In this section, we shall discuss the operation of mix as a permutation of circuits or hyperplanes of the two matroids involved in this operation.

Suppose that M_1 and M_2 are matroids on the same ground set E such that $r(M_1) = r(M_2)$ and

$$\xi_{M_1}(X,Y) + \xi_{M_2}(X,Y) \geq 1$$

for every partition $\{X,Y\}$ of E such that $\min\{|X|,|Y|\} \geq 1$.

(5.1) If X is a maximal subset of E such that $r_{M_1}(X) \neq r_{M_2}(X)$, then X is spanning in M_i and a hyperplane of M_{3-i}, for some $i \in \{1,2\}$.

Proof: Without loss of generality, we may suppose that $r_{M_1}(X) > r_{M_2}(X)$. As $r(M_1) = r(M_2)$, it follows that $X \neq E$. By the maximality of X, it follows that $r_{M_1}(X \cup \{e\}) = r_{M_2}(X \cup \{e\})$, for every $e \in E(\mathbf{M}) \setminus X$. Thus, $r_{M_1}(X) = r_{M_2}(X) + 1$, X spans e in M_1 and X does not span e in M_2. So, X is a spanning set of M_1 and a hyperplane of M_2. ■

We may write the dual of this result as:

(5.2) If X is a minimal subset of E such that $r_{M_1}(X) \neq r_{M_2}(X)$, then X is independent in M_i and a circuit of M_{3-i}, for some $i \in \{1, 2\}$. ■

We shall consider the following subsets of $\mathcal{X}$:

$$\begin{aligned} \mathcal{H} &= (\mathcal{H}(M_1) \cap \mathcal{S}(M_2)) \cup (\mathcal{H}(M_2) \cap \mathcal{S}(M_1)) \\ \mathcal{C} &= (\mathcal{C}(M_1) \cap \mathcal{I}(M_2)) \cup (\mathcal{C}(M_2) \cap \mathcal{I}(M_1)) \end{aligned}$$

A sequence $v_0, v_1, v_2, \ldots, v_n$ of vertices of a graph G is said to be a *path* of G, when $v_i v_{i+1} \in E(G)$ for every $i \in \{0, 1, 2, \ldots, n-1\}$. When $v_0, v_1, v_2, \ldots, v_n$ and $w_0, w_1, w_2, \ldots, w_m$ are paths of G such that $v_n = w_0$, we say that

$$v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m$$

is the *concatenation* of these paths.

(5.3) If $X, Y \in \mathcal{C} \cup \mathcal{H}$ belong to the same connected component of G_{M_1,M_2}, then there is a path joining X to Y in G_{M_1,M_2} which is the concatenation of paths of the following types:

(i) C, H, for $C \in \mathcal{C}$ and $H \in \mathcal{H}$;

(ii) H, C, for $C \in \mathcal{C}$ and $H \in \mathcal{H}$;

(iii) H, A, B, H', for $H, H' \in \mathcal{H}$ and $A, B \in \mathcal{X}$ satisfying $A \subseteq H, B \subseteq H', AB \in E(G_{M_1,M_2})$ and $|A \setminus B| = |B \setminus A| = 1$.

Proof: Suppose that

$$X = X_0, X_1, X_2, X_3, \ldots, X_{n-1}, X_n = Y$$

is a path that joins X to Y in G_{M_1,M_2}. We shall insert vertices in this path in the following way:

(i) When $X_i \subset X_{i+1}$ or $X_{i+1} \subset X_i$, then, by **(5.1)**, there is an $H_i \in \mathcal{H}$ such that $X_j \subseteq H_i$, for $j \in \{i, i+1\}$. Insert H_i between X_i and X_{i+1}.

(ii) When H_i and H_{i+1} do not exist, then, by **(5.1)**, there is a $K_{i+1} \in \mathcal{H}$ such that $X_{i+1} \subseteq K_{i+1}$. Insert K_{i+1}, X_{i+1}, in this order, between X_{i+1} and X_{i+2}.

(iii) When H_0 does not exist, then, by **(5.1)**, there is a $K_0 \in \mathcal{H}$ such that $X_0 \subseteq K_0$. Insert X_0, K_0, in this order, in the begining of the path.

(iv) When H_{n-1} does not exist, then, by **(5.1)**, there is a $K_n \in \mathcal{H}$ such that $X_n \subseteq K_n$. Insert K_n, X_n, in this order, in the end of the path.

When H_{i-1} and H_i exist, then, by **(5.2)**, there is a $C_i \in \mathcal{C}$ such that $C_i \subseteq X_i$. Replace X_i by C_i. Oberve that the path at the end of these constructions has the desired properties — when H_0 exists and X_0 is a hyperplane, then $X_0 = H_0$ and we ignore X_0; we may need to ignore X_n also. ∎

Now, we shall define two new sets:

$$\mathcal{H}' = \mathcal{H}(M_1) \triangle \mathcal{H}(M_2) \text{ and } \mathcal{C}' = \mathcal{C}(M_1) \triangle \mathcal{C}(M_2).$$

Observe that $\mathcal{C} \subseteq \mathcal{C}'$ and that $\mathcal{H} \subseteq \mathcal{H}'$. By **(5.1)** and **(5.2)**, when $C' \in \mathcal{C}' \setminus \mathcal{C}$ and $H' \in \mathcal{H}' \setminus \mathcal{H}$, then there is a $C \in \mathcal{C}$ such that $C \subset C'$ and a $H \in \mathcal{H}$ such that $H' \subset H$.

Let H_{M_1,M_2} be a graph whose vertex set is equal to $\mathcal{C}' \cup \mathcal{H}'$ and XY is an edge of this graph when

(i) $X, Y \in \mathcal{C}'$ and $X \subset Y$;

(ii) $X, Y \in \mathcal{H}'$ and $X \subset Y$;

(iii) $X \in \mathcal{C}, Y \in \mathcal{H}$ and $X \subseteq Y$;

(iv) $X, Y \in \mathcal{H}, M_1|(X \cap Y) = M_2|(X \cap Y)$ and there is $e \in X \setminus Y$ and $f \in Y \setminus X$ such that $\{e, f\} \subseteq cl_{M_1}(X \cap Y) \triangle cl_{M_2}(X \cap Y)$.

(5.4) If $X \in V(H_{M_1,M_2}) \setminus V(G_{M_1,M_2})$, then $d_{H_{M_1,M_2}}(X) = 1$.

Proof: As $X \notin \mathcal{X}$, it follows that $X \in (\mathcal{C}' \setminus \mathcal{C}) \cup (\mathcal{H}' \setminus \mathcal{H})$ and $r_{M_1}(X) = r_{M_2}(X)$. When $X \in \mathcal{C}(M_i)$, for some $i \in \{1, 2\}$, then X contains just one circuit of $\mathcal{C}(M_{3-i})$, since X has the same rank in both M_1 and M_2. Hence $d_{H_{M_1,M_2}}(X) = 1$. A similar argument holds when $X \in \mathcal{H}(M_i)$, for some $i \in \{1, 2\}$. ∎

(5.5) Suppose that X and Y belong to $V(H_{M_1,M_2}) \cap V(G_{M_1,M_2})$. X and Y belong to the same connected component of H_{M_1,M_2} if and only if X and Y belong to the same connected component of G_{M_1,M_2}.

Proof: First, we shall prove the "only if" part of this result. Suppose that

$$X = X_0, X_1, X_2, X_3, \ldots, X_{n-1}, X_n = Y$$

is a simple path which joins X and Y in H_{M_1,M_2}. By **(5.4)**, it follows that $X_i \in V(G_{H_1,H_2})$ for every i. If $X_i X_{i+1}$ is not an edge of G_{H_1,H_2}, then $X_i, X_{i+1} \in \mathcal{H}, M_1|(X_i \cap X_{i+1}) = M_2|(X_i \cap X_{i+1})$ and there are $e \in X_i \setminus X_{i+1}$ and $f \in X_{i+1} \setminus X_i$ such that $\{e, f\} \subseteq cl_{M_1}(X_i \cap X_{i+1}) \triangle cl_{M_2}(X_i \cap X_{i+1})$. So, $((X_i \cap$

$X_{i+1}) \cup \{e\})((X_i \cap X_{i+1}) \cup \{f\})$ is an edge of G_{M_1,M_2}. When this happens, we insert between X_i and X_{i+1} the vertices $(X_i \cap X_{i+1}) \cup \{e\}, (X_i \cap X_{i+1}) \cup \{f\}$ in this order. Repeat this operation for every i in this situation. The final path is a paht in G_{M_1,M_2} which joins X to Y.

Now, we shall prove the "if" part of this result. Suppose that

$$X = X_0, X_1, X_2, X_3, \ldots, X_{n-1}, X_n = Y$$

is a simple path which joins X and Y in G_{M_1,M_2}. By **(5.4)**, we may choose this path such that it is a concanation of paths of the following types: (i) C, H, for $C \in \mathcal{C}$ and $H \in \mathcal{H}$; (ii) H, C, for $C \in \mathcal{C}$ and $H \in \mathcal{H}$; (iii) H, A, B, H', for $H, H' \in \mathcal{H}$ and $A, B \in \mathcal{X}$ satisfying $|A \setminus B| = |B \setminus A| = 1$. Observe that (i) and (ii) are paths of H_{M_1,M_2}. So, we shall replace every path of type (iii) by H, H', when $M_1|(H \cap H') = M_3|(H \cap H')$, or by H, X, H', for some $X \in \mathcal{C}$ such that $X \subseteq H \cap H'$. At the end, we get a path from H_{M_1,M_2} which joins X to Y. ■

As a consequence of **(5.5)**, we have the following:

(5.6) $\{M_3, M_4\}$ is a mix of $\{M_1, M_2\}$ if and only if there is a partition $\{\mathcal{A}, \mathcal{B}\}$ of $V(H_{M_1,M_2})$ such that $AB \notin E(H_{M_1,M_2})$, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and

$\mathcal{C}(M_3) = (\mathcal{C}(M_1) \setminus \mathcal{A}) \cup (\mathcal{C}(M_2) \cap \mathcal{B})$ and $C(M_4) = (\mathcal{C}(M_1) \cap \mathcal{A}) \cup (\mathcal{C}(M_2) \setminus \mathcal{B})$. ■

A similar result holds for hyperplanes. Now we shall prove theorem 2.

Proof of theorem 2: By **(5.5)** and **(4.3)**, a s-mix is determineted by a partition $\{\mathcal{A}, \mathcal{B}\}$ of the vertex set of H_{M,M^*} such that $\mathcal{A} \neq \emptyset, \mathcal{B} \neq \emptyset$ and $AB \notin E(H^s_{M,M^*})$, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, where H^s_{M,M^*} is the graph obtained from H_{M,M^*} by adding the edges XY, for every $X \in V(H_{M,M^*})$ and $Y = E(M) \setminus X$.

Now, we shall consider the following graph:

$$G_1 = H^s_{M,M^*}/(E(H^s_{M,M^*}) \setminus E(H_{M,M^*})).$$

Observe that we can label naturaly the vertices of G_1 using the elements of $\mathcal{C}'$, since every edge that was contracted joins a vertex belonging to $\mathcal{C}'$ to its complement which belongs to $\mathcal{H}'$. Moreover, there is a natural bijection between the connected components of G_1 and H^s_{M,M^*}.

As $\mathcal{A} \cap V(G_1)$ is a feaseble set, it follows that it must be admissible and (**A2**) must hold. So, the graph G_2 obtained from G_1 by adding all the edges XY such that $X, Y \in \mathcal{C}'$ and $|X \cap Y| = 1$ has the same number of connected components as G_1.

Now, we shall prove that the graph G_3 obtained from G_2 by deleting all the edges XY of H_{M,M^*} such that $X, Y \in \mathcal{H}, M|(X \cap Y) = M^*|(X \cap Y)$ and there is $e \in X \setminus Y$ and $f \in Y \setminus X$ such that $\{e, f\} \subseteq cl_M(X \cap Y) \bigtriangleup cl_{M^*}(X \cap Y)$ has the same number of connected components of G_2.

Without loss of generality, we may suppose that X is a hyperplane of M. Observe that $f \in cl_{M^*}(X \cap Y)$, otherwise there is a circuit C_1 of M such that $f \in C_1 \subseteq (X \cap Y) \cup \{f\}$ and $C_1 \cap (E(M) \setminus X) = \{f\}$. So, there is a cocircuit D of M such that $f \in D \subseteq (X \cap Y) \cup \{f\}$. Hence, by hypothesis, there is a circuit C of M such that $e \in C \subseteq (X \cap Y) \cup \{e\}$. Thus, by orthogonality, Y is a hyperplane of M^*.

We shall prove that $C, D \in \mathcal{C}$. Suppose that $C \notin \mathcal{C}$ or $D \notin \mathcal{C}$, say $C \notin \mathcal{C}$. As $M|(X \cap Y) = M^*|(X \cap Y)$, it follows that C is a cocircuit of M or C contains a cocircuit D_1 of M such that $e \in D_1$. In both cases, we arrive at a contradiction, by orthogonality, since $(E(M) \setminus Y) \cap C = \{e\}$.

By orthogonality, we have that $C' = E(M) \setminus Y$ and $D' = E(M) \setminus X$ both belong to $\mathcal{C}'$. Let C'' and D'' be elements of $\mathcal{C}$ such that $C'' \subseteq C'$ and $D'' \subseteq D'$. Observe that $C''C'$, when $C'' \subset C'$, $D''D'$, when $D'' \subset D'$, $C''D$, $D''C$, $C'C$ and $D'D$ are edges of G_2. As XY is the edge $C'D'$ of G_2, we do not increase the number of connected components of G_2 when we delete it.

Now, we shall describe the edges of G_3: (i) XY such that $X \in \mathcal{C}, Y \in \mathcal{C}'$ and $X \subset Y$; (ii) XY such that $X, Y \in \mathcal{C}$ and $X \cap Y = \emptyset$; and (iii) XY such that $X, Y \in \mathcal{C}'$ and $|X \cap Y| = 1$. So, theorem 2 follows. $\blacksquare$

6 References

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