On the connectivity function of a binary matroid*

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Abstract

In this paper, we shall consider the following problem: up to duality, is a connected matroid reconstructible from its connectivity function? Cunningham conjectured that this question has an affirmative answer, but Seymour gave a counter-example for it. In the same paper, Seymour proved that a connected binary matroid is always reconstructible from its connectivity function, up to duality, in the class of binary matroids. In this paper, we prove that when M is a connected binary matroid and N another matroid with the same connectivity function as N, then $M = N$ or $M = N^*$.

Also, we shall discuss the implications of ours results on the following problem: when it exists, is the decomposition of the rank function of a 2-polimatroid as the sum of the rank functions of matroids unique?

1 Introduction

The matroid theory terminology used in this paper will follow Oxley [7]. The *connectivity function* of a matroid M is defined as

$$\xi_M(X,Y) = r_M(X) + r_M(Y) - r(M) + 1,$$

where $\{X, Y\}$ is a partition of $E(M)$. This function is invariant under duality, since it may be written as

$$\xi_M(X,Y) = r_M(X) + r_{M^*}(X) - |X| + 1.$$

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In particular, a matroid and its dual have the same connectivity function. When $M = M_1 \oplus M_2$, then

$$\xi_M(X,Y) + 1 = \xi_{M_1}(X \cap E(M_1), Y \cap E(M_1)) + \xi_{M_2}(X \cap E(M_2), Y \cap E(M_2)).$$

Thus, $M_1 \oplus M_2$ and $M_1 \oplus M_2^*$ have the same connectivity function.

In this paper, we shall study matroids with the same connectivity function. By the last identity, we may restrict our attention to connected matroids. In this case, Cunningham conjectured that, up to duality, a connected matroid is reconstructed from its connectivity function, that is:

Cunningham conjecture. If M and N are connected matroids on the same ground set having the same connectivity function, then $M = N$ or $M = N^*$.

In [11], Seymour proved the Cunningham conjecture for the class of binary matroids, that is:

Theorem 1. (3.3 of Seymour[11]) If M and N are connected binary matroids on the same ground set having the same connectivity function, then $N = M$ or $N = M^*$. ■

In the same paper, Seymour gave a counter-example for it in general. In [1], Lemos proved that this conjecture is true provided that $r(M) \neq r(M^*)$. In this paper, we shall prove that a connected binary matroid is reconstructible, not only in the class of binary matroids, but in the class of all matroids. Our main result is:

Theorem 2. Suppose that M is a connected binary matroid. If N is a matroid having the same connectivity function as M and $E(N) = E(M)$, then $N = M$ or $N = M^*$.

Note that

$$r_M(X) + r_{M^*}(X) = r_N(X) + r_{N^*}(X),$$

for every $X \subseteq E$, when M and N are matroids over the same ground set E having the same connectivity function. In general, we say that $\mathbf{M} = (M_1, M_2, M_3, M_4)$ is a *quad* when M_1, M_2, M_3 and M_4 are matroids over the same ground set, which is denoted by $E(\mathbf{M})$, such that

$$r_{M_1}(X) + r_{M_2}(X) = r_{M_3}(X) + r_{M_4}(X),$$

for every $X \subseteq E(\mathbf{M})$.

As one can guess from the definition of quads, this problem is related to the following problem which was proposed by Murty and Simon [6]: characterize polimatroid rank functions which are sums of rank functions of matroids? We also may ask: when this decomposition exists is it unique? It follows from ours results that a connected 2-polimatroid that is decomposed as the sum of rank functions of two binary matroids M_1 and M_2 cannot have another decomposition as the sum of rank functions of two other matroids M_3 and M_4, unless (M_1, M_2, M_3, M_4) is a well described quad. Moreover, when the decomposition is not unique, then there is only two ways of doing it. In [1], Lemos proved that the decomposition of the rank function ρ of a connected 2-polimatroid, defined on E, as the sum of the rank function of two matroids is unique, when it exists, provided that $\rho(E)$ is odd. We shall settle the problem of uniqueness of decomposition of the rank function of a connected 2-polimatroid in another paper, using the theory developed here.

We are setting a different notation from Lemos [1] to get a more compact description of the results. When $\mathbf{X}$ denotes a quad, X_i denotes the i-th coordinate of this quad, for $i \in \{1, 2, 3, 4\}$.

We say that two quads $\mathbf{M}$ and $\mathbf{N}$ are *equivalent*, and denote this by $\mathbf{M} \equiv \mathbf{N}$, when

$$\{\{M_1, M_2\}, \{M_3, M_4\}\} = \{\{N_1, N_2\}, \{N_3, N_4\}\}.$$

We say that two quads $\mathbf{M}$ and $\mathbf{N}$ are *isomorphic*, and denote this by $\mathbf{M} \simeq \mathbf{N}$, when there is a quad $\mathbf{H}$ such that $\mathbf{H} \equiv \mathbf{M}$ and a bijection $f : E(\mathbf{H}) \to E(\mathbf{N})$ such that f is an isomorphism between H_i and N_i, for every $i \in \{1, 2, 3, 4\}$.

For a quad $\mathbf{M}$ and disjoint subsets X and Y of $E(\mathbf{M})$, we define a *minor* of $\mathbf{M}$ as

$$\mathbf{M}\backslash X/Y = (M_1\backslash X/Y, M_2\backslash X/Y, M_3\backslash X/Y, M_4\backslash X/Y)$$

and the *dual* of $\mathbf{M}$ as $\mathbf{M}^* = (M_1^*, M_2^*, M_3^*, M_4^*)$. We shall use $\mathbf{M}|X$ to denote $\mathbf{M}\backslash(E(\mathbf{M}) \setminus X)$. In [1], Lemos have proved that

(1.1) (1.1, 1.2 and 1.3 of [1]) $\mathbf{M}\backslash X/Y$ and $\mathbf{M}^*$ are quads. ■

When $\mathbf{M}$ and $\mathbf{N}$ are quads such that $E(\mathbf{M}) \cap E(\mathbf{N}) = \emptyset$, we denote by $\mathbf{M} \oplus \mathbf{N}$ the quad

$$(M_1 \oplus N_1, M_2 \oplus N_2, M_3 \oplus N_3, M_4 \oplus N_4).$$

Case $N_1 = N_2 = N_3 = N_4 = N$, $\mathbf{M} \oplus \mathbf{N}$ is also denoted by $\mathbf{M} \oplus N$. For a matroid N such that $E(N) \cap E(\mathbf{M}) = \{e\}$, $\mathbf{M} \oplus_2 N$ denotes

$$(M_1 \oplus_2 N, M_2 \oplus_2 N, M_3 \oplus_2 N, M_4 \oplus_2 N),$$

which was proved to be a quad by Lemos in [1].

In [1], Lemos proved the next result which characterizes a quad and it was proved by Seymour for a class of quads, in [11]:

(1.2) (1.4 and 1.5 of [1]) Suppose that M_i is a matroid having E as its ground set, for every $i \in \{1, 2, 3, 4\}$. (M_1, M_2, M_3, M_4) is a quad if and only if

$$\mathcal{C}(M_1) \cup \mathcal{C}(M_2) = \mathcal{C}(M_3) \cup \mathcal{C}(M_4) \text{ and } \mathcal{C}(M_1) \cap \mathcal{C}(M_2) = \mathcal{C}(M_3) \cap \mathcal{C}(M_4). \quad ■$$

Now, we outline the proof of theorem 2: when M and N satisfies the hypothesis of theorem 2 and its conclusion does not hold, then (M, M^*, N, N^*)

belongs to the class $\mathcal{S}$ of quads $\mathbf{M}$ satisfying: $M_1 \neq M_3$, $M_1 \neq M_4$ and both M_1 and M_2 binary matroids. The "irreducible" quads of $\mathcal{S}$ shall be constructed in section four and all of them have exactly one disconnected matroid as one of its coordinates. So, none of them can be equivalent to (M, M^*, N, N^*). The construction of the "irreducible" quads of $\mathcal{S}$ shall be done by induction and the splitter-type theorem for quads is fundamental in this construction — its statement and proof can be found in section three: essentially, it says that an "irreducible" quad can be obtained from a well described "irreducible" quad by a sequence of one element lift or extension without leaving the class of "irreducible" quads.

2 Connectivity results for quads

In this section, we shall prove some connectivity results for quads which will be fundamental in the proof of the splitter-type theorem for quads which shall be presented in the next section. Here, the notions of connectivity and 3-connectivity for matroids are extended for quads.

Let $\mathbf{M}$ be a quad. For a partition $\{X, Y\}$ of $E(\mathbf{M})$, we denote by

$$\xi_{\mathbf{M}}(X, Y) = \xi_{M_1}(X, Y) + \xi_{M_2}(X, Y) = \xi_{M_3}(X, Y) + \xi_{M_4}(X, Y).$$

When $\xi_{\mathbf{M}}(X, Y) = 2$, then $\mathbf{M} = (\mathbf{M}|X) \oplus (\mathbf{M}|Y)$. So, we can restrict our attention to quads satisfying $\xi_{\mathbf{M}}(X, Y) \geq 3$, for every partition $\{X, Y\}$ of $E(\mathbf{M})$ such that $\min\{|X|, |Y|\} \geq 1$. Such quads are called *connected*. A quad $\mathbf{M}$ is said to be *non-trivial*, when $\{M_1, M_2\} \neq \{M_3, M_4\}$. In [1], Lemos proved the following result:

(2.1) Suppose that $\mathbf{M}$ is a non-trivial connected quad. Then,

(i) (2.6 of [1]) $\xi_{\mathbf{M}}(X, Y) \geq 4$, for every partition $\{X, Y\}$ of $\mathbf{M}$ such that $\min\{|X|, |Y|\} \geq 1$.

(ii) (2.7 of [1]) When $\xi_{M_1}(X, Y) = \xi_{M_2}(X, Y) = \xi_{M_3}(X, Y) = \xi_{M_4}(X, Y) = 2$, for a partition $\{X, Y\}$ of $E(\mathbf{M})$ such that $\min\{|X|, |Y|\} \geq 2$, then there is a matroid N such that $E(N) = X \cup \{e\}$ or $E(N) = Y \cup \{e\}$, for an element $e \notin E(\mathbf{M})$, and a quad $\mathbf{N}$ such that $E(\mathbf{N}) = (E(\mathbf{M}) \setminus E(N)) \cup \{e\}$ satisfying $\mathbf{M} = \mathbf{N} \oplus_2 N$. ■

As a consequence of **(2.1)**(i), we have the following:

(2.2) If $\mathbf{M}$ is a non-trivial connected quad, then M_i is both loopless and coloopless, for every $i \in \{1, 2, 3, 4\}$. ■

This is a particular instance of a more general result of Lemos [1]:

(2.3) (2.3 of [1]) Let $\mathbf{M}$ be a connected quad. If $\{X, Y\}$ is a 1-separation for both M_1 and M_3 such that $M_1|X = M_3|X$, then $\mathbf{M}$ is trivial. $\blacksquare$

(2.4) (theorem 2 of [1]) If $\mathbf{M}$ is a non-trivial connected quad, then

$$r(M_1) = r(M_2) = r(M_3) = r(M_4). \quad \blacksquare$$

By this theorem, for a non-trivial connected quad $\mathbf{M}$, we can define its *rank*, which is denoted by $r(\mathbf{M})$, as the rank of M_i, for $i \in \{1, 2, 3, 4\}$.

We say that $\{X, Y\}$ is a *k-separation* for a quad $\mathbf{M}$, when $\{X, Y\}$ is an exact k-separation for M_i, for every $i \in \{1, 2, 3, 4\}$. By **(2.1)**(ii), when, $\{X, Y\}$ is a 2-separation for a non-trivial connected quad $\mathbf{M}$, then $\mathbf{M} = \mathbf{N} \oplus_2 N$, where $\mathbf{N}$ is a non-trivial quad and N is a matroid such that $E(N) = Z \cup \{e\}$ and $e \in E(N) \cap E(\mathbf{N})$, for some $Z \in \{X, Y\}$. We say that Z is the *dead* set associated with the 2-separation $\{X, Y\}$. A quad is said to be *irreducible*, when it does not have a 1-separation or a 2-separation. Observe that the deletion or the contraction of an element of an irreducible quad gives a connected quad.

(2.5) Let $\mathbf{M}$ be a non-trivial connected quad. If $\{X, Y\}$ is a 2-separation for $\mathbf{M}$, then, for every $Z \in \{X, Y\}$, we have that

$$r_{M_1}(Z) = r_{M_2}(Z) = r_{M_3}(Z) = r_{M_4}(Z).$$

Moreover, when $|X| = 2$, X is contained in a series class of M_i, for every $i \in \{1, 2, 3, 4\}$, or X is contained in a parallel class of M_i, for every $i \in \{1, 2, 3, 4\}$.

Proof: Without loss of generality, we may suppose that Y is the dead set of this 2-separation. Thus $r_{M_1}(Y) = r_{M_2}(Y) = r_{M_3}(Y) = r_{M_4}(Y)$. Hence $r_{M_1}(X) = r_{M_2}(X) = r_{M_3}(X) = r_{M_4}(X)$, because $r_{M_i}(X) + r_{M_i}(Y) - r(M_i) = 1$ and $r(M_i) = r(\mathbf{M})$, for every $i \in \{1, 2, 3, 4\}$. Moreover, when $|X| = 2$, X is independent in all these matroids or X is dependent in all these matroids and the result follows. $\blacksquare$

(2.6) If $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ are different 2-separations for a non-trivial connected quad $\mathbf{M}$ such that X_1 and X_2 are maximal dead sets, then $X_1 \cap X_2 = \emptyset$.

Proof: Suppose that $X_1 \cap X_2 \neq \emptyset$. Since both X_1 and X_2 are maximal dead sets, we have that $Y_1 \cap X_2 \neq \emptyset$ and $X_1 \cap Y_2 \neq \emptyset$.

For $i \in \{1, 2\}$, as X_i is a dead set, it follows that $\mathbf{M} = \mathbf{N}_i \oplus_2 N_i$, where $\mathbf{N}_i = (N_{i1}, N_{i2}, N_{i3}, N_{i4})$ is a quad such that $E(\mathbf{N}_i) = Y_i \cup \{e_i\}$ and N_i is a matroid such that $E(N_i) = X_i \cup \{e_i\}$.

First, we shall prove also that $Y_1 \cap Y_2 \neq \emptyset$. Suppose that $Y_1 \cap Y_2 = \emptyset$. In this case, we shall prove that $M_i = M_j$, for every i and j such that $\{i, j\} \subset \{1, 2, 3, 4\}$, and we arrive at a contradiction. To demonstrate this, by symmetry, it is enough to prove that every circuit C of M_i is also a circuit of M_j. We may suppose that $C \cap Y_k \neq \emptyset$, for both $k \in \{1, 2\}$, otherwise C is a circuit of $N_k \backslash e_k$, and hence C would be also a circuit of M_j. For $k \in \{1, 2\}$, it follows that $(C \cap X_k) \cup \{e_k\}$ is a circuit of N_k. Hence, there is a circuit C' of M_j such that $C' \cap X_1 = C \cap X_1$ and $C' \cap Y_1 \neq \emptyset$. As $Y_2 \subseteq X_1$ and $C \cap Y_2 \neq \emptyset$, it follows that $(C' \cap Y_2) \cup \{e_2\} = (C \cap Y_2) \cup \{e_2\}$ is a circuit of N_{2j}. Since $(C \cap X_2) \cup \{e_2\}$ is a circuit of N_2, we have that $C = (C \cap Y_2) \cup (C \cap X_2)$ is a circuit of M_j. So, $Y_1 \cap Y_2 \neq \emptyset$.

For $i \in \{1, 2\}$, we set $\{Z_i, W_i\} = \{X_i, Y_i\}$. Using submodularity, we arrive at the following:

$$\xi_{M_k}(Z_1 \cap Z_2, W_1 \cup W_2) + \xi_{M_k}(Z_1 \cup Z_2, W_1 \cap W_2) \leq \xi_{M_k}(X_1, Y_1) + \xi_{M_K}(X_2, Y_2) = 4,$$

for every $k \in \{1, 2, 3, 4\}$. Thus, for $\mathcal{F} = \{X_1 \cap X_2, X_1 \cap Y_2, Y_1 \cap X_2, Y_2 \cap X_2\}$,

$$\sum_{Z \in \mathcal{F}} \xi_{M_k}(Z, E(\mathbf{M}) \setminus Z) \leq 8. \tag{1}$$

By **(2.1)**(i), for every $\{i, j\}$ equal to $\{1, 2\}$ or $\{3, 4\}$,

$$\xi_{M_i}(Z_1 \cap Z_2, W_1 \cup W_2) + \xi_{M_j}(Z_1 \cap Z_2, W_1 \cup W_2) \geq 4.$$

When, we take the sum over $\mathcal{F}$, we get

$$\sum_{Z \in \mathcal{F}} \xi_{M_i}(Z, E(\mathbf{M}) \setminus Z) + \sum_{Z \in \mathcal{F}} \xi_{M_j}(Z, E(\mathbf{M}) \setminus Z) \geq 16. \tag{2}$$

Hence, we have equalities in **(1)** and **(2)**. So, for every $k \in \{1, 2, 3, 4\}$,

$$\xi_{M_k}(Z_1 \cap Z_2, W_1 \cup W_2) = 1 + s_k \text{ and } \xi_{M_k}(Z_1 \cup Z_2, W_1 \cap W_2) = 3 - s_k, \tag{3}$$

for some $s_k \in \{0, 1, 2\}$.

In this paragraph, we shall prove that $|Z| = 1$ or $\{Z, E(\mathbf{M}) \setminus Z\}$ is a 2-separation for $\mathbf{M}$, for every set $Z \in \mathcal{F}$. Suppose that $|Z| > 1$ and that $\{Z, E(\mathbf{M}) \setminus Z\}$ is not a 2-separation for M_k, say $k = 1$, for some $Z \in \mathcal{F}$. Without loss of generality, by **(3)**, we may suppose that $Z = Z_1 \cap Z_2$ and that $\xi_{M_1}(Z_1 \cap Z_2, W_1 \cup W_2) = 3$. Thus, by **(3)**, $\xi_{M_1}(Z_1 \cup Z_2, W_1 \cap W_2) = 1$. Hence $W_1 \cap W_2 = Y_1 \cap Y_2$, otherwise $\{W_1 \cap W_2, E(N_i) \setminus (W_1 \cap W_2)\}$ is a 1-separation for N_i, where $i \in \{1, 2\}$ satisfies $X_i = W_i$, and $\mathbf{M}$ is not connected. As $\mathbf{M}$ is non-trivial and connected, we have that $\xi_{M_2}(X_1 \cup X_2, Y_1 \cap Y_2) = 3$, by **(2.1)**(i). So, by **(3)**, $\xi_{M_2}(X_1 \cap X_2, Y_1 \cup Y_2) = 1$ and we arrive at a contradiction because $\{X_1 \cap X_2, E(N_i) \setminus (X_1 \cap X_2)\}$ is a 1-separation for N_i for both $i \in \{1, 2\}$.

We can use all the 2-separations found in the previous paragraph to decompose each M_i. So, there is a set $E = \{e_Z : Z \in \mathcal{F}\}$ such that $|E| = 4$ and

$E \cap E(\mathbf{M}) = \emptyset$, and there are matroids K_{iZ} and J_i such that $E(K_{iZ}) = Z \cup \{e_Z\}$, $E(J_i) = E$ and

$$M_i = J_i \oplus_2 K_{iX_1\cap X_2} \oplus_2 K_{iX_1\cap Y_2} \oplus_2 K_{iY_1\cap X_2} \oplus_2 K_{iY_1\cap Y_2},$$

for every $i \in \{1,2,3,4\}$ — when $|Z| = 1$, we take K_{iZ} as the empty matroid and set e_Z equal to the element belonging to Z. Observe that the 2-separations $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ induce 2-separations on J_i. Thus, J_i is a circuit or cocircuit, for every $i \in \{1,2,3,4\}$.

For every $i \in \{1,2,3,4\}$, $J_i = H_i \oplus L_i$, where H_i and L_i are matroids such that $E(H_i) = \{e_1, e_{X_1\cap X_2}, e_{X_1\cap Y_2}\}$ and $E(L_i) = \{e_1, e_{Y_1\cap X_2}, e_{Y_1\cap Y_2}\}$. Observe that

$$N_1 = H_i \oplus_2 K_{iX_1\cap X_2} \oplus_2 K_{iX_1\cap Y_2},$$

for every $i \in \{1,2,3,4\}$. Thus,

$$K_{1X_1\cap Z_2} = K_{2X_1\cap Z_2} = K_{3X_1\cap Z_2} = K_{4X_1\cap Z_2},$$

for every $Z \in \{X, Y\}$, and $H_1 = H_2 = H_3 = H_4$. From the last identity, we have that $J_1 = J_2 = J_3 = J_4$. Similarly, we have that

$$K_{1Z_1\cap X_2} = K_{2Z_1\cap X_2} = K_{3Z_1\cap X_2} = K_{4Z_1\cap X_2},$$

for both $Z \in \{X, Y\}$. Hence $\mathbf{M}$ is equal to

$$(K_{1Y_1\cap Y_2}, K_{2Y_1\cap Y_2}, K_{3Y_1\cap Y_2}, K_{4Y_1\cap Y_2}) \oplus_2 N,$$

where $N = J_i \oplus_2 K_{iX_1\cap X_2} \oplus_2 K_{iX_1\cap Y_2} \oplus_2 K_{iY_1\cap X_2}$, for every $i \in \{1,2,3,4\}$. We arrive at a contradiction because $X_1 \cup X_2$ cannot be a dead set. ■

By **(2.6)**, we can decompose a non-trivial connected quad $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{H} \oplus_2 N_1 \oplus_2 N_2 \oplus_2 \cdots \oplus_2 N_k,$$

where $\mathbf{H}$ is a non-trivial irreducible quad and, for $i \in \{1,2,\ldots,k\}$, N_i is a connected matroid such that $E(N_i) \cap E(\mathbf{H}) = \{e_i\}$, for $e_i \notin E(\mathbf{M})$, $E(N_i) \cap E(N_j) = \emptyset$ and $e_i \neq e_j$, for every $i \neq j$. As this decomposition is unique, up to the labels of the base points of the 2-sums, by **(2.6)**, we say that $\mathbf{H}$ is the *core* of $\mathbf{M}$ and that $N_1, N_2, \ldots, N_k$ are the *leaves* of $\mathbf{M}$. We refer to it as *the canonical decomposition* for $\mathbf{M}$. We shall associate the following function with $\mathbf{M}$:

$$\gamma(\mathbf{M}) = \bigcup_{i=1}^{k} E(N_i) \setminus \{e_i\}.$$

Observe that $\gamma(M) = \emptyset$, when $\mathbf{M}$ is irreducible.

During the rest of this section, we shall study the deletion and contraction of elements belonging to a non-trivial irreducible quad. First, we shall give some definitions. For an element e of a non-trivial irreducible quad $\mathbf{M}$, we define $del(e)$ as the number of leaves of $\mathbf{M}\backslash e$, when $\mathbf{M}\backslash e$ is non-trivial, and as ∞ otherwise. Similarly, we define $con(e)$ as the number of leaves of $\mathbf{M}/e$, when $\mathbf{M}/e$ is non-trivial, and as ∞ otherwise. We also set

$$del(\mathbf{M}) = \min\{del(e) : e \in E(\mathbf{M})\} \text{ and } con(\mathbf{M}) = \min\{con(e) : e \in E(\mathbf{M})\}.$$

When $del(\mathbf{M}) = con(\mathbf{M}) = \infty$, we say that $\mathbf{M}$ is an *almost trivial* quad. Now, we shall present the characterization of the almost trivial quads. To do this, we need to set more notation: when $C_1, C_2, \ldots, C_n$ are different circuit-hyperplanes of a matroid M, we denote by $M_{C_1,C_2,\ldots,C_n}$ the matroid obtained from M after the relaxation of all the circuit-hyperplanes $C_1, C_2, \ldots, C_n$.

Now, we need the following corollary of a result of Mills, that appears in [5]:

(2.7) Let M and N be different matroids both loopless and coloopless defined in the same ground set. If, for every $e \in E(M)$, $M\backslash e = N\backslash e$ or $M/e = N/e$, then one of M and N is a relaxation of the other. ∎

(2.8) If $\mathbf{M}$ is almost-trivial, then there is a matroid M having a pair of circuit-hyperplanes X and Y such that $\{X, Y\}$ is a partition of $E(\mathbf{M})$ and $\mathbf{M} \equiv (M_{X,Y}, M, M_Y, M_X)$. This quad is denoted by $\mathbf{Q}(M, X, Y)$.

Proof: By hypothesis, for every $e \in E(\mathbf{M})$, we have that

$$\{M_1\backslash e, M_2\backslash e\} = \{M_3\backslash e, M_4\backslash e\} \text{ and } \{M_1/e, M_2/e\} = \{M_3/e, M_4/e\}.$$

For $i \in \{1,2\}$ and $j \in \{3,4\}$, when $M_i\backslash e = M_j\backslash e$, then $M_i/e = M_{7-j}/e$, otherwise $M_i/e = M_j/e$ and hence $M_i = M_j$ and $\mathbf{M}$ is trivial, a contradiction. Thus, there is a partition $\{X, Y\}$ of $E(\mathbf{M})$ such that

$$M_1\backslash x = M_3\backslash x, M_1/x = M_4/x, M_1/y = M_3/y \text{ and } M_1\backslash y = M_4\backslash y,$$

for every $x \in X$ and $y \in Y$.

We denote by $\mathcal{C}_{ij}$ the set $\mathcal{C}(M_i) \triangle \mathcal{C}(M_j)$. By **(1.2)** and **(2.7)**, we have that

$$\mathcal{C}_{13} = \mathcal{C}_{24} = \{X\} \cup \{X \cup \{y\} : y \in Y\} \text{ and}$$

$$\mathcal{C}_{14} = \mathcal{C}_{23} = \{Y\} \cup \{Y \cup \{x\} : x \in X\}.$$

To conclude the proof of this lemma, we observe that there is an $i \in \{1, 2, 3, 4\}$, say $i = 2$, such that X and Y are both circuit-hyperplanes of M_2, which is taken to be M. Thus, $M_1 = M_{X,Y}$. Note that both X and Y cannot be a circuit of M_j, for $j \in \{3, 4\}$, otherwise $M_2 = M_j$ and $\mathbf{M}$ is trivial. Hence, by **(1.2)**, we may suppose that X is a circuit of M_3 and Y is a circuit of M_4. So, $M_3 = M_Y$ and $M_4 = M_X$. ∎

(2.9) If $\mathbf{M}$ is a non-trivial connected quad, then $\mathbf{M}$ has an almost-trivial quad as a minor.

Proof: Suppose that this corollary is not true. Let $\mathbf{M}$ be a minimal counter-example for it. Observe that $\mathbf{M}$ is irreducible, otherwise, by the choice of $\mathbf{M}$, the core $\mathbf{N}$ of $\mathbf{M}$ has an almost-trivial quad as a minor which must be a minor of $\mathbf{M}$ as well. By its choice, $\mathbf{M}$ is not almost-trivial, and there is an element e of $E(\mathbf{M})$ such that $\mathbf{M}\backslash e$ or $\mathbf{M}/e$ is non-trivial, say $\mathbf{M}\backslash e$. As $\mathbf{M}\backslash e$ is connected because $\mathbf{M}$ is irreducible, it follows that $\mathbf{M}\backslash e$ has an almost-trivial quad as a minor. We arrive at a contradiction. ∎

We say that a quad $\mathbf{M}$ is *binary*, when, for every $i \in \{1,2,3,4\}$, M_i is a binary matroid.

(2.10) There exists no non-trivial connected binary quad.

Proof: By **(2.9)**, every non-trivial connected binary quad must have an almost-trivial quad $\mathbf{N}$ as a minor. Thus, $\mathbf{N}$ must be binary. By **(2.8)**, $\mathbf{N} \equiv \mathbf{Q}(N; X, Y)$, for some matroid N. As N_X is a binary matroid, it follows that $|X| = |Y| = 2$, otherwise $N_X \backslash Y'/X'$ is isomorphic to $U_{2,4}$, for some $X' \subset X$ and $Y' \subset Y$ such that $|X'| = |Y'| = r(N) - 2$. Thus, $N = N|X \oplus N|Y$ and $N_{X,Y}$ must be isomorphic to $U_{2,4}$. So, $\mathbf{N}$ cannot exist and the result follows. ∎

Seymour's theorem that was stated in the introduction is a consequence of this result, since (M, M^*, N, N^*) must be a non-trivial binary connected quad, when M and N are connected binary matroids having the same connectivity function such that $M \neq N$ and $M \neq N^*$.

We say that a quad $\mathbf{M}$ is *semi-binary*, when M_1 and M_2 are binary matroids or M_3 and M_4 are binary matroids. Thus, the proof of **(2.10)** can be easily adapted to prove the following result:

(2.11) A semi-binary almost-trivial quad has four elements. ∎

Our goal is to construct all the semi-binary irreducible non-trivial quads, to conclude that they cannot be equivalent to the core of (M, M^*, N, N^*), for a connected binary matroid M having the same connectivity function as N. This shall be done in the last section of this paper.

Now, we shall prove a result that is not needed in this paper, but is fundamental to study the uniqueness of the decomposition of the rank function of a 2-polimatroid as the sum of rank functions of matroids. We shall state and prove it here for future reference.

(2.12) If $\mathbf{M}$ is a non-trivial connected quad, then there is at most one $i \in \{1,2,3,4\}$ such that M_i is not connected. Moreover, if M_i is not connected, then it has two connected components.

Proof: Suppose that this result is not true and choose a minimal counterexample $\mathbf{M}$. Clearly $\mathbf{M}$ is irreducible. By **(2.8)**, $\mathbf{M}$ cannot be an almost-trivial quad. So, $\mathbf{M}$ has an element e such that $\mathbf{M}\backslash e$ or $\mathbf{M}/e$ is non-trivial, say $\mathbf{M}\backslash e$. As $\mathbf{M}$ is irreducible, it follows that $\mathbf{M}\backslash e$ is connected. Thus, by the choice of $\mathbf{M}$, there is at most one i such that $M_i\backslash e$ is not connected. So, e must be a loop or coloop of some M_j, for $j \neq i$. But this is contrary to **(2.2)**. ■

3 A splitter-type theorem for quads

In this section, we shall prove a splitter-type theorem for irreducible quads. It says that it is possible to get a non-trivial irreducible quad from an almost-trivial quad or a wheel-whirl quad, which is going to be defined below, by a sequence of one elements lifts or extensions without leaving the class of irreducible quads. This theorem plays a central hole in the construction of all semi-binary irreducible quads, since permits an inductive proof.

Let N be a matroid having a circuit-hyperplane C and let U be a disconnected matroid defined on the same ground set as N such that $U|C \simeq U_{|C|-1,|C|}$ and $U|(E(N) \setminus C) \simeq U_{1,|E(N)\setminus C|}$. By **(1.2)**, (N, U_C, N_C, U) is a quad, which is denoted by $\mathbf{Q}(N;C)$. When $N \simeq M(W_n)$, for $n \geq 2$, $\mathbf{Q}(N;C)$ is denoted by $\mathbf{W}_n$ and is called the *wheel-whirl* quad of rank n. Now, we state the main result of this section.

(3.1) If $\mathbf{M}$ is a non-trivial irreducible quad, then

(i) there is $e \in E(\mathbf{M})$ such that $\mathbf{M}\backslash e$ is non-trivial and irreducible; or

(ii) there is $e \in E(\mathbf{M})$ such that $\mathbf{M}/e$ is non-trivial and irreducible; or

(iii) $\mathbf{M}$ is an almost-trivial quad; or

(iv) $\mathbf{M}$ is a wheel-whirl quad.

Before the proof of **(3.1)**, we shall give the definition of a chain of triangles and triads of a matroid with a small modification from the one which appears in Oxley [8]: for a matroid M and integer $m \geq 4$, the sequence $z_1, z_2, \ldots, z_m, z_{m+1}, z_{m+2}$ of elements of M is an *admissible sequence* of M, when (i) $z_1, z_2, \ldots, z_m$ are all distinct; (ii) for $i \in \{1, 2, \ldots, m\}$, $Z_i = \{z_i, z_{i+1}, z_{i+2}\}$ is a triangle when i is odd and a triad when i is even. We have the following theorem:

(3.2) Suppose that $z_1, z_2, \ldots, z_m, z_{m+1}, z_{m+2}$ is an admissible sequence of a matroid M.

(i) (6.1 of Seymour [10]) When M is connected, $z_{m+1} = z_1$ and $z_{m+2} = z_2$, then $m = 2n$ and M is isomorphic to $\mathcal{W}_n$ or $M(W_n)$.

(ii) (see Tutte [12] or Oxley and Reid [9]) When M is 3-connected and $z_{m+2} = z_1$, then $m = 2n-1$ and M is isomorphic to a $\mathcal{W}_n$ or $M(W_n)$. ∎

We say that $Z_1, Z_2, \ldots, Z_m$ is a *chain* of M. This chain is said to be *closed*, when $Z_1 \cap Z_m \neq \emptyset$. In this case, we have (i) or (ii) of **(3.2)**.

Proof of (3.1): We may suppose that $0 < \min\{del(\mathbf{M}), con(\mathbf{M})\} < \infty$, otherwise the result follows: (a) when $del(\mathbf{M}) = 0$, we have (i); (b) when $con(\mathbf{M}) = 0$, we have (ii); (c) when $del(\mathbf{M}) = con(\mathbf{M}) = \infty$, then $\mathbf{M}$ is almost-trivial and we have (iii).

As the classes of almost-trivial quads and wheel-whirl quads are both closed under duality, we can replace $\mathbf{M}$ by $\mathbf{M}^*$, when necessary, and we may suppose that

$$\gamma = \min_{e \in E(\mathbf{M})} \{|\gamma(\mathbf{M}\backslash e)| : del(e) < \infty\} \leq \min_{e \in E(\mathbf{M})} \{|\gamma(\mathbf{M}/e)| : con(e) < \infty\}.$$

Choose an element e belonging to $\{f \in E(\mathbf{M}) : \gamma = |\gamma(\mathbf{M}\backslash f)|\}$ such that $del(e)$ is maximum. So, for $k = del(e)$,

$$\mathbf{M}\backslash e = \mathbf{H} \oplus_2 N_1 \oplus_2 N_2 \oplus_2 \cdots \oplus_2 N_k,$$

where $\mathbf{H}$ is a non-trivial and irreducible quad and N_i is a connected matroid, for each $i \in \{1, 2, \ldots, k\}$. Moreover, let e_i be the element belonging to both $E(\mathbf{H})$ and $E(N_i)$, for every $i \in \{1, 2, \ldots, k\}$. We also set $E_i = E(N_i) \setminus \{e_i\}$, for every $i \in \{1, 2 \ldots, k\}$.

(3.3) For every $i \in \{1, 2, \ldots, k\}$ and $f \in E_i$, $N_i\backslash f$ is not connected.

Suppose that $N_i\backslash f$ is connected, for some $f \in E_i$ and $i \in \{1, 2, \ldots, k\}$, say $i = k$. So,

$$\mathbf{M}\backslash\{e, f\} = \mathbf{H} \oplus_2 N_1 \oplus_2 N_2 \oplus_2 \cdots \oplus_2 N_{k-1} \oplus_2 N_k\backslash f$$

is the canonical decomposition of $\mathbf{M}\backslash\{e, f\}$. Hence $\mathbf{M}\backslash\{e, f\}$ is non-trivial and connected. Thus, $\mathbf{M}\backslash f$ is also non-trivial and connected.

By the decomposition of $\mathbf{M}\backslash\{e, f\}$, we have that: if $\{X, Y\}$ is a 2-separation for $\mathbf{M}\backslash\{e, f\}$, then $X \subset E_k$ or $Y \subset E_k$ or $E_k \setminus \{f\} \subseteq X$ or $E_k \setminus \{f\} \subseteq Y$.

In this paragraph, we shall prove that the dead sets of $\mathbf{M}\backslash f$ are contained in $(E_k \setminus \{f\}) \cup \{e\}$. Suppose that $\{X \cup \{e\}, Y\}$ is a 2-separation for $\mathbf{M}\backslash f$. First, we shall prove that $|X| \geq 2$. Suppose that $|X| = 1$, say $X = \{x\}$. By **(2.5)**, we have two cases to deal with: (a) $\{e, x\}$ is contained in a parallel class of $M_i\backslash f$, for every $i \in \{1, 2, 3, 4\}$. In this case, we arrive at a contradiction, because $\{\{e, x\}, E(\mathbf{M}) \setminus \{e, x\}\}$ is a 2-separation for $\mathbf{M}$; or (b) $\{e, x\}$ is contained in a series class of $M_i\backslash f$, for every $i \in \{1, 2, 3, 4\}$. Hence, x is a coloop of $M_i\backslash\{e, f\}$ and we arrive at a contradiction, since $\mathbf{M}\backslash\{e, f\}$ is connected. Thus, $|X| \geq 2$ and $\{X, Y\}$ is a 2-separation for $\mathbf{M}\backslash\{e, f\}$. Note that $E_k \setminus \{f\} \not\subseteq Y$, otherwise

$\{X \cup \{e\}, Y \cup \{f\}\}$ is a 2-separation for $\mathbf{M}$ because $E_k \setminus \{f\}$ spans f in M_i, for every $i \in \{1,2,3,4\}$. Similarly, we have that $E_k \setminus \{f\} \not\subseteq X$. So, $X \subset E_k$ or $Y \subset E_k$, by the previous paragraph. Hence the dead set of this 2-separation must be contained in $(E_k \setminus \{f\}) \cup \{e\}$.

So, by the choice of e,

$$|\gamma(\mathbf{M}\backslash e)| \leq |\gamma(\mathbf{M}\backslash f)| \leq |E_k| = |\gamma(\mathbf{M}\backslash e)| - \sum_{i=1}^{k-1} |E_i|.$$

Thus, $k = 1$ and $|\gamma(\mathbf{M}\backslash e)| = |\gamma(\mathbf{M}\backslash f)|$. Hence, by the choice of e, $\mathbf{M}\backslash f$ has just one leave, which must have as its ground set $Z = (E_1 \setminus \{f\}) \cup \{e\}$. We arrive at a contradiction because $\{Z, E(\mathbf{M}) \setminus (Z \cup \{f\})\}$ is a 2-separation for $\mathbf{M} \setminus f$ and Z spans f in M_i, for every $i \in \{1,2,3,4\}$.

(3.4) For every $i \in \{1,2,3,4\}$ and $j \in \{1,2,\ldots,k\}$, $cl_{M_i}(E(\mathbf{M}) \setminus (E_j \cup \{e\})) \cap E_j = \emptyset$

If $f \in cl_{M_i}(E(\mathbf{M}) \setminus (E_j \cup \{e\})) \cap E_j$, then f must be in parallel with e_j in N_j and hence $N_j\backslash f$ is connected, which is contrary to **(3.3)**.

(3.5) $k = 1$ and $|E_1| = 2$.

Without loss of generality, we may suppose that

$$|E_1| \leq |E_2| \leq \cdots \leq |E_k|.$$

If $f \in E(N_k) \setminus \{e_k\}$, then, by **(3.3)**, N_k/f is connected and the canonical decomposition of $\mathbf{M}\backslash e/f$ is the following

$$\mathbf{M}\backslash e/f = \mathbf{H} \oplus_2 N_1 \oplus_2 N_2 \oplus_2 \cdots \oplus_2 N_{k-1} \oplus_2 N_k/f.$$

Now, we shall study the maximal dead sets of $\mathbf{M}/f$. Suppose that $\{X \cup \{e\}, Y\}$ is a 2-separation for $\mathbf{M}/f$. We shall prove that its dead set is equal to: (a) $X \cup \{e\}$ and $|X| = 1$; or (b) $X \cup \{e\}$ and $X \subseteq E_i \setminus \{f\}$, for some $i \in \{1,2,\ldots,k\}$.

If $|X| = 1$, then $X \cup \{e\}$ is its dead set. Thus, we may suppose that $|X| \geq 2$. So, $\{X, Y\}$ is a 2-separation for $\mathbf{M}\backslash e/f$. Now, we have two cases to deal with: (c) $Y \subseteq E_i \setminus \{f\}$, for some $i \in \{1,2,\ldots,k\}$ or (d) $X \subseteq E_i \setminus \{f\}$, for some $i \in \{1,2,\ldots,k\}$. When (d) happens, we have (b). So, we may suppose that (c) happens. Thus, Y must span f in N_i, for some $i \in \{1,2,3,4\}$, otherwise $\{X \cup \{e,f\}, Y\}$ is a 2-separation for $\mathbf{M}$. By **(3.4)**, this cannot happens when $i < k$. So, $i = k$ and let C be a circuit of N_k such that $f \in C \subseteq Y \cup \{f\}$. As $N_k\backslash g$ is not connected, for every $g \in C$, by **(3.3)**, it follows that $C\backslash\{f\}$ contains a 2-cocircuit D of N_k, by a result of Oxley (lemma 10.2.1 of [7]). Observe that D is a 2-cocircuit of M_i/f, for every $i \in \{1,2,3,4\}$, since X spans e in M_i/f.

Hence D is a 2-cocircuit of M_i, for every $i \in \{1,2,3,4\}$, and we arrive at a contradiction.

By **(2.6)**, the maximal dead sets are disjoint. So, $\mathbf{M}/f$ has just one dead set. Hence

$$2(k-1) + |E_k| \le \sum_{i=1}^{k} |E_i| = |\gamma(\mathbf{M}\backslash e)| \le |\gamma(\mathbf{M}\backslash f)| \le |E_k| + 1.$$

Thus, $k = 1$.

Now, we shall prove that $|\gamma(\mathbf{M}\backslash e)| = 2$. When $|\gamma(\mathbf{M}\backslash e)| > 2$, we must have $|E_1| > 2$ and $(E_1 \setminus \{f\}) \cup \{e\}$ as a dead set of $\mathbf{M}/f$, otherwise $\{e, x\}$ is a dead set for $\mathbf{M}/f$ and we arrive at a contradiction by the choice of e. As $\{(E_1 \setminus \{f\}) \cup \{e\}, E(\mathbf{M}) \setminus (E_1 \cup \{e\})\}$ cannot be a 2-separation for $\mathbf{M}/f$, since $E(\mathbf{M}) \setminus (E_1 \cup \{e\})$ does not span f in M_l, for every $l \in \{1,2,3,4\}$, by **(3.4)**, we arrive at a contradiction and $|E_1| = 2$.

Consider the following two sets:

$$Z = \{e \in E(\mathbf{M}) : |\gamma(\mathbf{M}\backslash e)| = 2\} \text{ and } W = \{e \in E(\mathbf{M}) : |\gamma(\mathbf{M}/e)| = 2\}.$$

By **(3.5)**, $Z \cup W \neq \emptyset$. Thus, for each $e \in Z$ and $f \in W$, we have that

$$\mathbf{M}\backslash e = \mathbf{H}_e \oplus_2 N_e \text{ and } \mathbf{M}/f = \mathbf{H}_f \oplus_2 N_f,$$

where $\mathbf{H}_e$ and $\mathbf{H}_f$ are non-trivial irreducible quads and N_e is a matroid isomorphic to $U_{2,3}$ and N_f is a matroid isomorphic to $U_{1,3}$. We set $W_e = E(\mathbf{M}) \cap E(N_e)$ and $Z_f = E(\mathbf{M}) \cap E(N_f)$. Observe that $W_e \subseteq W, Z_f \subseteq Z$ and $|W_e| = |Z_f| = 2$. Moreover,

(3.6) For each $i \in \{1,2,3,4\}, e \in Z$ and $f \in W$, W_e or $W_e \cup \{e\}$ is a cocircuit of M_i and Z_f or $Z_f \cup \{f\}$ is a circuit of M_i.

(3.7) $f \in W_e$ if and only if $e \in Z_f$.

(3.8) When $W_e \cup \{e\}$ is a triad of M_i, for $e \in Z$ and $i \in \{1,2,3,4\}$, then $Z_{f_1} \cup \{f_1\}, W_e \cup \{e\}, Z_{f_2} \cup \{f_2\}$ is a chain of M_i, where $W_e = \{f_1, f_2\}$.

First, we shall prove that $Z_f \cup \{f\}$ is a triangle of M_i, for every $f \in W_e$. When $Z_f \cup \{f\}$ is not a triangle of M_i, then Z_f is a circuit of M_i, by **(3.6)**. Observe that $e \in Z_f \cap (W_e \cup \{e\})$, by **(3.7)**. Thus, by orthogonality, we have that $Z_f = \{e, g\}$, where $g \in W_e \setminus \{f\}$. As $\{e\}$ is a loop of M_i/g and $\mathbf{M}/g$ is connected, by **(2.2)**, it follows that $\mathbf{M}/g$ is trivial, a contradiction. Hence $Z_f \cup \{f\}$ is a triangle of M_i.

Now, we shall prove that $|(Z_f \cup \{f\}) \cap (W_e \cup \{e\})| = 2$. When $X = Z_f \cup \{f\} = W_e \cup \{e\}$, it follows that $\{X, E(\mathbf{M}) \backslash X\}$ is a 2-separation for M_i. As $g \in Z_f \cap W_e$, it follows that X is a triad of M_j, for every $j \in \{1,2,3,4\}$, otherwise, by **(3.6)**, W_e is a cocircuit of M_j, for some $j \in \{1,2,3,4\}$, and f is a coloop of $M_j \backslash g$, which is contrary to the fact that $\mathbf{M}\backslash g$ is non-trivial and connected, by **(2.2)**. Similarly, we have that X is a triangle of every M_j, for every $j \in \{1,2,3,4\}$. Hence $\{X, E(\mathbf{M}) \setminus X\}$ is a 2-separation for $\mathbf{M}$.

Now, we shall state the dual of this result:

(3.9) When $Z_f \cup \{f\}$ is a triangle of M_i, for $f \in W$ and $i \in \{1,2,3,4\}$, then $W_{e_1} \cup \{e_1\}, Z_f \cup \{f\}, W_{e_2} \cup \{e_2\}$ is a chain of M_i, where $Z_f = \{e_1, e_2\}$.

By **(3.5)** and **(3.6)**, for some $e \in E(\mathbf{M})$, it follows that $W_e \cup \{e\}$ is a triad of M_1 or M_2, say M_1, since $\mathbf{M}$ is irreducible. By **(1.2)**, it must be a triad of M_3 or M_4, say M_3. Thus, by **(3.8)** and **(3.9)**, this triad is contained in a chain $\mathcal{C}$ of both M_1 and M_3 which must close as in (i) of **(3.2)**. Let R be the set of elements of $E(\mathbf{M})$ belonging to two triangles of $\mathcal{C}$ and let S be the set of elements of $E(\mathbf{M})$ belonging to two triads of $\mathcal{C}$. We define $X = R \cup S$. Thus, by **(3.2)**(i), $\{X, E(\mathbf{M}) \setminus X\}$ is a 1-separation for both M_1 and M_3 and $M_1|X$ and $M_3|X$ are equal to a wheel or a whirl having rim $R \subseteq Z$ and set of spokes $S \subseteq W$. Thus, one is a whirl and the other to a wheel, otherwise $\mathbf{M}$ is trivial, by **(2.3)**. As $M_2|X$ and $M_4|X$ cannot be connected components of M_2 and M_4, respectively, equal to the same wheel or whirl, by **(2.3)**, it follows that $\mathcal{C}$ is not a chain for both M_2 and M_4 and W_e and Z_f is a cocircuit and a circuit, respectively, for every $e \in S$ and $f \in R$, by **(3.6)**. As S is a circuit of M_2 or M_4, by **(1.2)**, it follows that S is circuit which is a connected component of one of these matroids, since S is contained in one of its series classes. Similarly, R is a cocircuit which is a connected component of M_2 or M_4. As $\mathbf{M}|X$ is a non-trivial connected quad, it follows that $r_{M_l}(X) = r(\mathbf{M}|X)$ is independent of l. So, R and S are connected components of the same matroid M_i, for $i \in \{2,4\}$, since S is a basis for X in the other. Thus, $\{X, E(\mathbf{M}) \setminus X\}$ is a 1-separation for $\mathbf{M}$. and it follows that $X = E(\mathbf{M})$. We have (iv) of theorem 1. ■

4 Irreducible semi-binary quads

The main result of this section gives a description of all the semi-binary irreducible quads and from it we derive theorem 2. We need the following lemma which is similar to ones of Lemos [3] or Oxley [6]:

(4.1) Let N be a matroid having a circuit-hyperplane C without parallel elements.

(i) If $\{X, Y\}$ is a 1-separation for N, then $\{X, Y\} = \{C, E(N) \setminus C\}$.

(ii) If $\{X, Y\}$ is a 2-separation for N, then $\{X, Y\} = \{C, E(N) \setminus C\}$ or $Z \subset C$

is contained in a series class of N, for some $Z \in \{X, Y\}$.

(iii) If $|C| \geq 3$ and N is binary and not 3-connected, then C contains a non-trivial series class of N.

Proof: (i) Observe that $\{X, Y\} = \{C, E(N) \setminus C\}$ because C is a circuit of N and $E(N) \setminus C$ is a cocircuit of N.

(ii) For $Z \in \{X, Y\}$, we have that

$$r(Z) \geq \min\{|C| - 1, |Z \cap C|\} + \min\{1, |Z \setminus C|\}.$$

When $C \cap Z \neq \emptyset$, for both $Z \in \{X, Y\}$, we have that

$$|C| + 1 = r(N) + 1 = r(X) + r(Y) \geq |C| + \min\{1, |X \setminus C|\} + \min\{1, |Y \setminus C|\}.$$

So, $Z \setminus C = \emptyset$, for some $Z \in \{X, Y\}$ and the result follows. When $C \cap Z = \emptyset$, for some $Z \in \{X, Y\}$, say $Z = X$, we have that $Z = C$, otherwise $r(Y) = r(N)$ and X is a parallel class of N.

(iii) From (i) and (ii), we have (iii) unless $\{C, E(N) \setminus C\}$ is the only 2-separation for N and it is exact. In this case, we can write $N = N_1 \oplus_2 N_2$, where N_1 and N_2 are binary matroids such that $E(N_2) = C \cup \{e\}$, for a new element e. As $r^*(N_2) = 2$ and $|E(N_2)| = |C| + 1 \geq 4$, it follows that N_2 has a non-trivial series class which must be contained in C. ■

(4.2) $\mathbf{M}$ is a non-trivial semi-binary irreducible quad if and only if there is a 3-connected binary matroid N having a circuit-hyperplane C such that $\mathbf{M} \equiv \mathbf{Q}(N; C)$ or $\mathbf{M} \simeq \mathbf{W}_2$.

Proof: We shall prove the "only if" part of the theorem, since the "if" is trivial. Suppose that it is not true and choose a counter-example $\mathbf{M}$ which is minimal. By **(2.11)**, an almost-trivial matroid cannot be semi-binary, unless it is isomorphic to $\mathbf{W}_2$. So, by **(3.1)**, (i) $\mathbf{M}$ is isomorphic to a wheel-whirl quad. In this case, we may take N equal to the wheel and C to be its rim and the result follows; or (ii) $\mathbf{M}$ has an element e such that $\mathbf{M}\backslash e$ or $\mathbf{M}/e$ is irreducible. As this theorem is invariant under duality, we may suppose that $\mathbf{M}$ has an element e such that $\mathbf{M}\backslash e$ is irreducible. So, by the choice of $\mathbf{M}$, $\mathbf{M}\backslash e \equiv \mathbf{Q}(N; C)$, for a binary 3-connected matroid N, or $\mathbf{M}\backslash e \simeq \mathbf{W}_2$. When the first possibility occurs, we suppose that $M_1\backslash e = N$ and $M_3\backslash e = N_C$. We divide the proof of this theorem in some lemmas. At the end, we shall arrive at a contradiction.

(4.3) $\mathbf{M}\backslash e \equiv \mathbf{Q}(N; C)$ and $\min\{r(N), r(N^*)\} \geq 3$.

By the next lemma, up to isomorphism, there is just one non-trivial semi-binary irreducible quad having rank equal to 2: $\mathbf{W}_2$. So, $\mathbf{M}\backslash e$ cannot be isomorphic to it, otherwise, by **(4.4)**, $\mathbf{M}$ must also be isomorphic to $\mathbf{W}_2$. Thus, $\mathbf{M}\backslash e \equiv \mathbf{Q}(N; C)$ and both $r(N)$ and $r(N^*)$ are at least three.

(4.4) If $\mathbf{H}$ is a non-trivial semi-binary irreducible quad such that $r(\mathbf{H}) = 2$, then $\mathbf{H} \simeq \mathbf{W}_2$.

Proof: It is enough to prove that $\mathbf{W}_2$ does not have an irreducible semi-binary extension. So, suppose that $\mathbf{H}\backslash e$ is isomorphic to $\mathbf{W}_2$ and that $E(\mathbf{H}) = \{a, b, c, d, e\}$. By **(1.2)**, we may suppose that $\{a, b\}$ and $\{c, d\}$ are circuits of H_2, $\{a, b\}$ is a circuit of H_3 and $\{c, d\}$ is a circuit of H_4. As H_3 and H_4 do not have a minor isomorphic to $U_{2,4}$, since $\mathbf{H}$ is semi-binary, it follows that $\{e, e_i\}$ is a circuit of H_i, for $i \in \{3, 4\}$. Observe that $e_3 \neq e_4$, otherwise $\{\{e, e_3\}, E(\mathbf{H}) \setminus \{e, e_3\}\}$ is a 2-separation for $\mathbf{H}$, by **(1.2)**. Thus, by **(1.2)**, $\{e, e_3\}$ or $\{e, e_4\}$ is a circuit of H_2, otherwise $\{e_3, e_4\}$ is a circuit of H_1. So, H_2 has a parallel class P with three elements. P must be a parallel class of H_3 or H_4 because $\mathbf{H}/e$ is connected and trivial, since it has loops, by **(2.2)**. So, $H_2/e = H_i/e$, for some $i \in \{3, 4\}$, and P is a parallel class of M_i. This cannot occurs, since $e_3 \in \{a, b\}$ if and only if $e_4 \in \{c, d\}$. That is, both M_3 and M_4 have a three element parallel class or both does not have it. ■

(4.5) C does not span e in M_1.

Suppose that C spans e in M_1. Choose a circuit C' of M_i, for $i \in \{1, 2\}$, such that $e \in C' \subseteq C \cup \{e\}$ and $|C' \cap C|$ is minimum. First, we shall prove that $|C \setminus C'| \geq 2$. As M_1 is binary and C spans e in M_1, there are circuits C_1 and C_2 of M_1 such that $C_1 \cap C_2 = \{e\}$ and $C_1 \triangle C_2 = C$. We may suppose that $|C_1| \leq |C_2|$. Thus, $|C'| \leq |C_1|$ and $|C \setminus C'| \geq |C \setminus C_1| = |C \cap C_2| \geq 2$, since $|C| \geq 3$, by **(4.3)**.

In this paragraph, we prove that C' is a circuit of M_i, for every $i \in \{1, 2, 3, 4\}$. Observe that $\mathbf{M}/(C' \cap C)$ is non-trivial, since $|C \setminus C'| \geq 2$ and hence

$$M_3\backslash e/(C' \cap C) = N_C/(C' \cap C) = [N^*_{E(N)\backslash C}\backslash(C' \cap C)]^*$$

is non-binary. So, $\mathbf{M}/(C' \cap C)$ is not connected, by **(2.2)**, because $M_i/(C' \cap C)$ has a loop, for some $i \in \{1, 2\}$. By the dual of **(4.1)**(i), $M_1\backslash e/(C' \cap C) = N/(C' \cap C)$ is connected. So, the only 1-separation for $\mathbf{M}/(C' \cap C)$ has $\{e\}$ as one of its sets. Thus, e is a loop of $M_i/(C' \cap C)$, for every $i \in \{1, 2, 3, 4\}$, otherwise it would be a coloop of M_i, for some $i \in \{1, 2, 3, 4\}$, which is contrary to **(2.2)**. Thus, by **(1.2)**, C' is a circuit of M_i, for every $i \in \{1, 2, 3, 4\}$.

Observe that $|C'| \geq 3$, otherwise C' is a parallel class of M_i, for every $i \in \{1, 2, 3, 4\}$ and $\mathbf{M}$ is not irreducible. Now, $C'' = (C \setminus C') \cup \{e\}$ is a circuit of M_1. Similarly to the previous paragraph, we have that $\mathbf{M}/(C'' \cap C)$ is a non-trivial quad, since $|C \setminus C''| = |C \cap C'| \geq 2$ and $M_3/(C'' \cap C)$ has $U_{2,4}$ as a minor. Observe that the only 1-separation for $M_1/(C'' \cap C)$ has $\{e\}$ as one of its sets, by the dual of **(4.1)**(i) applied to $N/(C'' \cap C) = M_1/(C'' \cap C)\backslash e$. But $M_2/(C'' \cap C)$ does not have a 1-separation having $\{e\}$ as one of its sets, because there is just one circuit of M_2 contained in $C \cup \{e\}$, namely C'. We arrive at a contradiction, by **(2.2)**, since $\mathbf{M}/(C'' \cap C)$ is a non-trivial connected quad having a matroid with a loop.

(4.6) M_1 is a 3-connected matroid.

By **(2.2)**, it is enough to prove that $\{e, f\}$ is not a circuit of M_1 for every $f \in E(\mathbf{M}) \setminus \{e\}$. Suppose that $\{e, f\}$ is a circuit of M_1. By **(4.5)**, $f \notin C$. By **(1.2)**, we have two cases to consider: (i) $\{e, f\}$ is a circuit of M_3; or (ii) $\{e, f\}$ is a circuit of M_4. In both cases, we shall arrive at a contradiction proving that $\{e, f\}$ is a circuit of M_i, for every $i \in \{1, 2, 3, 4\}$.

First, we shall prove that $\{e, f\}$ cannot be a circuit of M_4. If $\{e, f\}$ is a circuit of M_4, then $E(\mathbf{M}) \setminus C$ is a parallel class of M_4. As $|E(\mathbf{M}) \backslash C| \geq 3$, by **(4.3)**, we can choose different elements a and b of $E(M) \setminus (C \cup \{e\})$. By **(1.2)**, it follows that $\{e, a\}$ or $\{e, b\}$ is a circuit of M_2, otherwise $\{a, b\}$ is a parallel class of M_1. Hence e is in parallel with f in M_2, since both a and b are parallel to f in this matroid. By **(1.2)**, $\{e, f\}$ is also a circuit of M_3. We arrive at a contradiction.

When $\{e, f\}$ is a circuit of M_3, $C \cup \{e\}$ is also a circuit of M_3, since e and f are parallel in this matroid. As $C \cup \{e\}$ is not a circuit of M_1, by **(4.5)**, it follows that $C \cup \{e\}$ is a circuit of M_2, by **(1.2)**. Hence $(C \cup \{e\}) \triangle (C \cup \{f\}) = \{e, f\}$ is also a circuit of M_2, since M_2 is binary. Again, we arrive at a contradiction using **(1.2)**.

(4.7) For every element $f \in E(\mathbf{M}) \setminus (C \cup \{e\})$, there is a triad T_f of M_1 such that $f \in T_f$ and $T_f \setminus \{f\} \subseteq C$.

First, we shall prove that $M_1 \backslash f$ is not 3-connected. When $M_1 \backslash f$ is 3-connected, it follows that $\mathbf{M} \backslash f$ is an irreducible non-trivial semi-binary quad. Thus, by the choice of $\mathbf{M}$, $\mathbf{M} \equiv \mathbf{Q}(N'; C')$, for some 3-connected binary matroid N'. So, $C = C'$. Hence $C \cup \{e\}$ is also a circuit of M_3 and e is not spanned by M_1. That is, C is a circuit-hyperplane of M_1 and $M_3 = (M_1)_C$. We arrive at a contradiction and $M_1 \backslash f$ is not 3-connected.

Let $\{X \cup \{e\}, Y\}$ be a 2-separation for $M_1 \backslash f$. Now, we shall prove that $|X| \geq 2$. If $|X| = 1$, say $X = \{x\}$, then $\{x, e\}$ is a series class of $M_1 \backslash f$, by **(4.6)**. So, $\{e, f, x\}$ is a triad of M_1 and $M_1 \backslash e = N$ is not 3-connected, a contradiction. Hence $|X| \geq 2$ and $\{X, Y\}$ is a 2-separation for $M_1 \backslash \{e, f\} = N \backslash f$. By **(4.1)**(iii), it follows that there is $Z' \subseteq Z \subseteq C$, for some $Z \in \{X, Y\}$, such that $|Z'| = 2$ and Z' is contained in a series class of $M_1 \backslash \{e, f\} = N \backslash f$. As X spans e in M_1, it follows that $Z = Y$, by **(4.5)**, and Y' is contained in a series class of $M \backslash f$. Thus, we can take $T_f = \{f\} \cup Y'$.

(4.8) If $g \in T_f \cap C$, for some $f \in E(\mathbf{M}) \setminus (C \cup \{e\})$, then there is a triangle T_g of M_1 such that $\{f, g\} \subseteq T_g$ and $T_g \cap C = \{g\}$.

First, we shall prove that M_1/g is not 3-connected. If M_1/g is 3-connected, then $\mathbf{M}/g$ is a non-trivial quad because $M_1 \backslash e/g \neq M_3 \backslash e/g$, by **(4.3)**. So,

$\mathbf{M}/g$ is an irreducible non-trivial semi-binary quad and, by the choice of $\mathbf{M}$, $\mathbf{M}/g \equiv \mathbf{Q}(N', C')$ for a 3-connected binary matroid N'. Thus, $C' = C \setminus \{g\}$ and $C \cup \{e\}$ or $(C \setminus \{g\}) \cup \{e\}$ is a circuit of M_3. Hence $C'' = (C \setminus \{g\}) \cup \{e\}$ is a circuit of M_3, otherwise $M_3 = (M_1)_C$. As $C'' \cap T_f = \{g'\}$, where $T_f \cap C = \{g, g'\}$, it follows that T_f is not a triad of M_3. Now, we shall prove that T_f is a triad of $M_3 \backslash e$. Observe that T_f is a triad of $M_1 \backslash e = N$, since this matroid is 3-connected. Thus, by the dual of **(1.2)**, T_f is a triad of $M_3 \backslash e$ or $M_4 \backslash e$. As, by orthogonality, T_f cannot be a triad of $M_4 \backslash e$, it follows that T_f is a triad of $M_3 \backslash e$. Hence $T_f \cup \{e\}$ is a cocircuit of M_3. So, $T_f \cup \{e\}$ is a cocircuit of M_2, by the dual of **(1.2)**. We arrive at a contradiction, by orthogonality, because $P = E(\mathbf{M}) \setminus (C \cup \{e\})$ is contained in a parallel class of M_2 and $P \cap (T_f \cup \{e\}) = \{f\}$. Thus, M_1/g is not 3-connected.

Let $\{X \cup \{e\}, Y\}$ be a 2-separation for M_1/g. When $|X| = 1$, say $X = \{x\}$, then $\{x, e\}$ is a parallel class of M_1/g, by **(4.6)**. So, we can take $T_g = \{e, g, x\}$, since M_1 is 3-connected and $x \notin C$, otherwise C spans e in M_1, which is contrary to **(4.5)**. Hence, we may suppose that $|X| \geq 2$. Thus, $\{X, Y\}$ is a 2-separation for $M_1 \backslash e/g = N/g$. By the dual of **(4.1)**(iii), it follows that there is $Z \in \{X, Y\}$ and $Z' \subseteq Z$ such that $Z' \cap C = \emptyset$, $|Z'| = 2$ and Z' is contained in a parallel class of $M_1 \backslash e/g = N/g$. Thus, we can take $T_g = \{g\} \cup Z'$.

By **(4.7)** and **(4.8)**, we can construct a chain of M_1 having its rim contained in C and spokes in $E(\mathbf{M}) \setminus C$. This chain must close, since the only element of $E(\mathbf{M}) \setminus C$ which may not belong to a triad of M_1 is e. It may happens that this chain starts and ends with triangles that contain e. But, by **(4.6)**, M_1 is 3-connected, and by **(3.2)**(ii), it follows that M_1 must be a wheel or whirl. We arrived at the final contradiction. ∎

Now, we shall prove the main result of this paper:

Proof of theorem 2: Suppose that theorem 2 fails. Choose a pair of matroids M and N which contradict it. Observe that $\mathbf{M} = (M, M^*, N, N^*)$ is a semi-binary non-trivial connected quad having only connected matroids as coordinates. Thus, the core $\mathbf{N}$ of $\mathbf{M}$ must be semi-binary. As $\mathbf{N}$ is irreducible, by **(4.2)**, it must be equivalent to some $\mathbf{Q}(H; C)$, for a 3-connected binary matroid H or it is isomorphic to $\mathbf{W}_2$. We arrive at a contradiction because these quads has exactly one matroid which is not connected as a coordinate. ∎

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