Improved Bound for the Carathéodory Rank of the Bases of a Matroid*

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Abstract

Let M be a matroid on m elements and let r be its rank. We show that any vector in the integer cone of the incidence vectors of bases of M can be written as nonnegative integer combination of at most $m+r-1$ incidence vectors of bases of M, and these can be determined in oracle-polynomial time.

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1 Introduction

Let H be a set of vectors in $\mathbb{R}^m$. We define

$$\begin{aligned}
\text{cone}(H) &:= \Big\{\sum_{h\in H}\lambda_h h \mid \lambda_h \in \mathbb{R}_+ \ (h\in H)\Big\};\\
\text{lat}(H) &:= \Big\{\sum_{h\in H}\lambda_h h \mid \lambda_h \in \mathbb{Z} \ (h\in H)\Big\};\\
\text{int.cone}(H) &:= \Big\{\sum_{h\in H}\lambda_h h \mid \lambda_h \in \mathbb{Z}_+ \ (h\in H)\Big\}.
\end{aligned}$$

So, cone(H), lat(H), and int.cone(H) are the *cone*, the *lattice*, and the *integer cone* generated by the vectors in H, respectively. We have the following containment:

$$\text{int.cone}(H) \subseteq \text{cone}(H) \ \cap \ \text{lat}(H). \tag{1}$$

The set H is called a *Hilbert base* if equality holds in (1). Thus, a set of vectors forms a Hilbert basis if any vector which can be written both as their nonnegative combination and their integer combination can also be written as their nonnegative integer combination.

The *Carathéodory rank* of a Hilbert base H is the least integer t such that any vector in the integer cone generated by the vectors in H can be written as a nonnegative integer combination of at most t vectors in H.

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Tutte [22] and, independently, Nash-Williams [17] established a necessary and sufficient condition under which a given graph has k edge-disjoint spanning trees. Their theorem reads as follows.

Theorem 1 (Tutte [22], Nash-Williams [17]) *A graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if, for every partition $(V_1, V_2, \ldots, V_s)$ of V, the number of edges joining vertices in distinct parts is at least $k(s-1)$.* ∎

It is a consequence of Theorem 1 that the incidence vectors of spanning trees of a graph form a Hilbert Base.

If M is a matroid, we denote its set of elements by $E(M)$, and the rank of a subset $S \subseteq E(M)$ by $r(S)$, or $r_M(S)$ in case of ambiguity. We sometimes write $r(M)$ for $r(E(M))$. It follows from Edmonds' generalization of Theorem 1, the so-called Matroid Partition Theorem, that incidence vectors of the bases of a Matroid do form a Hilbert base.

Theorem 2 (Matroid Partition Theorem [7]) *A matroid M has k disjoint bases if and only if*

$$kr(S) + |E(M) - S| \geq kr(M)$$

for every subset S of $E(M)$. ∎

The problem we consider is related to a question posed by Cunningham [6]. Let M be a matroid. What is the least integer t such that any vector w in the integer cone of incidence vectors of bases of M can be written as a nonnegative integer combination of at most t incidence vector of bases of M? That is, what is the least upper bound t for the Carathéodory rank of any Hilbert base formed by incidence vectors of bases of a matroid? Cunningham [6] showed that his algorithm for testing membership in matroid polyhedra gives an upper bound of m^4+1 for t, where m is the number of elements in M.

Cook, Fonlupt and Schrijver [5] have proved the following integer analogue of Carathéodory's Theorem (for undefined terms see Schrijver [19]).

Theorem 3 (Cook, Fonlupt, and Schrijver [5]) *Let $H \subset \mathbb{Z}^m$ be a Hibert base generating a pointed cone. If $w \in$ int.cone(H) then w is the nonnegative integer combination of at most $2m-1$ vectors in H.* ∎

The previous theorem implies that the Carathéodory rank of any Hilbert base H generating a pointed cone is at most $2m-1$ (if the hypothesis of the cone(H) be pointed is dropped then there is no upper bound for the Carathéodory rank of H in terms of the dimension m of the space (cf. [5])). Sebő [21] has shown that this upper bound of $2m-1$ can be improved to $2m-2$ and conjectured that the actual upper bound is m (this bound is the best possible). Sebő's conjecture is also called the *Integral Carathéodory Conjecture* and it has been disproved by Bruns, Gubeladze, Henk, Martin, and Weismantel [3]. (For an updated discussion on this and others related conjectures see Bouvier and Gonzalez-Springberg [1], Bruns and Gubeladze [2], and Firla and Ziegler [10]). In particular, as the incidence vectors of the bases of a matroid form a Hilbert base generating a pointed cone, Sebő's result implies the upper bound of $2m-2$ for Carathéodory rank of any these Hilbert bases arising from bases of a matroid.

A related result has been proved by Chaourar [4]. A set $H \subset \mathbb{Z}^m$ of vectors is said a *strong Hilbert base* if for every face F of cone(H) there exists an element $h_F \in F \cap H$ with the following property: if w is a vector in the integer cone generated by H and F is the minimal face of cone(H) containing w, then $w - h_F \in$ cone(H). It is easy to see that for strong Hilbert bases the Integral Carathéodory Conjecture does hold. Chaourar [4] has shown that incidence vectors of bases of matroids without certain minors (the cycle matroid of the complete graph on four vertices is one of them) form a strong Hilbert base. So, for those matroids, the Carathéodory rank of the incidence vectors of their bases is m.

The following theorem constitutes our main result. We are assuming that a matroid M is given by an oracle returning $r(S)$ for each query $S \subseteq E$. For a discussion on the running time of oracle algorithms, see Grötschel, Lovász, and Schrijver [15].

Theorem 4 *Let M be a matroid on a set E with m elements and let w be a vector in the integer cone generated by the incidence vectors of the bases of M. Then w can be written as a nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M, and these can be determined in oracle-polynomial time.*

This paper is structured as follows. In Section 2, we define all necessary terminology and we describe some auxiliary results. The proof of Theorem 4 is given in Section 3. Finally, in Section 4 we make a few last remarks.

2 Definitions and preliminaries

We assume knowledge of elementary matroid theory; for an introduction and undefined terms, see Oxley [18] and Welsh [23].

Let M be a matroid on a set E. If X and Y are disjoint subsets of E then $M/X \setminus Y$ denotes the matroid obtained from M by contracting X and deleting Y. By $M(X)$ we denote the restriction of M to X or the deletion of $E \setminus X$ from M.

Let w be a vector in $\mathbb{R}^E$. For $S \subseteq E$ we denote $\sum_{e \in S} w_e$ by $w(S)$. We denote by cone(M), lat(M), and int.cone(M) the cone, the lattice, and the integer cone generated by the incidence vector of bases of M, respectively. For a subset S of E, the *incidence vector* χ^S of S is defined by the rule that $\chi^S_e = 1$ if $e \in S$, and $\chi^S_e = 0$ otherwise.

It follows from the Edmonds' description of the matroid polytope by linear inequalities (cf. [8, 9]) that a vector $x \in \mathbb{R}^E$ belongs to cone(M) if and only if

$$\begin{array}{rcll} x_e & \geq & 0 & \text{for all } e \in E, \\ x(S) & \leq & (x(E)/r(M))r(S) & \text{for all } S \text{ subset of } E. \end{array} \tag{2}$$

For each base B of M and $w \in$ cone(M) let

$$\mu_{w,B} := \max\{\mu \in \mathbb{R}_+ \mid w - \mu\chi^B \in \text{cone}(M)\}.$$

If $w \in \mathbb{R}^E$ we call a subset S of E w-tight if $w(S) = (w(E)/r(M))r(S)$. (So, if $w \in$ cone(M) and $w_e > 0$ for every $e \in E$ then the w-tight sets are in one to one correspondence with the faces of cone(M) containing w.)

Let cl be the closure operator of M, that is cl is the function from 2^E to 2^E defined, for every $S \subseteq E$, by $\text{cl}(S) := \{e \in E \mid r(S \cup \{e\}) = r(S)\}$.

Proposition 5 *Let M be a matroid on a set E and let $w \in cone(M)$ with $w_e > 0$ for every $e \in E$. If S is a w-tight subset of E, then S is closed (a flat).*

Proof. Suppose that S is not closed. Then there exists a $e \in \mathrm{cl}(S) \setminus S$. Thus,

$$\begin{aligned} w(S \cup \{e\}) &= w(S) + w(e) \\ &= (w(E)/r(M))r(S) + w_e \quad (\text{as } S \text{ is } w\text{-tight}) \\ &= (w(E)/r(M))r(S \cup \{e\}) + w_e \quad (\text{as } e \in \mathrm{cl}(S)) \\ &> (w(E)/r(M))r(S \cup \{e\}) \quad (\text{as } w_e > 0). \end{aligned}$$

This contradicts the hypothesis that $w \in \mathrm{cone}(M)$. ∎

Let f be a set-function on a set E, i.e., a function defined on the class of all subsets of E. The function f is called *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y), \tag{3}$$

for all subsets X and Y of E. Similarly, f is called *supermodular* if $-f$ is submodular, i.e. if f satisfies (3) with the opposite sign. f is modular if f is both submodular and supermodular, i.e. if f satisfies (3) with equality. The following proposition is a well-know consequence of the submodularity of the rank function of a matroid.

Proposition 6 *Let M be a matroid on a set E and let $w \in cone(M)$ with $w_e > 0$ for every $e \in E$. If B is a base of M such that $\chi^B(S) = r(S)$ for every w-tight subset S of E, then $\mu_{w,B} > 0$. In particular, if E contains no w-tight proper subset, then $\mu_{w,B} > 0$ for every base B of M.*

Proof. Suppose that $\mu_{w,B} = 0$. By (2), as $w_e > 0$ for every $e \in B$, we have that there exists a w-tight set S such that

$$w(S) - \mu\chi^B(S) \quad > \quad (w(E)/r(M) - \mu)r(S) \quad \text{for all } \mu > 0.$$

This implies that $\chi^B(S) < r(S)$ contradicting the hypothesis. ∎

We shall make use of the Discrete Separation Theorem of Frank [11, 12] in order to prove the main result in the next section.

Theorem 7 (Discrete Separation Theorem [11, 12]) *Let p and b be super- and submodular set-function on a set E, respectively. There is a finite modular set-function m for which $p(S) \leq m(S) \leq b(S)$ for all $S \subseteq E$ if and only if $p(S) \leq b(S)$ for all $S \subseteq E$. If p and b are integer-valued, then m can be chosen to be integer-valued. Moreover, if such a m exists, it can be found in oracle-polynomial time, provided that a polynomial-time subroutine is available for minimizing a submodular set-function.* ∎

We say that $S \subseteq E$ minimizes a set-function f if $f(S)$ is as small as possible. Throughout, we assume that a submodular function f is given by an oracle returning $f(S)$ for each query $S \subseteq E$. Minimizing a submodular set-function can be done in oracle-polynomial time via the ellipsoid algorithm, as shown by Grötschel, Lovász and Schrijver [14] (see also [15]). Recently, Iwata, Fleischer, and Fujishige [16] and, independently, Schrijver [20] have presented a non-ellipsoidal algorithm to minimize a submodular set-function.

Given a matroid M by its rank oracle, a base B of M and $w \in \text{cone}(M)$ one can compute $\mu_{w,B} = \max\{\mu \in \mathbb{R}_+ \mid w - \mu\chi^B \in \text{cone}(M)\}$ in oracle-polynomial time. For $\mu \in \mathbb{R}_+$, let f_μ be the set function on E defined as $f_\mu(S) := (w(E)/r(M) - \mu)r(S) - w(S) + \mu\chi^B(S)$ for all S subset of E. So, f_μ is submodular, provided that $\mu \leq w(E)/r(M)$. Deciding for a given $\mu \in \mathbb{R}_+$ whether or not $w - \mu\chi^B \in \text{cone}(M)$ is equivalent to deciding whether $w_e \geq \mu$ for all $e \in E$ and $f_\mu \geq 0$ (or, equivalently, finding $S \subseteq E$ with $f_\mu(S) < 0$, if any such set exists). Since we can minimize a submodular function in oracle polynomial-time, we can find by binary search the value $\mu_{w,B}$ (using the fact that this value $\mu_{w,B}$ has encoding length bound by the encoding length of w and M).

3 Proof of Theorem 4

In this section, we prove Theorem 4. We first establish a key lemma from which our main result follows easily. If M is a matroid on E and w is a vector in $\mathbb{R}^E$ then by $d(w)$ we denote the minimum dimension of a face of cone(M) containing w. Roughly, this key lemma states that if w is a non-null vector in int.cone(M) and B_1 is an arbitrary base of M such that $d(w - \lfloor\mu_{w,B_1}\rfloor\chi^{B_1}) = d(w)$ (i.e. μ_{w,B_1} is not an integer) then there exists a base B_2 of M such that $d(w - \lfloor\mu_{w,B_1}\rfloor\chi^{B_1} - \chi^{B_2}) < d(w)$. So, in order to 'bring' a vector of int.cone(M) to a face of cone(M) of smaller dimension one needs at most two incidence vectors of bases of M.

Lemma 8 *Let M be a matroid on a set E and let $w \in$ int.cone(M) with $w_e > 0$ for every $e \in E$ and such that E contains no w-tight proper subset.*

- *(i) Suppose B_1 is a base of M such that $w - \chi^{B_1} \notin$ cone(M). Then one can find in oracle-polynomial time a base B_2 of M and a proper subset X of E such that $w - \chi^{B_2} \in$ int.cone(M) and X is $(w - \chi^{B_2})$-tight. In particular $\mu_{w,B_2} = 1$.*
- *(ii) Suppose B_1 is a base of M such that μ_{w,B_1} is not an integer and let $v := w - \lfloor\mu_{w,B_1}\rfloor\chi^{B_1}$. Then one can find in oracle-polynomial time a base B_2 of M and a proper subset X of E such that $v - \chi^{B_2} \in$ int.cone(M) and X is $(v - \chi^{B_2})$-tight. In particular $\mu_{v,B_2} = 1$.*

Proof. One can easily derive (ii) from (i). So, we shall prove only (i).

By the hypothesis and Proposition 6 we have that $0 < \mu_{w,B_1} < 1$. Let $k := w(E)/r(E) - 1$. In order to prove the present lemma we shall proceed as follows:

- (a) Firstly, we prove that one can find in oracle-polynomial time a base B of M such that $w - \chi^B$ can be written as a nonnegative integer combination of k incidence vectors of bases of M (i.e. $w - \chi^B \in$ int.cone(M)).
- (b) Secondly, we show how one can in oracle-polynomial time obtain from B_1 and B a base B_2 of M and a closed proper subset X of E fulfilling (i).

Proof of (a). Define $p : 2^E \to \mathbb{Z}$ by $p(S) := w(S) - kr(S)$. The set-function p is supermodular. Using the oracle for r we can build a polynomial time subroutine for returning $p(S)$ for each query $S \subseteq E$.

We have that

$$p(S) = w(S) - kr(S) \leq (k+1)r(S) - kr(S) \leq r(S)$$

for all $S \subseteq E$. Therefore, by the Discrete Separation Theorem, one can find in oracle-polynomial time an integer-valued modular set-function $m : 2^E \to \mathbb{Z}$ such that

$$p(S) \leq m(S) \leq r(S) \quad \text{for all } S \subseteq E. \tag{4}$$

The above inequalities imply that $p(\emptyset) = m(\emptyset) = r(\emptyset) = 0$ and thus

$$m(S) = \sum_{e \in S} m(\{e\}) \quad \text{for all } S \subseteq E.$$

Inequalities (4) also imply that $p(E) = m(E) = r(E)$.

We claim that $m(\{e\}) \geq 0$ for all $e \in E$. Suppose not. Let $E_- := \{e \in E \mid m(\{e\}) < 0\} \neq \emptyset$ and let $E_+ := \{e \in X \mid m(\{e\}) > 0\}$. Then

$$\begin{aligned} r(E_+) &\geq m(E_+) \\ &= m(E) - m(E_-) \\ &= r(E) - m(E_-) \\ &> r(E), \end{aligned}$$

which contradicts the fact that r is the rank function of a matroid. So, as m is integer-valued and $m(\{e\}) \leq r(\{e\}) \leq 1$ we conclude that $m(\{e\}) \in \{0, 1\}$ for all $e \in X$.

Let $B := \{e \in E \mid m(\{e\}) = 1\}$. B is a base of M. Indeed,

$$\begin{aligned} \chi^B_e = m(\{e\}) &\geq 0 && \text{for all } e \in E, \\ \chi^B(S) = m(S) &\leq r(S) && \text{for all } S \subseteq E \end{aligned}$$

implying that χ^B is an integer vector in the matroid polytope of M and therefore B is an independent set of M (cf. [8, 9]) with $|B| = m(E) = r(E)$. Hence, B is a base of M.

Finally, we verify that $w - \chi^B \in \text{int.cone}(M))$. For all $S \subseteq E$ we have that

$$\begin{aligned} kr(S) + w(E \setminus S) - \chi^B(E \setminus S) &= kr(S) + w(E) - w(S) - \chi^B(E) + \chi^B(S) \\ &= w(E) - \chi^B(E) + \chi^B(S) - (w(S) - kr(S)) \\ &= kr(E) + \chi^B(S) - (w(S) - kr(S)) \\ &= kr(E) + m(S) - p(S) \\ &\geq kr(E). \end{aligned}$$

Thus, by the Matroid Partition Theorem we conclude that $w - \chi^B$ is the sum of k incidence vectors of bases of M. This completes the proof of (a). ∎

Proof of (b). Let $\{e_1, \ldots, e_l\} := B_1 \setminus B$ and $\{f_1, \ldots, f_l\} := B \setminus B_1$. We may assume that $T_i := B_1 \setminus \{e_1, \ldots, e_i\} \cup \{f_1, \ldots, f_i\}$ is a base of M for $i = 0, 1, \ldots, l$ (this order can found making $O(l^2)$ queries to the oracle for r). Since $0 < \mu_{w,B_1} < 1$ and $\mu_{w,B} \geq 1$ then there exists a index $j \in \{1, \ldots, l\}$ such that $\mu_{w,T_{j-1}} < 1$ and $\mu_{w,T_j} \geq 1$. This index j can be found in oracle-polynomial time since $\mu_{w,T}$ can be computed in oracle-polynomial for any base T of M. By Proposition 6 we have that $\mu_{w,T_{j-1}} > 0$.

As $\mu_{w,T_{j-1}}$ is not an integer then by (2) there exists a $(w - \mu_{w,T_{j-1}}\chi^{T_{j-1}})$-tight proper subset X of E, that is

$$w(X) - \mu_{w,T_{j-1}}\chi^{T_{j-1}}(X) = (k + 1 - \mu_{w,T_{j-1}})r(X).$$

and

$$w(X) - \chi^{T_{j-1}}(X) \geq kr(X) + 1. \tag{5}$$

By Proposition 5 we know that X is a closed set. (Such a set X can be found in oracle-polynomial time by minimizing the submodular function $(k+1-\mu_{w,T_{j-1}})r(X) - w(X) + \mu_{w,T_{j-1}}\chi^{T_{j-1}}(X)$.)

On the other hand, as $\mu_{w,T_j} \geq 1$, we have that

$$w(X) - \chi^{T_j}(X) \leq kr(X). \tag{6}$$

Combining (5) and (6) we obtain that

$$\begin{aligned} kr(X) &\geq w(X) - \chi^{T_j}(X) \\ &= w(X) - \chi^{T_{j-1}}(X) + \chi^{e_j}(X) - \chi^{f_j}(X) \\ &\geq kr(X) + 1 + \chi^{e_j}(X) - \chi^{f_j}(X). \end{aligned}$$

So, we conclude that $e_j \notin X$, $f_j \in X$ and $w(X) - \chi^{T_j} = kr(X)$. One can also verify that $w - \chi^{T_j} \in \text{int.cone}(X)$. Hence, $B_2 := T_j$ is a base of M such that $w - \chi^{B_2} \in \text{int.cone}(M)$ and X is a closed $(w - \chi^{B_2})$-tight proper subset of E. ∎

We are now ready to prove Theorem 4.

Proof of Theorem 4. The proof is algorithmic. By preprocessing M and w we may assume that M is simple, $w_e > 0$ for all $e \in E$ and that E contains no w-tight proper subset (the preprocessing of w consists essentially in minimizing the submodular function $(w(E)/r(M))r(S) - w(S)$ over $2^E - \{\emptyset, E\}$).

The algorithm meant is defined recursively on $m + r(M)$. First suppose that $r(M) = 1$ (this can be checked making one query to the oracle for r). We can trivially write w as a nonnegative integer combination of m incidence vectors of bases of M. So, we assume $r(M) > 1$.

By making no more than m queries to r we can obtain a base B_1 of M. Let $v := w - \lfloor \mu_{w,B_1} \rfloor \chi^{B_1}$ (μ_{w,B_1} can be computed in oracle-polynomial time). By Proposition 6 we have that $\mu_{w,B_1} > 0$. We split into two cases.

Case 1. μ_{w,B_1} is an integer.

From the description (2) of cone(M) by linear inequalities there exists some $e \in E$ such that $v_e = 0$ or there exists a proper subset X of E which is v-tight.

Let us first suppose that $E_0 := \{e \in E \mid v_e = 0\}$ is nonempty. Let $E' := E \setminus E_0$ and let v' be the restriction of v to E'. By our induction hypothesis we can write v' as a nonnegative integer combination of at most $m - |E_0| + r(E') - 1 \leq m + r(M) - 2$ incidence vectors of bases of $M(E')$. Hence, we can write w as a nonnegative combination of at most $m + r(M) - 1$ incidence vectors of bases of M.

So, we may assume that $E_0 = \emptyset$. Let X be a maximal proper v-tight set of E (X can be found by minimizing $(v(E)/r(M))r(S) - v(S)$ over $2^E - \{\emptyset, E\}$). From Proposition 5 we know that X is closed (so M/X is a simple matroid). Let $M(X_1), \ldots, M(X_p)$ be the connected components of $M(X)$ (i.e. $M(X)$ is the direct sum of $M(X_1), \ldots, M(X_p)$). Let $k := w(E)/r(M) - \lfloor \mu_{w,B_1} \rfloor$, $Y := E \setminus X$, let v_Y be the restriction of v to Y and let v_{X_i} be the restriction of v to X_i $(i = 1, \ldots, p)$. One can verify that:

(a) v_Y is the sum of k incidence vectors of bases of M/X ($v_Y \in \text{int.cone}(M/X)$);

(b) v_{X_i} is the sum of k incidence vectors of bases of $M(X_i)$ ($i = 1, \dots, p$) ($v_{X_i} \in \text{int.cone}(M(X_i))$ ($i = 1, \dots, p$));

(c) $r(M/X) < r(M)$ and $r(M(X_i)) < r(M)$ ($i = 1, \dots, p$).

Thus, by recursion we can write v_Y as a nonnegative integer combination of at most $|Y| + r(M/X) - 1$ incidence vectors of bases of M/X and write v_{X_i} as a nonnegative integer combination of at most $|X_i| + r(M(X_i)) - 1$ incidence vectors of bases of $M(X_i)$ ($i = 1, \dots, p$). (All these recursion can be carried out in oracle-polynomial time.) These combinations can be glued together to form a nonnegative integer combination of v of at most

$$|Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p \quad = \quad m + r(M) - 2p - 1 \tag{7}$$

incidence vector of bases of M. This glueing can be done as follows. Let $B_{Y,1}, \dots, B_{Y,k}$ be bases of M/X such that:

- $v_Y = \chi^{B_{Y,1}} + \cdots + \chi^{B_{Y,k}}$;
- there exist indexes $0 = l_{Y,0} < l_{Y,1} < \ldots < l_{Y,k_Y} = k$ with $k_Y < |Y| + r(M/X) - 1$ such that
$$B_{Y,l_{Y,j}+1} = B_{Y,l_{Y,j}+2} = \ldots = B_{Y,l_{Y,j+1}} \text{ for } j = 0, \dots, k_Y - 1.$$

For $i = 1, \dots, p$ let $B_{X_i,1}, \dots, B_{X_i,k}$ be bases of $M(X_i)$ such that:

- $v_{X_i} = \chi^{B_{X_i,1}} + \cdots + \chi^{B_{X_i,k}}$;
- there exist indexes $0 = l_{X_i,0} < l_{X_i,1} < \ldots < l_{X_i,k_{X_i}} = k$ with $k_{X_i} < |X_i| + r(M(X_i)) - 1$ such that
$$B_{X_i,l_{X_i,j}+1} = B_{X_i,l_{X_i,j}+2} = \ldots = B_{X_i,l_{X_i,j+1}} \text{ for } j = 0, \dots, k_{X_i} - 1.$$

Then, by (a) and (b) above, $B_i := B_{Y,i} \cup B_{X_1,i} \cup \ldots \cup B_{X_p,i}$ is a base of M ($i = 1, \dots, k$) and the sum

$$\chi^{B_1} + \cdots + \chi^{B_k}$$

shows that v can be written as a nonnegative integer combination of at most

$$k_Y + \sum_{i=1}^{p} k_{X_i} - p \quad \le \quad |Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p$$

bases of M.

Therefore, we can write w as a nonnegative integer combination of at most $m + r(M) - 2p \le m + r(M) - 2$ incidence vectors of bases of M.

Case 2. μ_{w,B_1} is not an integer.

From Lemma 8 it follows that one can find in oracle-polynomial time a base B_2 of M and a closed proper subset X of E such that $\mu_{v,B_2} = 1$ and X is $(v - \chi^{B_2})$-tight. Now, proceeding similarly as in the Case 1, simply replacing w by v and B_1 by B_2, one can write v as a nonnegative integer combination of at most

$$m + r(M) - |E_0| - 2p \quad \leq \quad m + r(M) - 2,$$

incidence vectors of bases of M, where $E_0 = \{e \in E \mid v_e - \chi_e^{B_2} = 0\}$ and p is the number of connected components of $M(X)$.

Thus, we can write w as a nonnegative integer combination of at most $m+r(M)-1$ incidence vectors of bases of M. This completes the proof of Theorem 4. ∎

4 Final Remarks

Using methods of the present paper it might be possible to derive a sharper upper bound for the least number t needed in order to write any vector in the integer cone formed by the incidence vectors of bases of a matroid as a nonnegative integer combination of at most t of these incidence vectors. If one can show that for matroids with rank lesser than or equal to k we have that $t \leq m$ then the bound of $m + r(M) - 1$ in Theorem 4 can be replaced by $m + r(M) - k$. In the proof of Theorem 4 it was somehow stressed the fact that if the tight set X found has $p \geq 2$ connected components then one would need fewer distinct incidence vectors in the nonnegative integer combination derived for w. The 'bad' X's sets for the proof are precisely the connected hyperplanes of M. If one can show that for any $w \in$ int.cone(M), there exists a base B of M such that the tight X set obtained in Theorem 4 from $(w - \chi^B)$ is not a connected hyperplane, then one could prove that m is a upper bound for t.

We would like also to mention the NP-completeness of a related problem. Consider the problem: *given:* a matroid M, a vector $w \in$ int.cone(M), and an integer t; *question:* can w be written as a nonnegative integer combination of at most t incidence vectors of bases of M? This problem is NP-complete even restricted to graphic matroids. Indeed, let $a_1, \ldots, a_n$ be an instance of the PARTITION problem such that $a_1 + \cdots + a_n = 2M$ (cf. Garey and Johnson [13]). Let $G = (V, E)$ be a graph with vertex set $V = \{u, v, w\}$ and edge set $E = \{a, b, e_1, \ldots, e_n\}$ such that a and b connect u and v and e_i connects v and w $(i = 1, \ldots, n)$. Let $w \in \mathbb{R}^E$ such that $w_a = w_b = M$ and $w_{e_i} = a_i$ $(i = 1, \ldots, n)$. One can verify that w is a vector in int.cone$(M(G))$, where $M(G)$ is the cycle matroid of G, and that w can be written as a nonnegative integer combination of at most n incidence vectors of bases of $M(G)$ if and only if the answer to the given instance of PARTITION is 'yes.'

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