On the Structure of Bull-Free Perfect Graphs, 2: The Weakly Triangulated Case

Celina M. H. de Figueiredo[1]

Frédéric Maffray[2]

Oscar Porto[3]

Abstract. A bull is a graph obtained by adding a pendant vertex at two vertices of a triangle. A graph is perfectly orderable if it admits an ordering such that the greedy sequential method applied on this ordering produces an optimal coloring for every induced subgraph. Chvátal conjectured that every bull-free graph with no odd hole or antihole is perfectly orderable. In a previous paper we studied the structure of general bull-free perfect graphs, and reduced Chvátal's conjecture to the case of weakly triangulated graphs. Here we focus on weakly triangulated graphs. Our method lays out the structure of all bull-free weakly triangulated graphs. Its results have been used recently by Hayward to establish Chvátal's conjecture in full.

[1]Universidade Federal do Rio de Janeiro, Instituto de Matemática, Caixa Postal 68530, 21944 Rio de Janeiro, RJ, Brasil. `celina@cos.ufrj.br`. Partially supported by CNPq, CAPES and COFECUB.

[2]CNRS, Laboratoire Leibniz, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France. `frederic.maffray@imag.fr`. Partially supported by CNPq, CAPES and COFECUB (Brazil) and by the Air Force Office for Scientific Research under grant number F49620-93-1-0041 to Rutgers University Center for Operations Research (New Jersey).

[3]PUC-Rio, Dept. de Engenharia Elétrica, Rua Marquês de São Vicente 225, Predio Cardeal Leme, Sala 401, CEP 22453, Rio de Janeiro, RJ, Brasil. `oscar@ele.puc-rio.br`

1 Introduction

The bull is the graph shown on Figure 1. Following Berge [1], a graph is *perfect* if the vertices of every induced subgraph H can be colored with $\omega(H)$ colors, where $\omega(H)$ is the maximum clique size in H. Chvátal and Sbihi [4] proved that all bull-free graphs with no odd hole or odd antihole are perfect. Here a *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complement of a hole. Graphs with no odd hole and no odd antihole have become known as *Berge* graphs.

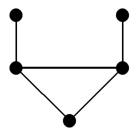

Figure 1: The bull

In a previous paper [6], we gave a new proof of the perfectness of bull-free Berge graphs. Some problems on the structure of bull-free Berge graphs remained open, in particular their relation with the class of so-called perfectly orderable graphs. Recall that a graph is *perfectly orderable* if it admits a linear ordering Ω of its vertices such that, for every induced subgraph H, applying the greedy sequential coloring algorithm on the vertices of H along the ordering Ω yields an optimal coloring of H. Chvátal [2] introduced perfectly orderable graphs and proved that they are perfect. A graph is called weakly triangulated if it contains no hole and no antihole. Hayward [7] proved that all weakly triangulated graphs are perfect. Our goal is a proof of the following conjectures.

Conjecture 1 (Chvátal [3]) *Every bull-free Berge graph with no antihole is perfectly orderable.*

Conjecture 2 *Every bull-free weakly triangulated graph is perfectly orderable.*

Clearly, Conjecture 2 is a special case of Conjecture 1. In [6], we proved the converse, i.e., the validity of Conjecture 2 implies that of Conjecture 1. Our purpose in this paper is to examine this second conjecture.

A classical result on perfectly orderable graphs deals with the notion of *homogeneous set* (i.e., a proper subset of at least two vertices such that every vertex not in the subset is adjacent to either all or none of the vertices in the set) [9]:

> *If every proper subgraph of a graph with a homogeneous set is perfectly orderable then so is the graph.*

Thus to prove that all graphs in a class are perfectly orderable it is sufficient to consider those with no homogeneous set.

In [6] we proved that every bull-free Berge graph either is perfectly orderable or has a homogeneous set or is weakly triangulated. Our main tool is the so-called Box Partition whose exact technical definition will be recalled in Section 2. Roughly, the Box Partition consists in a bipartite skeleton for the graph. The Box partition plays a crucial role in our study because of the following lemma.

Lemma 1 (The Perfectly Orderable Box Partition Lemma [6]) *Let G be a bull-free perfect graph with no antihole and no homogeneous set. If G admits a box partition then G is perfectly orderable.*

Our main results are the following theorems, whose proofs will be given in Section 3. A P_6 is a chordless path on six vertices.

Theorem 1 *Let G be a bull-free weakly triangulated graph that contains no homogeneous set. If G contains a P_6 then G has a box partition.*

Theorem 2 *Let G be a bull-free weakly triangulated graph that contains no homogeneous set. If G contains no P_6 then G or $\overline{G}$ has a box partition.*

After a first version of this paper appeared [5], Hayward proved (using Theorem 2):

Theorem 3 (Hayward [8]) *Let G be a bull-free weakly triangulated graph that contains no homogeneous set and no P_6. If $\overline{G}$ admits a box partition then G is perfectly orderable.*

Now, Theorems 1, 2 and 3 together imply the validity of Conjecture 2 and thus of Conjecture 1.

2 Boxes and Perfect Orderings

We will usually say "sees" (resp. "misses") instead of "is adjacent to" (resp. "is not adjacent to"). We say that a graph is trivial when it has only one vertex. The subgraph of a graph G induced by a subset of vertices S will be denoted by $G[S]$, but sometimes also by S when there is no ambiguity. For a subset S of vertices, $N(S)$ denotes the set of vertices of $G-S$ that have at least one neighbour in S. A chordless cycle, resp. path, on k vertices is denoted by C_k, resp. P_k. We will always say "component" for "connected component".

Given a subset of vertices S of a graph G, a vertex of $G - S$ is *partial* on S if it has at least one neighbour and at least one non-neighbour in S; in the opposite case it is *impartial on* S. For any subset B of vertices in a graph G, we let $M(B)$ denote the set of vertices of $G - B$ that are partial on B. In particular, a proper subset H of at least two vertices is homogeneous if and only if every vertex of $G - H$ is impartial on H, i.e., $M(H) = \emptyset$. Note that if H is a homogeneous set of G then it is also a homogeneous set of the complement graph $\overline{G}$. Note also that G is bull-free if and only if $\overline{G}$ is bull-free.

We call *broom* the graph consisting of a P_4, plus a fifth vertex adjacent to all vertices of the P_4, plus a sixth vertex adjacent to the fifth vertex only; see Figure 2. The next lemma generalizes a result of Reed and Sbihi [10] and will be useful.

Lemma 2 (The Broom Lemma [6]) *If a bull-free, C_5-free graph contains a broom then it has a homogeneous set which contains the P_4 of the broom.* □

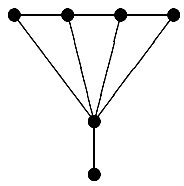

Figure 2: The broom

Definition (The Box Partition) *Let G be a graph with vertex set V. We call Box Partition any partition of V into at least two disjoint non-empty subsets called the boxes, inducing connected subgraphs which satisfy the following properties:*

(i) Each box is labelled either "odd" or "even" (each vertex will be labelled odd or even accordingly), and there is no edge between two odd boxes or between two even boxes;

(ii) For each box B such that $M(B) \neq \emptyset$, there exists in $V - B$ two auxiliary adjacent vertices a_B and a'_B, such that a_B sees all of B and misses all of $M(B)$, while a'_B sees all of $M(B)$ and misses all of B.

Remark *When G is a bull-free graph that admits a box partition with the above notation, the fact that a'_B sees every vertex of $M(B)$ is a consequence of the other facts given in Property (ii).* Indeed, suppose a'_B missed a vertex x of $M(B)$. By the definition of $M(B)$ and the connectedness of B there exist adjacent vertices u, v in B such that x sees u and misses v. But then a_B, u, v, x, a'_B would be a bull.

Lemma 3 (Additional properties of the boxes) *Let G be a bull-free graph which admits a box partition and contains no homogeneous set. Then:*

(iii) Every box is P_4-free;

(iv) If G contains no $\overline{C}_6$, then for every box B, any two adjacent vertices of B have comparable neighbourhoods in $M(B)$;

(v) If G contains no C_6 and no $\overline{C}_6$, then for every box B, any two vertices of B have comparable neighbourhoods in $M(B)$.

Proof. To prove (iii), suppose that a box B contains a P_4. If $M(B) = \emptyset$ then B is a homogeneous set, a contradiction. If $M(B) \neq \emptyset$ then Property (ii) implies the existence of the auxiliary vertices a_B, a'_B. Now the subgraph formed by these two vertices and a P_4 in B is a broom, and then by the Broom Lemma G should contain a homogeneous set, a contradiction.

To prove (iv), suppose on the contrary that some two adjacent vertices u, v in a box B have incomparable neighbourhoods in $M(B)$. So there exist a vertex x in $M(B) \cap N(u) - N(v)$ and a vertex y in $M(B) \cap N(v) - N(u)$. Since $M(B) \neq \emptyset$ the auxiliary vertices a_B, a'_B exist for B; note that a_B sees u and v and misses x and y, while a'_B sees x and y and misses u and v. If xy is not an edge of G then a_B, u, v, x, y is a bull. If xy is an edge then a_B, u, v, x, y, a'_B is a $\overline{C}_6$.

To prove (v), suppose on the contrary that some two vertices u, v in a box B have incomparable neighbourhoods in $M(B)$. So there exists a vertex x in $M(B) \cap N(u) - N(v)$ and a vertex y in $M(B) \cap N(v) - N(u)$. By (iv) we may assume that u, v are non-adjacent. Recall the auxiliary vertices a_B, a'_B for B, so that a_B sees u and v and misses x and y, while a'_B sees x and y and misses u and v. If xy is an edge of G then a'_B, x, y, u, v is a bull. So let us assume that x misses y. Since B is connected, there exists a path from u to v inside B; since B is P_4-free by (iii), we can write this path as uwv. Note that a'_B misses w. Vertex x misses w or else x, u, w, v, a'_B is a bull. Likewise y misses w or else y, v, w, u, a'_B is a bull. But now u, w, v, y, a'_B, x induce a C_6, a contradiction. So (v) is proved. □

Property (v) implies that, given a box B, the relation defined by the inclusion between the sets $M(B) \cap N(u)$ $(u \in B)$ is a total (but not necessarily strict) order on the vertices of B.

The Perfectly Orderable Box Partition Lemma stated in the Introduction was proved in [6]. For the sake of completeness let us recall the algorithm used to construct a perfect orientation. So let G be a bull-free perfect graph with no even antihole, and suppose that G admits a box partition and contains no homogeneous set. By additional property (iii) each box is P_4-free. Let $\mathrm{O}(B)$ be an arbitrary linear ordering of the vertices of B, for each box B. All edges of G are oriented according to the following rules:

- *Rule V0.* If an edge has its two endpoints in different boxes, orient it from its even extremity to its odd extremity.
- *Rule E1.* If x, y are in an even box B and x has strictly more neighbours than y in $M(B)$ then orient the edge xy from x to y.
- *Rule E2.* If x and y are in an even box B and have the same neighbourhood in $M(B)$ then orient xy as in $\mathrm{O}(B)$.
- *Rule O1.* If x, y are in an odd box B and x has strictly more neighbours than y in $M(B)$ then orient xy from y to x.
- *Rule O2.* If x, y are in an odd box B and have the same neighbourhood in $M(B)$, and if there exists a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$, then orient the edge xy from x to y.
- *Rule O3.* If x, y are in an odd box B and do not satisfy the hypotheses of Rules O1 and O2, then orient xy according to $\mathrm{O}(B)$.

Using the definition and additional properties of the box partition, it was proved in [6] that these rules are consistent and produce a perfect orientation for G.

3 Boxes in Bull-free Weakly Triangulated Graphs

Let us recall some technical results about bull-free graphs.

Observation 1 *Let $u_1u_2\cdots u_k$ be a chordless path P with $k \geq 5$ in a bull-free graph weakly triangulated graph G. Let x be a vertex of $G - P$. Then $N(x) \cap P$ is one of the following sets:*

- $\emptyset$*;*
- P*;*
- $\{u_i, u_{i+1}, u_{i+2}\}$ *for some i with $1 \leq i \leq k-2$;*
- $\{u_1, u_2, u_3, u_5\}$*, or, symmetrically,* $\{u_{k-4}, u_{k-2}, u_{k-1}, u_k\}$*;*
- $\{u_1, u_2\}$ *or* $\{u_1, u_2, u_4\}$*, or, symmetrically,* $\{u_{k-1}, u_k\}$ *or* $\{u_{k-3}, u_{k-1}, u_k\}$*;*
- $\{u_i, u_{i+2}, u_{i+4}, \ldots, u_{i+2r}\}$ *for some ir such that $1 \leq i \leq i + 2r \leq k$.*

The proof of this observation is easy and we only sketch it. If x sees at least four consecutive vertices of P then x must see all of P or else a bull can be found in $P \cup \{x\}$. If x sees three but not four consecutive vertices then it cannot see any other vertex of P, for otherwise a bull is found, except in the case described in the fourth item of the Observation. If x sees two but not three consecutive vertices then these two must be u_1, u_2 or, respectively, u_{k-1}, u_k or else a bull is found; then x may also see u_4, resp. u_{k-3}, but no other vertex of P or else again a bull appears. Finally when x does not see two consecutive vertices of P then $N(x) \cap P$ is the last item described in the Observation, or else an odd or even hole is found in $P \cup \{x\}$.

3.1 Proof of Theorem 1

This proof parallels the proof given in [6] which established the existence of a box partition in a bull-free perfect graph with no homogeneous set provided that the graph contains an even hole: the box partition was "built around" a shortest even hole. Here the path P_6 will play a role similar to that even hole.

So let G be a bull-free weakly triangulated graph containing a P_6 but containing no homogeneous set. In particular G is broom-free, by the Broom Lemma. Let k be the length of a longest path in G. So there exists an "inflated P_k", i.e., k disjoint non-empty sets $V_1, \ldots, V_k$ such that, for $i = 1, \ldots, k-1$, every vertex in V_i sees every vertex in V_{i+1} and there are no other edges between different V_j's.

Now we partition the rest of $G - P$. Since G is broom-free, we may assume that no vertex sees all of P.

- Let Z be the set of vertices that have no neighour on P;
- Let Y_1 be the set of vertices of $G - P$ that have at least one neighbour in each of V_1, V_2, V_4, and see no vertex of $P - (V_1 \cup V_2 \cup V_4)$. Let Y_k be the obvious symmetric set.
- Let Y_2 be the set of vertices that have at least one neighbour in each of V_1, V_2, V_3, V_5, and have no neighbour in $P - (V_1 \cup V_2 \cup V_3 \cup V_5)$. Let Y_{k-1} be the obvious symmetric set.
- For each i, r such that $1 \leq i \leq i + 2r \leq k$, let $A_{i,i+2r}$ be the set of vertices that have at least one neighbour in each of $V_i, V_{i+2}, \ldots, V_{i+2r}$ and have no neighbour in $P - (V_i \cup V_{i+2} \cup \cdots \cup V_{i+2r})$.
- Define A_1 as the union of all $A_{i,i+2r}$'s with i even, and, symmetrically, A_2 as the union of all $A_{i,i+2r}$'s with i odd.

A routine application of the Observation yields the following lemma.

Lemma 4 *Every vertex of G belongs to exactly one of the sets $V_1, \dots, V_k, Y_1, Y_2, Y_{k-1}, Y_k, A_1, A_2, Z$.*

Now let us present some further facts about this partition. Here we will call symmetry the fact that any statement that holds for V_i (resp. Y_i, A_j) also holds for V_{k+1-i} (resp. Y_{k+1-i}, A_{3-j}), whenever this is obvious. We will let v_i stand for an arbitrary vertex in V_i. It will be convenient to let k', resp. k'', stand for the largest odd, resp. even, integer smaller than or equal to k, so that $\{k', k''\} = \{k-1, k\}$.

Lemma 5 *The above sets satisfy the following properties:*

5.1. Every vertex in Y_1 sees all of V_2, and by symmetry, every vertex of Y_k sees all of V_{k-1};

5.2. Every vertex of Y_2 sees all of $V_1 \cup V_3$, and by symmetry every vertex of Y_{k-1} sees all of $V_{k-2} \cup V_k$;

5.3. Every vertex of Y_1 sees all of Y_2, and by symmetry every vertex of Y_{k-1} sees all of Y_k;

5.4. For any i and $r \geq 2$, every vertex of $A_{i,i+2r}$ sees all of $V_{i+2} \cup \dots \cup V_{i+2r-2}$;

5.5. For any $i \geq 3$ and any r such that $i + 2r \leq k$, every vertex of $A_{i,i+2r}$ is impartial on each component of V_i;
by symmetry, for any $j \leq k-2$ and any r such that $1 + 2r \leq j$, every vertex of $A_{j-2r,j}$ is impartial on each component of V_j;

5.6. For each i such that $3 \leq i \leq k-2$, the set V_i is stable;

5.7. There is no edge between A_1 and Y_1;

5.8. There is no edge between A_2 and Y_2;

5.9. By symmetry, there is no edge between A_1 and $Y_{k'}$ or between A_2 and $Y_{k''}$;

5.10. There is no edge between Z and $Y_1 \cup Y_2 \cup Y_{k-1} \cup Y_k$.

Proof. If 5.1 fails, we can find vertices $y_1 \in Y_1$, $u_1 \in N(y_1) \cap V_1$, $u_4 \in N(y_1) \cap V_4$, $u_2 \in V_2 - N(y_1)$, but then y_1, u_1, u_2, v_3, u_4 induce a C_5. Similarly, if 5.2 fails, a bull or a C_5 is easily found.

To prove 5.3, suppose on the contrary that some y_1 in Y_1 misses some y_2 in Y_2. By the definition of these sets there exist vertices $u_1 \in V_1 \cap N(y_1)$, $u_4 \in V_4 \cap N(y_1)$, $u_5 \in V_5 \cap N(y_2)$; moreover y_1 misses u_5 and y_2 misses u_4. Note that y_2 sees u_1 by 5.2. But then u_1, y_1, u_4, u_5, y_2 form a C_5, a contradiction.

Next, if 5.4 fails an even hole is easily found.

Next, if 5.5 is false there exist adjacent vertices u, v in V_i and a vertex x in $A_{i,i+2r}$ that sees u and misses v, but then $x, u, v, v_{i-1}, v_{i-2}$ form a bull.

To establish 5.6, it suffices to prove that every non-trivial connected component B of V_i is homogeneous. So suppose on the contrary that there exist adjacent vertices u, v in B and a vertex x in $G - B$ that sees u and misses v. The definition of the partition and 5.5 imply that up to the symmetry only two cases are possible for this situation: either $i = 4$ and $x \in Y_1$; or $i = 5$ and $x \in Y_2$. However, in the first case we see that x, u, v, v_5, v_6 form a bull. Similarly, in the second case x, u, v, v_6, v_7 form a bull, except if v_7 does not exist, i.e., if $k = 6$; but then $i = k - 1$ and so we have nothing to prove.

To prove 5.7 suppose on the contrary that there exists an edge a_1y_1 with $a_1 \in A_1, y_1 \in Y_1$. By the definition of these sets, there exist a neighbour u_1 of y_1 in V_1 and a neighbour u_4 of y_1 in V_4, and a_1 sees no vertex in the V_i's with odd i. Recall from 5.1 that y_1 sees every v_2. Hence a_1 too must see every v_2 of else y_1, u_1, v_2, v_3, a_1 would be a bull. Now a_1 must have a neighbour in V_4 for otherwise 5.5 implies that a_1 sees no vertex in the V_i's with i even and $i \geq 4$, and then we could add a_1 to V_1 and augment the inflated P_k. If a_1 sees u_4 then y_1, u_4, a_1, v_5, u_1 is a bull. So we may assume that a_1 misses u_4, sees another vertex w_4 in V_4, and that y_1 misses w_4. But then u_4 misses w_4 by 5.6 and the vertices v_2, y_1, a_1, u_4, w_4 form a bull, a contradiction. So 5.7 is established.

To prove 5.8 suppose on the contrary that there exists an edge y_2a_2 with $y_2 \in Y_2, a_2 \in A_2$. By the definition of these sets there exists a neighbour u_2 of y_2 in V_2 and a neighbour u_5 of y_2 in V_5, and a_2 sees no vertex in the V_i's with even i. Recall from 5.2 that y_2 sees each v_1 and each v_3. Hence a_2 must see every v_3 or else y_2, u_2, v_3, v_4, a_2 would be a bull. Then a_2 sees every v_1 or else y_2, a_2, v_3, v_4, v_1 would be a bull. Now, a_2 must have a neighbour in V_5, for otherwise 5.5 implies that a_2 sees no vertex in the V_i's with i odd and $i \geq 5$, and then we would be able to augment the inflated P_k by adding a_2 to V_2. If a_2 sees u_5 then a_2, y_2, u_5, u_2, v_6 is a bull. So we may assume that a_2 misses u_5, sees another vertex w_5 of V_5, and that y_2 misses w_5. If u_5 misses w_5 then v_3, y_2, a_2, u_5, w_5 is a bull. If u_5 sees w_5 then $v_3, y_2, a_2, v_4, u_5, w_5$ form a $\overline{C}_6$, a contradiction. So 5.8 is established.

Finally, suppose that there is an edge zy with $z \in Z, y \in Y_1 \cup Y_2 \cup Y_{k-1} \cup Y_k$. If $y \in Y_1$ there exists a neighbour u_1 of y in V_1, and y_1 sees every v_2 and misses every v_3. But then y, u_1, v_2, v_3, z is a bull, a contradiction. The proof is the same if $y \in Y_2$ with all subscripts shifted by $+1$, and for $y \in Y_{k-1} \cup Y_k$ by symmetry. So 5.10 holds. □

Now we define the boxes for the whole graph as follows. Write $W_i = V_i \cup Y_i$ for $i = 1, 2, k-1, k$ and $W_i = V_i$ for $3 \leq i \leq k-2$. Then define

$$\begin{aligned} L_1 &= W_1 \cup W_3 \cup W_5 \cup \cdots \cup W_{k'} \\ L_2 &= W_2 \cup W_4 \cup W_6 \cup \cdots \cup W_{k''} \cup A_2. \end{aligned}$$

Note that $L_2 = N(L_1)$ by the definition of the partition and by 5.7 and 5.9. Then, for $j \geq 3$, define

$$L_j = N(L_{j-1}) - L_{j-2}.$$

In other words, the L_j's are the levels of a breadth-first search decomposition whose first level is chosen as L_1. We define as odd boxes the connected components of the L_j's with odd j, and as even boxes the connected components of the L_j's with even j.

Lemma 6 *These boxes form a box partition.*

Proof. It is clear that Property (i) of the Box Partition holds. Let us prove now Property (ii).

Claim 1 *Property (ii) of the Box Partition holds for each box B in L_1.*

Proof. If B is a connected component of W_i for some odd i with $3 \leq i \leq k' - 2$ then, by 5.6, B is trivial and we have nothing to prove.

Now assume that $B \subseteq W_1 = V_1 \cup Y_1$. We know that any v_2 sees all of B and that v_3 misses all of B. Moreover, the definition of the partition and Facts 5.2, 5.3, 5.7, and 5.10 imply that $M(B) \subseteq A_2 \cup V_4$. It follows that v_2 misses all of $M(B)$, and then the Remark implies that v_3 sees all of $M(B)$. So the vertices $a_B = v_2$ and $a'_B = v_3$ satisfy Property (ii) for B.

If k is odd and B is in W_k then the proof is the same as for W_1, by symmetry. Finally, assume that k is even and B is in $W_{k-1} = V_{k-1} \cup Y_{k-1}$. We know that any v_{k-2} sees all of B and that any v_{k-3} misses all of B. Moreover, the definition of the partition and established properties imply that $M(B) \subseteq A_2 \cup V_{k-4}$. Hence v_{k-2} misses all of $M(B)$, and v_{k-3} sees all of $M(B)$ by the Remark. So the vertices $a_B = v_{k-2}$ and $a'_B = v_{k-3}$ satisfy Property (ii) for B. □

Claim 2 *Property (ii) of the Box Partition holds for each box B in L_2, with a_B in an odd V_i.*

Proof. Let B be a box in L_2. Recall that there is no edge between A_2 and $L_2 - A_2$ by the definition of A_2 and by 5.8 and 5.9. If B is in $L_2 - A_2$ then B is in a W_i and the proof is the same as in the previous claim with odd and even subscripts shifted. We now assume that $B \subseteq A_2$. By the definition of A_2 every vertex of B has at least one neighour in an odd W_i. For any vertex x in an odd W_i we let $\tilde{x}$ denote an arbitrary neighbour of x in an even V_j; obviously such a vertex always exists, with $j = i \pm 1$, by the definition of the W_i's and by 5.1 and the symmetric fact. We claim that

If a, b are any two vertices in $N(B) \cap L_1$ then the sets $N(a) \cap B$ and $N(b) \cap B$ are comparable by inclusion. (1)

Suppose that this is false: there exist two vertices a, b in L_1 and two vertices u, v in B such that u sees a and misses b while v see b and misses a. By the connectivity of B, there exists a path Q from u to v in B. We choose these four vertices a, b, u, v so that this path is as short as possible.
First note that a misses b. Indeed, if a sees b then they must be in the same box B in L_1, and thus u, v are in $M(B)$. But then this violates Property (v) for B.
Now, since a misses b, there exists a path R from a to b whose vertices are alternately in odd and even V_i's. We may assume that R is chordless and even. We claim that u misses v. Indeed, if u sees v then $R + au + uv + vb$ is an odd cycle with at least five vertices, and since G has no odd hole this cycle must contain a triangle. Such a triangle can only be of the type xuv where x is a vertex in $R \cap L_1$. But then x, u, v, a, b form a bull.
Now, since u misses v, the path Q can be written as $uw \cdots v$. Vertex a misses w or else there will exist three consecutive vertices u', w', v' of Q such that a see u' and w' and misses v', and then $a, u', w', v', \tilde{a}$ will be a bull. Then b must miss w or else the four vertices a, b, u, w would contradict the choice of a, b, u, v. Vertex w has a neighbour c in L_1 by the definition of B. Vertex c must see u, for otherwise the four vertices a, c, u, w would contradict the choice of a, b, u, v. Then c must see all of Q for otherwise we would again find a bull $c, u', w', v', \tilde{c}$ for some three consecutive vertices u', w', v' of Q. This implies $Q = uwv$, for otherwise we could find a broom made up of $\tilde{c}, c$ and any four consecutive vertices of Q. Then a must see $\tilde{c}$ for otherwise either $c, u, w, a, \tilde{c}$ is a bull or $\tilde{c}, c, a, u, w, v$ is a broom. Likewise b must see $\tilde{c}$. But now $\tilde{c}, a, u, w, v, b$ induce a C_6. So (1) is established.

Now let a be a vertex of L_1 that sees the most vertices of B. It must be that a sees all of B, or else (1) would be contradicted by any b in $L_1 \cap N(B - N(a))$. We choose such an a with the following restriction: if possible, choose a in an odd V_i with $3 \leq i \leq k' - 2$. We claim that

$$\textit{Vertex } a \textit{ misses all vertices of } M(B). \tag{2}$$

Suppose on the contrary that a sees a vertex x of $M(B)$, i.e., x is in $G - B$ and there exist adjacent vertices u, v in B such that x sees u and misses v. Necessarily x is in L_1; in fact x and a must be in a non-trivial box of L_1, so they are either in W_1 or in $W_{k'}$. This and the restriction on the choice of a now implies that every vertex in an odd V_i with $3 \leq i \leq k' - 2$ must have a non-neighbour in B. Let us assume that a, x are in W_1; the argument will be obviously the same when they are in $W_{k'}$. Recall that a and x both see v_2 and miss v_3. Then v_3 must see v or else v_2, x, a, v, v_3 would be a bull. Then v_3 must see u or else v_3, v, u, x, v_2 would be a C_5. As pointed out above, v_3 has a non-neighbour w in B; any such w sees a and misses v_2. Notice that B is P_4-free, or else a P_4 of B plus a and v_2 would form a broom. Choose w so that the distance from w to u and v in B is minimized. If w see exactly one of u, v then v_3, u, v, w, v_2 would be a bull. If w misses both u, v then a shortest path from w to u, v in B can be written as wyu or wyv since

B is P_4-free; then v_3 sees y or else the choice of w is contradicted; but then v_3, u, y, w, v_2 or v_3, v, y, w, v_2 is a bull. So, it must be that w sees both u and v. Then w sees x or else a, v, w, x, v_3 is a bull; but now v_3, v, w, x, v_2 is a C_5. By this contradiction, (2) is proved.

Finally, recall that $\tilde{a}$ misses all of B since it is in $L_2 - A_2$. So $\tilde{a}$ sees all of $M(B)$ by the Remark following the definition of the box partition. In conclusion, $a_B = a$ and $a'_B = \tilde{a}$ satisfy the claim for B. $\square$

Claim 3 *Property (ii) of the Box Partition holds for each box B in L_j, $j \geq 2$, with $a_B \in L_{j-1}$. Moreover, we may assume that $a'_B \in L_{j-2}$ if $j \geq 3$.*

Proof. We prove this by induction on j. For $j = 2$ this is the preceding claim. Now assume $j \geq 3$.

First let us prove a technical fact:

> *If a, b are two non-adjacent vertices in L_{j-1}, there exists an even chordless path $R_{a,b}$ between them, whose vertices are alternately in odd L_i's and even L_i's and are all in $L_{j-1} \cup L_{j-2} \cup \cdots \cup L_1$. Moreover, if $j \geq 4$ all the interior vertices of $R_{a,b}$ miss all of L_j.* (3)

The proof of this fact is as follows. By the definition of the L_j's, there exists a chordless path $R_a = a_1 a_2 \cdots a_j$ such that $a_i \in L_i$ $(i = 1, \ldots, j)$ and $a_j = a$, and a chordless path $R_b = b_1 b_2 \cdots b_j$ such that $b_i \in L_i$ $(i = 1, \ldots, j)$ and $b_j = b$. If there exists a pair of such paths R_a, R_b for which $a_i = b_i$ for some i, then we pick the pair for which this i is largest, and take $R_{a,b} = a_j a_{j-1} \cdots a_i b_{i+1} \cdots b_j$. Note that there is no chord of the type $a_h b_{h+1}$ or $a_{h+1} b_h$ in $R_{a,b}$, for otherwise we could find a new pair of paths with an even larger i. Also there is no chord of the type $a_h b_h$, for otherwise these two vertices a_h, b_h should be in a box D in L_h, with obviously $h < j$, and then the four vertices $a_h, b_h, a_{h+1}, b_{h+1}$ would violate Property (v) for D which holds by the induction hypothesis on j. Hence R has the desired property.
Now assume that such a pair of paths cannot be found. Since a_1, b_1 are in L_1, either they are adjacent or there exists between them a path S whose interior vertices are alternately in odd V_i's (in L_1) and even V_i's (in L_2). We choose all these paths so that in either case the path $R_{a,b} = R_a + ab + R_b$ or $R_a + S + R_b$ is as short as possible.
If a_1 is adjacent to b_1, then these two vertices lie in a box D in L_1. Note that a_2 misses b_1 and a_1 misses b_2, for otherwise we could take $a_1 = b_1$ and go back to the case when R_a and R_b intersect. But now the vertices a_1, b_1, a_2, b_2 violate Property (v) for D. So a_1 misses b_1. Hence we may assume that S has even length. In consequence $R_{a,b}$ also has even length. Now $R_{a,b}$ is chordless, for otherwise it is a routine matter to check as above that any chord would yield an even shorter $R_{a,b}$ or a contradiction to Property (v) for some box in a level L_h with $h < j$. The last sentence in (3) is obviously true as $R_{a,b}$ may contain vertices of L_{j-1} only if $j - 1 = 2$. So (3) is established.

We claim that

If a, b are any two vertices in $N(B) \cap L_{j-1}$ then the sets $N(a) \cap B$ and $N(b) \cap B$ are comparable by inclusion. (4)

The proof of this fact is the same as the proof of (1) in the preceding claim, with the following adaptation: L_2 is replaced by L_j, and L_1 is replaced by L_{j-1}; the path R is replaced by the path $R_{a,b}$ given by (3); for each x in L_{j-1}, the role of $\tilde{x}$ can be played by any vertex in $N(x) \cap L_{j-2}$. So (4) is established.

Now, if a is any vertex of L_{j-1} that sees the most vertices of B, it must be that a sees all of B, or else (4) would be contradicted by any b in $L_{j-1} \cap N(B - N(a))$. Then we choose an arbitrary vertex a of L_{j-1} that sees all of B, with the following restriction: if $j = 3$ and if possible, we choose a in an even V_i with $4 \leq i \leq k'' - 2$. Now we claim that

Vertex a misses all vertices of $M(B)$. (5)

To prove this, suppose on the contrary that a sees a vertex x of $M(B)$, so that there exist adjacent vertices u, v in B such that x sees u and misses v. Necessarily x and a lie in a non-trivial box D of L_{j-1}. Recall from the induction hyopthesis that there exist auxiliary vertices $a_D \in L_{j-2}$ and a'_D such that a_D sees x and a and misses v, while a'_D sees a_D and v and misses x and a. The fact that a'_D sees a_D and v, which are respectively in L_{j-2} and in L_j, implies that a'_D is in L_{j-1}. In turn, this and the induction hypothesis imply $j - 1 = 2$, for otherwise a'_D should be in L_{j-3}. So D is a non-trivial box in L_2, and so D is in W_2 or in $W_{k''}$ or in A_2. In either case, this latter fact and the restriction on the choice of a implies that every vertex in an even V_i with $4 \leq i \leq k'' - 2$ misses at least one vertex of B. To simplify, we may continue the proof in the case $D \subseteq W_2$; the argument in the other two cases will be evidently similar. At this point however, the proof goes on exactly as the proof for (2) with all subscripts shifted by $+1$, and we omit it. Hence (5) is proved.

Finally, note again that $\tilde{a}$ misses all of B since it is in L_{j-2}, and so $\tilde{a}$ sees all of $M(B)$ by the Remark. In conclusion, $a_B = a$ and $a'_B = \tilde{a}$ satisfy the claim for B. $\square$

The three claims that precede imply that every box of G satisfies Property (ii). So we have obtained a Box Partition for G, and this concludes the proof of Theorem 1. $\square$

3.2 Proof of Theorem 2

To prove this theorem, we will again define a partition of the vertex set of the graph G. However there will be some cases in which this does not yield a box partition for G. In such cases we will look at the complement and find a box partition for $\overline{G}$.

Call F_0 the graph with six vertices a, b, c, d, e, f and seven edges $ab, ac, bc, bd, ce, de, ef$; see Figure 3.

Figure 3: The graph F_0 and its complement $\overline{F_0}$

Lemma 7 (Chvátal and Sbihi [4]) *For any graph G the following conditions are equivalent:*

- *G contains no bull, no F_0 and no $\overline{F_0}$;*
- *Every induced subgraph of G either contains a homogeneous set, or is triangle-free, or is co-triangle-free.*

Let G for a bull-free weakly triangulated graph which contains no homogeneous set and no P_6. We want to prove that G or $\overline{G}$ has a box partition. We can assume that G contains no $\overline{P}_6$, for otherwise Theorem 1 yields a box partition in $\overline{G}$.

If G contains no P_5 then G contains no F_0 and no $\overline{F_0}$, and it follows from Lemma 7 that G or $\overline{G}$ is bipartite, in which case it has a trivial box partition. Now we assume that G contains a P_5.

Since G contains a P_5, more generally there exist five non-empty disjoint subsets of vertices $V_1, \dots, V_5$ that form an "inflated P_5" in the sense that for $i = 1, \dots, 4$, every vertex in V_i is adjacent to every vertex in V_{i+1}, and there is no other edge between two different V_j's. We will write $P = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$. Among all possible such P's, we choose one that is maximal according to the following criteria. First, consider the P's that maximize $V_2 \cup V_4$ with respect to set-inclusion. Next, among all P's selected by the first criterion, consider those that maximize $V_1 \cup V_5$. Finally, among all P's selected by the second criterion, choose one that maximizes V_3.

Now let us define the following subsets of vertices:

- Let W be the set of vertices in $V - P$ that have at least one neighbour in $V_2 \cup V_4$ and have no neighbour in $V_1 \cup V_3 \cup V_5$;
- Let X be the set of vertices in $V - P$ that have at least one neighbour in V_3 and have no neighbour in $P - V_3$;
- Let Y_1 be the set of vertices in $V - P$ that see all of V_2, have at least one neighbour in V_1 and one neighbour in V_4, and have no neighbour in $V_3 \cup V_5$;

- Let Y_5 be the set of vertices in $V-P$ that see all of V_4, have at least one neighbour in V_5 and one neighbour in V_2, and have no neighbour in $V_1 \cup V_3$;
- Let T be the set of vertices in $V-P$ that see all of $V_1 \cup V_3 \cup V_5$ and miss at least one vertex in $V_2 \cup V_4$;
- Let A be the set of vertices in $V-P$ that see all of P;
- Let Z be the set of vertices in $V-P$ that see no vertex in P.

¿From now on we let v_i stand for an arbitrary of V_i $(i=1,\dots,5)$. These twelve sets will be called a P_5*-partition* of G. The word partition is justified by the following lemma.

Lemma 8 *Every vertex of G lies in exactly one of the twelve sets V_1, V_2, V_3, V_4, V_5, W, X, Y_1, Y_5, T, A, Z.*

Proof. It is a routine matter to check that these twelve sets are disjoint. So we only have to show that every vertex u of $V-P$ lies in one of the remaining seven sets. Denote by I_u the set of subscripts i for which $N(u) \cap V_i \neq \emptyset$. Obviously $0 \leq |I_u| \leq 5$. Also let u_i denote an arbitrary vertex of $V_i \cap N(u)$ if this set is non-empty; and let u'_i denote an arbitrary vertex of $V_i - N(u)$ if this set is non-empty.

If $|I_u| = 0$ then u lies in Z.

Assume $|I_u| = 1$. If $I_u = \{1\}$ then $uu_1v_2v_3v_4v_5$ is a P_6, a contradiction. By symmetry, the same contradiction appears if $I_u = \{5\}$. So it must be that I_u is equal to either $\{2\}, \{3\}$ or $\{4\}$, and then u lies in W or X.

Assume $|I_u| = 2$. If $I_u = \{1,2\}$, then u must actually see every vertex in V_2, or else $uu_1u'_2v_3v_4v_5$ is a P_6, a contradiction. But now the five sets $V_1 \cup \{u\}, V_2, V_3, V_4, V_5$ form an inflated P_5 which is as good as P for the first criterion and better for the second criterion, a contradiction. By symmetry, it cannot be that $I_u = \{4,5\}$.
If $I_u = \{2,3\}$, then u, u_2, u_3, v_1, v_4 is a bull. The same contradiction occurs if $I_u = \{3,4\}$.
If $I_u = \{1,3\}$ then, as above, it must actually be that u sees all vertices of V_3, or else a P_6 $uu_1v_2u'_3v_4v_5$ is found. But then, the five sets $V_1 \cap N(u), V_2 \cup \{u\}, V_3, V_4, V_5$ form an inflated P_5 which is better than P for the first criterion, a contradiction. The same contradiction happens if $I_u = \{3,5\}$.
If $I_u = \{2,4\}$ then u lies in W.
If $I_u = \{1,4\}$ (resp. $\{1,5\}$) then the vertices u, u_1, v_2, v_3, u_4 (resp. $u, u_1, v_2, v_3, v_4, u_5$) induce a C_5 (resp. a C_6), a contradiction. The same happens if $I_u = \{2,5\}$.

Assume $|I_u| = 3$. If $I_u = \{1,2,3\}$ then u must see all of V_1 or else u, u'_1, u_2, u_3, v_4 is a bull. If u also sees every vertex of V_3 then we could add u to V_2 and thus obtain an inflated P_5 that is better than P for the first criterion, a contradiction. Hence u must have a non-neighbour u'_3 in V_3. Then u must see all vertices in V_2 or else again

$uu_1u_2'u_3'v_4v_5$ is a P_6. But now the five sets $V_1 \cup \{u\}, V_2, V_3 - N(u), V_4, V_5$ form an inflated P_5 which is as good as P for the first criterion and better for the second, a contradiction. The same happens if $I_u = \{3, 4, 5\}$.
If $I_u = \{2, 3, 4\}$ then u must see all vertices of V_2 or else u, u_2', u_3, u_4, v_5 would form a bull. By symmetry u must see all of V_4. But now we could replace V_3 by $V_3 \cup \{u\}$ and obtain an inflated P_5 which is as good as P for the first two criteria and better for the third, a contradiction.
If $I_u = \{1, 2, 4\}$ then u must see all vertices in V_2, or else u, u_1, u_2', v_3, u_4 induce a C_5. So u lies in Y_1. By symmetry, if $I_u = \{2, 4, 5\}$ then u lies in Y_5.
If $I_u = \{1, 2, 5\}$ (resp. $I_u = \{1, 4, 5\}$) then a C_5 is found as in the case $I_u = \{2, 5\}$ (resp. $I_u = \{1, 4\}$) above.
If $I_u = \{2, 3, 5\}$ (resp. $I_u = \{1, 3, 4\}$) then a bull is found as in the case $I_u = \{2, 3\}$ (resp. $I_u = \{3, 4\}$) above.
If $I_u = \{1, 3, 5\}$ then u must see all vertices of V_3, for otherwise the vertices $u, u_1, v_2, u_3', v_4, u_5$ induce a hole. Then u must see all vertices in V_5, or else the five sets $V_1 \cap N(u), V_2 \cup \{u\}, V_3, V_4, V_5 - N(u)$ form an inflated P_5 that is better than P for the first criterion. By symmetry u must see all of V_1. So u is in T.

Assume that $|I_u| = 4$. If $I_u = \{1, 2, 3, 4\}$ then the vertices u, u_1, u_3, u_4, v_5 induce a bull. By symmetry it also cannot be that $I_u = \{2, 3, 4, 5\}$. If $I_u = \{1, 2, 4, 5\}$ then u, u_1, u_2, v_3, u_5 form a bull.
If $I_u = \{1, 3, 4, 5\}$, then u must see all vertices in V_3 or else u, u_1, v_2, u_3', u_4 induce a bull. Also u must see all vertices in V_5 or else u, u_1, u_3, u_4, u_5' induce a bull. Then u must see all vertices in V_1 for otherwise the five sets $V_1 - N(u), V_2, V_3, V_4 \cup \{u\}, V_5$ form an inflated P_5 which is better than P for the first criterion. Now u lies in T. A similar argument holds if $I_u = \{1, 2, 3, 5\}$.

Assume that $|I_u| = 5$. Then u must see all vertices in V_1 or else u, u_1', u_2, u_3, u_5 induce a bull. Likewise, u must see all vertices in V_5. Also, u must see all vertices in V_3 for otherwise u, u_1, u_2, u_3', u_5 induce a bull. Now u lies in either T or A. This concludes the proof of Lemma 8. □

Note that this proof of Lemma 8 yields a polynomial-time algorithmic procedure for finding such a partition. More precisely, we can start from an arbitrary P_5 and add the other vertices one by one; whenever a vertex cannot be added into one of the seven other sets, the proof shows that it is possible to obtain a better inflated P_5, and we restart the procedure from this new basis.

Now we would like to examine more closely the adjacency between the twelve sets. Let T' be the subset of vertices of T that have at least one neighbour in $V_2 \cup V_4$, and $T_0 = T - T'$, i.e., every vertex of T_0 misses all of $V_2 \cup V_4$. Let Z_1 be the set of vertices of Z that have a neighbour in $V - Z$, and $Z_2 = Z - Z_1$. Then,

Lemma 9 *The above sets satisfy the following properties:*

9.1. A is empty.

9.2. There is no edge between the three sets Y_1, Y_5, W.

9.3. There is no edge between T and X.

9.4. Every vertex of T sees every vertex of $Y_1 \cup Y_5$.

9.5. There is no edge between T_0 and T'.

9.6. There is no edge between Z and $W \cup Y_1 \cup Y_5 \cup T'$.

9.7. One of the sets X and T' is empty.

9.8. V_3 is a stable set.

9.9. T_0 is a stable set.

9.10. If $w \in W$ and $t \in T$ are non-adjacent vertices, then w misses all of V_4, t misses all of V_2 and sees all of V_4, and every vertex of T misses all of V_2 (or vice-versa with subscripts 2 and 4 swapped).

9.11. W is a stable set.

9.12. Z_2 is a stable set.

Proof. To prove 9.1, we first claim that A sees all of $V - A$. Indeed, if this is not true there are two non-adjacent vertices $a \in A$ and $u \in V - A$. Vertex u is not in $V_1 \cup \cdots \cup V_5$ by the definition of A. If u is in Y_1, Y_5, X, W, T_0, then it is a routine matter to check that a, u and some three vertices from $V_1 \cup \cdots \cup V_5$ induce a bull. If u is in Z then the vertices a, u, v_1, v_2, v_3, v_4 induce a broom in $\overline{G}$, a contradiction. Finally, if u is in T' then by symmetry we may assume that u has a neighbour u_2 in V_2 and a non-neighbour u_4 in V_4, but then a, u, u_2, v_3, u_4, v_5 induce a $\overline{P}_6$. So the claim is true. It follows that if $A \neq \emptyset$ the set $V - A$ is a homogenous set, a contradiction.

To prove 9.2, first suppose that there is an edge y_1y_5 with $y_1 \in Y_1$, $y_5 \in Y_5$. By the definition of Y_5, there exists a vertex u_2 in $V_2 \cap N(y_5)$, and a vertex u_5 in $V_5 \cap N(y_5)$. But then y_1, u_2, v_3, y_5, u_5 is a bull. Secondly, suppose that there is an edge y_1w with $y_1 \in Y_1$, $w \in W$. Let u_1 be a vertex in $V_1 \cap N(y_1)$. Vertex w must see every v_2 or else u_1, y_1, v_2, w, v_3 is a bull. Now w must have a neighbour w_4 in V_4, or else w could have been added to V_1 during the construction of the inflated P_5. Vertex w_4 misses y_1 or else w, y_1, w_4, u_1, v_5 would be a bull. Let u_4 be any vertex in $V_4 \cap N(y_1)$. This u_4 exists by the definition of Y_1, and we may assume that it misses w or else we could take $w_4 = u_4$. Then u_4 must see w_4 for otherwise v_5, w_4, w, y_1, u_4 form a hole; but then $y_1, v_2, w, u_4, w_4, v_3$ form a $\overline{C}_6$. Now 9.2 is established.

To prove 9.3 suppose on the contrary that there is an edge tx with $t \in T$, $x \in X$. By the definition of T, vertex t has a non-neighbour in $V_2 \cup V_4$, say t misses some $u_2 \in V_2$. But then x, v_3, t, u_2, v_5 is a bull.

To prove 9.4 suppose on the contrary and without loss of generality that a vertex $t \in T$ misses a vertex $y_1 \in Y_1$. By the definition of Y_1 there exists a vertex u_1 in $V_1 \cap N(y_1)$ and a vertex u_4 in $V_4 \cap N(y_1)$. Vertex t must miss v_2 or else v_3, v_2, t, y_1, v_5 is a bull, and t must see u_4 or else t, u_1, y_1, u_4, v_3 induce a hole. But then $t, v_3, u_4, u_1, v_2, y_1$ induce a $\overline{C}_6$.

To prove 9.5 suppose on the contrary that there exists such an edge t_0t' with $t_0 \in T_0, t' \in T'$. By the definition of T and T', vertex t' admits a neighbour in one of V_2, V_4 and a non-neighbour in the other one, say $u_2 \in V_2 \cap N(t')$ and $u_4 \in V_4 - N(t')$. But then t_0, v_5, t', u_4, u_2 is a bull.

To prove 9.6 let z be any vertex in Z, and first suppose that there is an edge zw with $w \in W$. We may assume that w has a neighbour u_2 in V_2 and a non-neighbour u_4 in V_4; but then $zwu_2v_3u_4v_5$ is a P_6, a contradiction. Now suppose that there is an edge zy_1 with $y_1 \in Y_1$. By the definition of Y_1 there exists a vertex u_1 in $V_1 \cap N(y_1)$. But then u_1, v_2, y_1, v_3, z induce a bull. Suppose finally that there exists an edge zt' with $t' \in T'$. By the definition of T' and by symmetry we may assume the existence of a vertex u_2 in $V_2 \cap N(t')$ and of a vertex u_4 in $V_4 - N(t')$. But then u_2, v_3, t', u_4, z induce a bull.

To prove 9.7 assume there is a vertex x in X and a vertex t in T'. By the definition of T' and by symmetry we may assume the existence of a vertex u_2 in $V_2 \cap N(t')$. By the definition of X vertex x has a neighbour u_3 in V_3. By 9.3 t misses x. But then u_2, u_3, t, x, v_5 is a bull.

To prove 9.8, let us show that every non-trivial connected component B of V_3 is homogeneous, which will be a contradiction. If B is not homogeneous, there exist adjacent vertices u, v in B and a vertex x of $G - B$ that sees u and misses v. By the definition of the partition x must belong to X. But then x, u, v, v_2, v_1 is a bull. (In a similar fashion it is possible to show that V_1 and V_5 are cliques, but we will not use this fact.)

To prove 9.9, as above suppose on the contrary that some connected component B of T_0 is not a homogeneous set, i.e., there exist adjacent vertices u, v in B and a vertex w of $G - B$ that sees u and misses v. By the definition of T and by 9.3, 9.4 and 9.1, vertex w must be in W or in Z. In either case, there exists a non-neighbour x of w in $V_2 \cup V_4$. But then w, u, v, v_3, x is a bull.

To prove 9.10, let w, t be such a pair. By symmetry we may assume that w has a neighbour u_2 in V_2. Vertex t must miss every such vertex u_2 in $N(w) \cap V_2$, for otherwise t, u_2, v_3, w, v_5 is a bull. Suppose w has a neighbour u_4 in V_4. If t sees u_4 then t, u_4, w, u_2, v_1 is a hole; if t misses u_4 then t, v_5, u_4, w, u_2, v_1 is a hole. Hence w misses all of V_4. Now w must have a non-neighbour in V_2 for otherwise we could add w to V_1 and enlarge the inflated P_5. So the sets $V_2 \cap N(w)$ and $V_2 - N(w)$ are both non-empty. There is no edge

between these two sets, for otherwise the endpoints of such an edge plus v_3, v_4, w would form a bull. Suppose that t has a neighbour w_2 in V_2. Then w_2 misses w as above, i.e., $w_2 \in V_2 - N(w)$, and so w_2 misses u_2; but then t, v_3, w_2, u_2, v_5 is a bull. So t misses all of V_2. Finally, suppose t has a non-neighbour u_4 in V_4; then w, u_2, v_1, t, v_5, u_4 is a P_6. Hence t sees all of V_4.
Now let us consider an arbitrary vertex t' of T, and suppose that t' has a neighbour in V_2. By the preceding point it must be that w sees t'. Since V_2 is the union of the two non-empty sets $V_2 \cap N(w)$ and $V_2 - N(w)$ with no edge between them, it follows that if t' does not see all of V_2 then necessarily there exists $u_2 \in V_2 \cap N(w)$ and $w_2 \in V_2 - N(w)$ such that t' sees exactly one of u_2, w_2; but then t, v_3, u_2, w_2, v_5 would be a bull. So t' must see all of V_2. Now t' must miss a vertex u_4 in V_4 for otherwise t' would be in A, not in T. Then t' must miss t or else t', t, v_1, w, u_4 is a bull. But then t', v_3, w_2, w, t is a bull. In conclusion, t' must miss all of V_2, and 9.10 is proved.

To prove 9.11, assume on the contrary that some connected component B of W is not homogeneous, i.e., there exist adjacent vertices u, v of B and a vertex x in $G - B$ that sees u and misses v. The definition of B and of the partition implies easily that x is in $V_2 \cup V_4 \cup X \cup T$.
First assume that x is in V_2. If v has a neighbour u_4 in V_4, then u_4 must miss u for otherwise u_4, v, u, x, v_5 is a bull; but then u_4, v, u, x, v_3 is a hole. Hence $N(v) \cap V_4 = \emptyset$. Now, by the definition of W, v must have a neighbour u_2 in V_2. Vertex u sees every v_4 or else v, u, x, v_3, v_4, v_5 would be a P_6. Also u misses u_2 or else v_1, u_2, u, v, v_4 is a bull. Then x sees v_2 or else x, v_3, v_2, v, u is a hole; but now x, u_2, v_3, v, v_4 is a bull. In consequence, x cannot be in V_2, and by symmetry x cannot be in V_4.
Now assume x is in X. Let u_3 be a neighbour of x in V_3. Note that u admits a non-neighbour u_2 in V_2 (or, by symmetry, in V_4), or else u could be added to V_3. Now either v, u, x, u_3, u_2, v_1 is a P_6 or v, u, x, u_3, u_2 is a hole, a contradiction.
Finally, assume x is in T. By symmetry we may assume that v has a neighbour u_2 in V_2 and then, by 9.10 applied to the non-adjacent vertices v and x, we may assume that v misses all of V_4, and that x sees all of V_4 and misses all of V_2. Vertex v must have a non-neighbour w_2 in V_2 for otherwise we could add v to V_1 and obtain a larger inflated P_5. Since we just proved that all vertices of $V_2 \cup V_4$ are impartial on B, it must be that u too misses all of $V_4 \cup \{w_2\}$. But then x, v_3, v_4, w_2, u is a bull. So 9.11 is established.

Facts 9.6 and 9.1 imply that if Z is not empty then $N(Z) \subseteq X \cup T_0$, and so $X \cup T_0$ is a cutset separating Z from the rest of the graph. To prove 9.12, as above let us suppose on the contrary that some non-trivial connected component of Z_2 is not homogeneous, i.e., there exist two adjacent vertices u, v in Z_2 and a vertex z in $G - Z_2$ that sees u and misses v. By the definition of Z_2, z is in Z_1. By the definition of Z_1, z has a neighbour x in $X \cup T_0$, and so x has a neighbour u_3 in V_3 and misses every v_2. But now $vuzxu_3v_2$ is a P_6, a contradiction. □

This is as much as we can say so far about the general structure of a bull-free weakly triangulated graph with no P_6 or $\overline{P}_6$. Unfortunately, it is not the case that this always yields a box partition. Let us nevertheless define as an *odd box* every connected component of each of the sets $V_1 \cup Y_1$, V_3, $V_5 \cup Y_5$, W, Z_1. The rest of the graph, i.e., the union of the sets V_2, V_4, X, T, and Z_2, will be considered the *even part* of G. In Lemma 10 we will show that the odd boxes satisfy Properties (i) and (ii) of the definition of the Box Partition. In Lemmas 12–15 we will see that under certain restrictions the connected components of the even part also satisfy the Properties of the Box Partition. Finally, when those restrictions do not apply, we will look at the complement $\overline{G}$ of G and show that $\overline{G}$ admits a Box Partition.

Lemma 10 *The odd boxes satisfy Properties (i) and (ii).*

In the proof of Lemma 10 and later we will repeatedly use a technical argument that we state here separately. Let us say that a set B is *under control* if there exist in $G - B$ two disjoint subsets of vertices C_0, C_1 such that: every vertex of B has a neighbour in C_0; every vertex of C_0 has a neighbour in C_1; there is no edge between B and C_1; if C_0 is not a stable set then $|C_1| = 1$ (say $C_1 = \{c_1\}$) and there exists a vertex c_2 that sees c_1 and misses all of $B \cup C_0$; if C_1 is not a clique then there exists a vertex c_2 that sees all of C_1 and misses all of $B \cup C$.

Lemma 11 *If B is connected and under control then B satisfies Property (ii) of the Box Partition.*

Proof. We only need show that C_0 contains a vertex a_B that sees all of B, for in that case we pick a vertex a'_B in $C_1 \cap N(a_B)$ and the vertices a_B, a'_B will satisfy the properties of the auxiliary vertices required for (ii), by the Remark and by the definition of control.

So let us take a vertex a of C_0 that sees the most vertices of B, and assume that a still misses some v in B. Let b be a neighbour of v in C_0. By the choice of a, there must exist a vertex u in $B \cap N(a) - N(b)$. Since B is connected there exists a shortest path P from u to v in B. Let us choose u, b, v so that this path is as short as possible. If a sees b then C_0 is not a stable set and so $C_1 = \{c_1\}$ and there exists c_2 as in the definition of control; but then c_1, a, b, u, c_2 is a bull. So a must miss b. Let c_1 be a vertex in $C_1 \cap N(a)$ and c'_1 be a vertex in $C_1 \cap N(b)$. We may assume that either $c_1 = c'_1$ or b misses c_1 and a misses c'_1. Since G is P_6-free, P has at most five vertices.
If $P = uv$ then $c_1 \neq c'_1$ for otherwise c_1, a, u, v, b is a hole. Then c_1 misses c'_1 for otherwise c_1, a, u, v, b, c'_1 is a hole. So C_1 is not a clique and c_2 exists. Then $c_2, c_1, a, u, v, b, c'_1$ is a hole.
If $P = uxv$ then a must miss x or else c_1, a, u, x, v is a bull. Likewise b must miss x or else c'_1, b, v, x, u is a bull. But then c_1, a, u, x, v, b is a hole or a P_6.
If $P = ux_1x_2v$, then a must miss x_1 for otherwise either c_1, a, u, x_1, x_2 or c_1, a, x_1, x_2, v form a bull. Then b misses x_1 or else b, u, x_1 would contradict the choice of b, u, v.

Likewise, a misses x_2. But then c_1, a, u, x_1, x_2, v is a P_6.
If $P = ux_1x_2x_3v$, then again a must miss x_1 for otherwise c_1, a and some three consecutive vertices of P form a bull as above. Symmetrically, b misses x_3. Then b misses x_1 or else b, u, x_1 contradicts the choice of b, u, v. Now x_1 must have a neighbour c in C_0, and $c \neq a, b$. Vertex c must see u or else c, u, x_1 contradict the choice of b, u, v. Let c_1'' be a vertex in $C_1 \cap N(c)$. Then c must see x_2 or else c_1'', c, u, x_1, x_2 is a bull, and c must see x_3 or else c_1'', c, x_1, x_2, x_3 is a bull. But now $c_1'', c, u, x_1, x_2, x_3$ form a broom. This contradiction completes the proof. □

Proof of Lemma 10. The absence of edges between two odd boxes is a consequence of their definition as connected components, of the definition of the sets Y_1, Y_5, W, Z_1, and of Facts 9.2 and 9.6.

Hence Property (i) holds for the odd boxes. Now let us prove (ii) for each odd box.

If B is a box in $V_1 \cup Y_1$, observe that $M(B) \subseteq V_4 \cup X$ because of established properties of the partition. We know that v_2 sees all of B and misses all of $M(B)$. Also, v_3 misses all of B. Therefore and by the Remark, Property (ii) holds for B with the auxiliary vertices $a_B = v_2$ and $a'_B = v_3$. By symmetry every box in $V_5 \cup Y_5$ satisfies (ii).

If B is a box in V_3 or W then B is trivial because of 9.8 and 9.11, so $M(B) = \emptyset$ and (ii) holds trivially.

Finally, consider a box B in Z_1. By the definition of Z_1, every vertex of B has a neighbour in $X \cup T_0$. In the case where B has a neighbour x in X then x must see all of B, for otherwise there would exist adjacent u, v in B such that x sees u and misses v, and then $vuxu_3v_2v_1$ would be a P_6, where u_3 is any vertex in $V_3 \cap N(x)$. So we can take $a_B = x$ and $a'_B = u_3$ and (ii) follows for B from the Remark. In case there is no edge between B and X then B is under control with $C_0 = T_0$, which is a stable set by 9.9, and $C_1 = \{v_3\}$. It follows from Lemma 11 that B satifies Property (ii). This ends the proof of Lemma 10. □

Lemma 12 *Every connected component of $X \cup Z_2$ satisfies Property (ii).*

Proof. Note that there is no edge between X and Z_2 by the definition of Z_2. Moreover, every component of Z_2 is trivial by 9.12. So let us only consider a component B of X. The definition of X implies that B is under control with $C_0 = V_3$, which is a stable set by 9.8, and $C_1 = \{v_2\}$. Hence Lemma 11 implies the desired conclusion. □

Lemma 13 *If $V_2 \cup V_4 \cup T$ is not connected then G admits a box partition.*

Proof. We define as an even box every connected component of the even part of G, which is $V_2 \cup V_4 \cup T \cup X \cup Z_2$. Note that there is no edge between $V_2 \cup V_4 \cup T$ and $X \cup Z_2$ by the definition of X and Z_2 and by 9.3. The components of $X \cup Z_2$ satisfy Property (ii) by Lemma 12. So let us only consider a component B of $V_2 \cup V_4 \cup T$. By the hypothesis there exists a vertex c_1 in another component of $V_2 \cup V_4 \cup T$. It follows that B is under control with $C_0 = V_3$, which is a stable set by 9.8, and $C_1 = \{c_1\}$. Hence every even box satifies Property (ii). This and Lemma 10 complete the proof. $\square$

Lemma 14 *If $X \cup T_0 \neq \emptyset$ then G admits a box partition.*

Proof. It suffices to prove that under this hypothesis $V_2 \cup V_4 \cup T$ is not connected, and to apply the preceding lemma. However, if $X \neq \emptyset$ then 9.7 implies that T is empty, and so $V_2 \cup V_4 \cup T = V_2 \cup V_4$, which is obviously disconnected. If $T_0 \neq \emptyset$ then the definition of T_0 and 9.5 and 9.9 imply that any vertex of T_0 is a component of $V_2 \cup V_4 \cup T$ different from any component intersecting V_2 or V_4, hence again $V_2 \cup V_4 \cup T$ is disconnected. $\square$

Lemma 15 *If $X \cup T_0 = \emptyset$ and some vertex in W misses a vertex in T then G admits a box partition.*

Proof. Under this hypothesis Z is empty, for otherwise the graph would be disconnected since $X \cup T_0$ is a cutset separating Z from the rest of the graph. Moreover 9.10 implies that there is no edge between T and V_2 (or between T and V_4, by symmetry). It then follows that $V_2 \cup V_4 \cup T$ is not connected, and the desired conclusion is given by Lemma 13. $\square$

Lemma 16 *If G has no box partition then $\overline{G}$ has a box partition.*

Proof. By Lemmas 14 and 15, we may assume that $X \cup T_0 = \emptyset$ and that every vertex in W sees every vertex in T. Note that Z is empty (for the same reason as in Lemma 15). So, apart from $V_1, \ldots, V_5$, only the sets Y_1, Y_5, W, T may be non-empty. By Lemma 13 we may assume that $T' \neq \emptyset$. Let t be a vertex in T'; by symmetry we may assume that the sets $V_2 \cap N(t)$ and $V_4 - N(t)$ are not empty. Note that $\overline{G}$ contains a P_5 made of t, a vertex from $V_4 - N(t)$, a vertex from $V_2 \cap N(t)$, v_5 and v_3. Since $\overline{G}$ is a weakly triangulated graph with no homogeneous set and no P_6, it admits a P_5-partition as in Lemma 8. We now show that it admits such a partition satisfying Lemma 13.

Lemma 17 *The following facts hold:*

17.1. Every vertex in Y_5 misses every vertex in $V_2 \cap N(t)$;

17.2. $(\{t\}, V_4 - N(t), V_2 \cap N(t), V_5 \cup Y_5, V_3)$ is an inflated P_5 in $\overline{G}$.

17.3. Every vertex in $V_4 \cap N(t)$ sees every vertex in $V_4 - N(t)$.

Proof. If some y_5 in Y_5 sees some v in $V_2 \cap N(t)$, then $y_5v_3u_5vu_4t$ is a P_6 in $\overline{G}$, with $u_5 \in V_5 \cap N(y_5)$ and $u_4 \in V_4 - N(t)$, a contradiction. So 17.1 holds. Now 17.2 is a consequence of the definition of the five sets and of 17.1. Finally, if 17.3 fails, some $u \in V_4 \cap N(t)$ misses some $v \in V_4 - N(t)$; but then t, v_1, v_3, u, v induce a bull. □

Now we can build a P_5-partition for $\overline{G}$ starting from the inflated P_5 given by 17.2. We denote by $P^* = (V_1^*, V_2^*, V_3^*, V_4^*, V_5^*)$ the maximal inflated P_5 that results from the construction. This partition has the additional sets $Y_1^*, Y_5^*, X^*, W^*, T^*, A^*, Z^*$, defined analogously to Lemma 8. These sets satisfy Lemma 9 with adjacency and non-adjacency in G reversed.

Lemma 18 *The following facts hold:*

18.1. $V_4 - N(t) \subseteq V_2^*$ *and* $V_5 \cup Y_5 \subseteq V_4^*$*;*

18.2. $V_1^* \subseteq T$*;*

18.3. $V_5^* \subseteq V_3 \cup Y_1 \cup W$*;*

18.4. $T^* \subset V_4$*;*

18.5. Every vertex in T^* *sees every vertex in* V_2^**.*

Proof. The five basic sets $V_1^*, V_2^*, V_3^*, V_4^*, V_5^*$ may change during the construction of the maximal P^*, but we know that V_2^* and V_4^* can only increase, thus 18.1 holds.

Next, a vertex in V_1^* must see all of V_4^*, and so, by 18.1, all of $V_5 \cup Y_5$, hence it must be in $V_4 \cup T$. Such a vertex must also miss all of V_2^*, so, by 18.1, all of $V_4 - N(t)$, hence, by 17.3, it cannot be in $V_4 \cap N(t)$. This implies 18.2.

Next, a vertex in V_5^* must miss all of V_4^* and so all of $V_5 \cup Y_5$, so it must be in $V_1 \cup V_2 \cup V_3 \cup Y_1 \cup W$. A vertex in V_5^* must see all of V_2^*, hence all of $V_4 - N(t)$, so it cannot be in V_1 or V_2. This implies 18.3.

Next, any vertex t^* in T^* misses all of $V_1^* \subseteq T$, hence it is in $V_2 \cup V_4$. Suppose $t^* \in V_2$. Consider any vertex $v_5^* \in V_5^*$. By 18.3 and the fact that $v_5^*t^*$ is not an edge, v_5^* must be in W. Consider any vertex $v_3^* \in V_3^*$. So v_3^* is not in $V_5 \cup Y_5$. Since v_3^* misses all of V_4^* it cannot be in $T \cup V_4$; since it sees $v_5^* \in W$, by Lemma 9 it cannot be in $V_1 \cup V_3 \cup Y_1 \cup W$; so v_3^* must be in V_2. But now $v_1^*, v_3^*, v_3, v_5, t^*$ form a bull. This implies 18.4.

To prove 18.5, assume on the contrary that some $t^* \in T^*$ misses some $u \in V_2^*$. By 18.4 we have $t^* \in V_4$. Since u must see all of V_4^* (in particular all of $V_5 \cup Y_5$), u must be in T or $V_4 \cap N(t)$.
First suppose $u \in T$. Consider any $v_3^* \in V_3^*$. Since v_3^* misses $u \in V_2^*$ and $u \in T$, v_3^* must be in $V_2 \cup V_4$; since it misses all of V_4^* it cannot be in V_4; so v_3^* is in V_2. The vertex v_3 is not in $V_1^* \cup V_3^* \cup V_5^* \cup Y_1^* \cup Y_5^*$ because it sees t^*; it cannot be in $X^* \cup Z^* \cup V_2^*$ because it

misses $v_5 \in V_4^*$; it cannot be in $T^* \cup V_4^*$ because it sees $v_3^* \in V_3^*$. So v_3 must be in W^*. This implies that v_3 sees v_5^*, which contradicts 18.3.
Now suppose $u \in V_4$. Vertex t sees t^*, for otherwise t^* would be in $V_4 - N(t) \subseteq V_2^*$. Then t sees u or else t, t^*, v_3, u, v_1 form a bull. Consider any vertex $v \in V_4 - N(t)$; by 17.3, v sees t^* and u. Now any vertex $v_5^* \in V_5^*$ misses t^* (because it is in T^*), and sees u, v (because they are in V_2^*). Likewise any $v_1^* \in V_1^*$ sees v_5^* and misses u, v, t^*. But then u, v, v_5^*, t^*, v_1^* form a bull. This contradiction completes the proof of the lemma. $\square$

Now we finish the proof of Lemma 16. Fact 18.5 implies that $V_2^* \cup V_4^* \cup T^*$ induces a disconnected subgraph of $\overline{G}$. So Lemma 13 yields a box partition for $\overline{G}$. $\square$

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References

[1] C. Berge. Les problèmes de coloration en théorie des graphes. *Publ. Inst. Stat. Univ. Paris*, 9:123–160, 1960.

[2] V. Chvátal. Perfectly ordered graphs. In Berge C. and V. Chvátal, editors, *Topics on perfect graphs*, pages 63–65. North-Holland, Amsterdam, 1984.

[3] V. Chvátal. A class of perfectly orderable graphs. Technical Report 89573-OR, Forschungsbereich für Diskrete Mathematik, Institut für Ökonometrie und Operations Research, Rheinische Friedrich Wilhelms Universität, Bonn, Germany, May 1989.

[4] V. Chvátal and N. Sbihi. Bull-free Berge graphs are perfect. *Graphs and Combinatorics*, 3:127–139, 1987.

[5] C. de Figueiredo, F. Maffray, and O. Porto. On the structure of bull-free perfect graphs, 2: The weakly triangulated case. RUTCOR Research Report 45-94, Rutgers University Center for Operations Research, New Brunswick, NJ, USA, 1994.

[6] C.M.H. de Figueiredo, F. Maffray, and O. Porto. On the structure of bull-free perfect graphs. *Graphs and Combinatorics*, 13:31–55, 1997.

[7] R. Hayward. Weakly triangulated graphs. *J. Comb. Theory B*, 39:200–209, 1985.

[8] R. Hayward. Bull-free weakly triangulated perfectly orderable graphs. Manuscript, 1998. Dept. of Math and Comp. Science, Lethbridge University, Canada.

[9] C.T. Hoàng and N. Khouzam. On brittle graphs. *J. Graph Theory*, 12:391–404, 1988.

[10] B.A. Reed and N. Sbihi. Recognizing bull-free perfect graphs. *Graphs and Combinatorics*, 11:171–178, 1995.