A Generalization of Tucker's proof for $(K_4 - e)$-free Perfect Graphs*

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Abstract

In this note, the authors generalize the ideas presented by A. Tucker in his proof of the SPGC for $K_4 - e$-free graphs in order to find a vertex v (called here a Tucker vertex) in G whose special neighborhood allows to extend a previous coloring of $G - v$. The search of such a vertex led us to define a property on the intersection of large cliques and a family of classes of graphs that satisfy this property. We prove that every graph G in this family has a Tucker vertex and we use this fact to give a polynomial-time algorithm to compute $\omega(G)$. We give a proof of the SPGC for a new class: graphs where every edge of a maximal clique of size at least 4 belongs to precisely that clique. The proof directly yields a combinatorial polynomial-time algorithm for coloring perfect graphs in this new class with the size of a maximum clique colors.
Key words: Perfect graphs, vertex-coloring algorithms, Tucker vertex.

1 Introduction

A graph G is perfect if, for each induced subgraph H of G, the chromatic number $\chi(H)$ of H is equal to the size $\omega(H)$ of a maximum clique of H. The class of perfect graphs and two conjectures about their structure were introduced by C. Berge in [4, 5]. One of these conjectures is still open and it is known as the *Strong Perfect Graph Conjecture*

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(SPGC for short). A *hole* in a graph G is an odd chordless cycle of length at least 5. An *antihole* is the complement of a hole. A graph is called *Berge* if it contains neither a hole nor an antihole as induced subgraph. The SPGC asserts that *A graph is perfect if and only if it is Berge.* In the last thirty years, the SPGC was confirmed true for several classes of graphs and all these proofs introduced, each time, concepts, methods and ideas that have been extended to several other branches of graph theory.

In this note, the authors generalize the ideas presented by A. Tucker in his proof of the SPGC for $K_4 - e$-free graphs [7] in order to find a vertex v in G whose special neighborhood allows to extend a previous coloring of $G - v$. The search of such a vertex led us to define a property on the intersection of large cliques, called here the Quite-Isolated p-Cliques property (p-QIC property), where p sets the minimum size of the cliques that we are interested in. As a byproduct of our search, we obtain a polynomial-time algorithm to compute $\omega(G)$ for any p-QIC graph G and for any fixed p. Moreover, we give a proof of the SPGC for a new class of graphs that strictly contains the line-graph of bipartite graphs, the chordal graphs, the $(K_4 - e)$-free graphs and K_4-free graphs. We also give a polynomial-time algorithm to optimally color them.

This paper is organized as follows. In the next section, we give the p-QIC property and we define the classes of $\mathcal{G}_p$ graphs, that is, the classes of graphs that, for a given p, satisfy the p-QIC property. In the third section, we prove that the Berge graphs of $\mathcal{G}_4$ are perfect and we give a polynomial-time algorithm to optimally color them. Finally, in the last section, we propose a conjecture whose validity implies the SPGC for bigger classes.

Before finishing this section, we recall some definitions that will be useful later. In this paper, all the graphs are finite and simple. A *clique* means a maximal clique. A *p-clique* means a clique of size at least p. The neighborhood of a vertex x will be denoted by $N(x)$. Given a path P and distinct vertices x and y on P, (x, P, y) denotes the segment of P going from x to y.

2 Quite-Isolated p-Cliques Property

We start this section by defining the p-QIC property:

Definition 1 (Quite-isolated p-Cliques (p-QIC)) *A graph G satisfies the p-QIC property if and only if for any p-clique K of G and every clique $K' \neq K$, we have that $|K \cap K'| \leq 1$.*

The p-QIC property implies that every edge of a p-clique does not belong to another clique. This definition allows us to define the following family of classes of graphs:

Definition 2 (The classes $\mathcal{G}_p$ of graphs) *A graph G belongs to $\mathcal{G}_p$ if and only if G satisfies p-QIC property.*

Notice that the hereditary class $\mathcal{G}_p$ can be completely characterized by a family of forbidden subgraphs $\mathcal{F}_p$:

$$\mathcal{F}_1 = \mathcal{F}_2 = \{K_4 - \text{ e}\}.$$

$$\mathcal{F}_p = \Big\{K_{p+1} - \ i \text{ adjacent edges, for all } i = 1, \ldots, p-2\Big\}.$$

Since each element of $\mathcal{F}_p$ has $p+1$ vertices and $\mathcal{F}_p$ is a finite family, we can recognize a graph of $\mathcal{G}_p$ in $O(n^{p+1})$ time.

Now, we are going to prove that every Berge graph of $\mathcal{G}_p$ has a p-Tucker vertex whose definition follows:

Definition 3 (p-Tucker vertex) *Given an integer $p > 2$, a vertex x is a p-Tucker vertex if and only if x is contained in at most two p-cliques.*

Theorem 1 *For a given $p > 2$, every Berge graph of $\mathcal{G}_p$ has a p-Tucker vertex.*

Proof. Let G be a graph of $\mathcal{G}_p$. We may assume that $\omega(G) \geq p$, for otherwise every vertex of G would be a p-Tucker vertex. Let us consider a maximal induced path $P = v_1, v_2, \ldots, v_k$ of G with respect to the following properties:

(i) For all i, the edge $v_i v_{i+1}$ belongs to a p-clique C_i (note that it follows that $v_i v_{i+1}$ belongs to a unique p-clique);

(ii) For all $j > i+1$, $C_i \cap C_j = \emptyset$.

We will prove that v_k is a p-Tucker vertex. Observe that P has at least one edge, since $\omega(G) \geq p$. Also, by the choice of P, every $w \in N(v_k) - C_{k-1}$ belonging to a p-clique K containing v_k satisfies one (perhaps both) of the following conditions:

(a) wv_i is an edge, for some $i < k-1$;

(b) $K \cap C_i$, for some $i < k-1$.

We claim that :

Claim 1 *Every vertex w adjacent to an edge $v_i v_{i+1}$ belongs to C_i.*

Indeed, in the opposite case, $C_i \cup \{w\}$ contains as induced subgraph an element of $\mathcal{F}_p$.

Let us now focus our attention on the neighborhood of v_k. Let K be any $(p-1)$-clique in $N(v_k) - C_{k-1}$ (if it exists). First, we will prove that K has at least one vertex satisfying (b). By contradiction, suppose that this is not true. By maximality of P, every vertex of K satisfies (a).

Notice that any vertex v_i $(i < k)$ is adjacent to at most one vertex of K. Indeed, in the opposite case by maximality of K, the subgraph induced by $\{v_i\} \cup K$ belongs to $\mathcal{F}_p$.

Let w be a vertex of K. Let $l(v_j) = k - j + 1$ denote the length of the path (v_j, P, v_k, w). If wv_j is an edge then, by Claim 1, $l(v_j)$ must be odd or else (v_j, P, v_k, w) would contain a hole. Now, let w' be a vertex of $K - w$ and $v_i w$ and $v_j w'$ be two edges. Since $l(v_i)$ and $l(v_j)$ are both odd, $|j - i + 1|$ is even. So the path (w, v_i, P, v_j, w') contains a hole. In any case, we obtain a contradiction.

Thus, we may assume that every $(p-1)$-clique in $N(v_k) - C_{k-1}$ contains a vertex satisfying (b).

Now, by contradiction, suppose there exist two distinct $(p-1)$-cliques K_1 and K_2 in $N(v_k) - C_{k-1}$. Let $w_1 \in K_1$ and $w_2 \in K_2$ be two vertices satisfying (b). Suppose that $w_1 \in C_r$ and $w_2 \in C_s$ (w.l.o.g. we may assume that $r \leq s$). Observe that w_1 and w_2 are non-adjacent, for otherwise $\{v_k\} \cup K_1 \cup K_2$ would contain an element of $\mathcal{F}_p$ as induced subgraph. Moreover, we have that $r \neq s$, for otherwise $\{w_1, w_2\} \cup C_r$ would contain as induced subgraph an element of $\mathcal{F}_p$.

By Claim 1 and by (ii), w_i $(i = 1, 2)$ is adjacent to no other edge of P than $v_r v_{r+1}$ and $v_s v_{s+1}$, respectively.

By the same arguments of the previous case, $l(v_{r+1})$ and $l(v_{s+1})$ must be both odd. Thus (v_{r+1}, P, v_s) has an odd length. Hence, the path $(v_{r+1}, P, v_s, w_2, v_k, w_1)$ contains a hole. This contradiction ends the proof of the theorem. □

Observe that the proof of Theorem 1 yields a polynomial-time algorithm to find a Tucker vertex in a p-QIC graph, for a fixed integer p. The general lines of this algorithm are given below.

For a given Berge graph in $\mathcal{G}_p$, one can test for the existence of a p-clique K in $O(n^p)$ time. If such a p-clique K exists, then choose any edge $v_1 v_2$ of K and start to build the path described in the proof of Theorem 1 with $C_1 = K$. At step i, we can in $O(n^{p+1})$

time either extend the path with an edge $v_i v_{i+1}$ and a p-clique C_i, or the vertex v_i is a p-Tucker vertex. Finally, for a fixed p and a Berge graph of $\mathcal{G}_p$, we can find in $O(n^{p+2})$ time a p-Tucker vertex and the at most two p-cliques containing it. Thus, we can obtain an elimination ordering $(v_1, v_2, \ldots, v_n)$ on the set of vertices of G, where v_i is a p-Tucker vertex in the graph G_i induced by $\{v_i, \ldots, v_n\}$, in $O(n^{p+3})$ time.

Now, for a fixed p, we determine $\omega(G)$ in polynomial time : First, observe that we can compute $\omega(N(v))$ in $O(n^{p+1})$ time if v is a p-Tucker vertex (since such a vertex is in at most two vertex-disjoint p-cliques with no edge between them). Thus, using the elimination ordering and the following induction formula

$$\omega(G_{i+1}) = \text{Max}\{\omega(G_i - v_i), \omega(N(v_i)) + 1\},$$

we can determine $\omega(G)$ in $O(n^{p+2})$ time.

3 $\mathcal{G}_4$ verifies the Strong Perfect Graph Conjecture

Theorem 1 yields a simple induction proof of our main result. The main idea of our proof is to apply a 3-chromatic exchange in a component that contains a 4-Tucker vertex of a $\mathcal{G}_4$ Berge graph (using Tucker's algorithm to color K_4-free Berge graphs [8]). Several works used this idea to optimally color perfect graphs [1, 3, 2, 6].

Theorem 2 *A Berge graph in $\mathcal{G}_4$ is $\omega(G)$ colorable.*

Proof. Let G be a Berge graph in $\mathcal{G}_4$ and let v be a 4-Tucker vertex of G (by Theorem 1, such a vertex exists). Our proof works by induction on the number of vertices of G. If G has one vertex, then the theorem trivially holds. Assume that $|V(G)| \geq 2$. By induction hypothesis, $G - v$ can be colored with $\omega(G - v)$ colors. Now, let $\mathcal{C} = \{S_1, \ldots, S_{\omega(G-v)}\}$ be an optimal coloring of $G - v$. If one color among $\{1, \ldots, \omega(G)\}$ does not occur in $N(v)$, then assign this color to v in order to produce an $\omega(G)$-coloring of G. In this case, the theorem holds.

Now, suppose that all colors occur in $N(v)$ and let K_1 and K_2 (each K_i may be empty) be two 4-cliques containing v. By definition of $\mathcal{G}_4$, there is no edge between K_1 and K_2. Moreover, $|K_i - v| < \omega(G)$, for $i = 1, 2$. Hence there exist a color α and a color β (may be $\alpha = \beta$) such that $K_1 \cap S_\alpha = \emptyset$ and $K_2 \cap S_\beta = \emptyset$. Choose an arbitrary third color, let us say γ. We have that the subgraph H induced by $\{v\} \cup S_\alpha \cup S_\beta \cup S_\gamma$ is a K_4-free Berge graph. So, Tucker's algorithm for coloring K_4-free Berge graphs can be applied to H. The coloring of H extends $\mathcal{C}$ to an ω-coloring of the whole G. $\square$

To convert the proof of Theorem 2 into a polynomial-time algorithm to optimally color a $\mathcal{G}_4$ Berge graph, we suppose that we have an elimination ordering $(v_1, v_2, \ldots, v_n)$ on the set of vertices of G, where v_i is a 4-Tucker vertex in the graph G_i induced by $\{v_i, \ldots, v_n\}$. As first step, we determine $\omega(G)$ in polynomial time (see the previous section).

Now, to obtain an $\omega(G)$-coloring of a Berge graph G in $\mathcal{G}_4$, it is sufficient to proceed like in the proof of Theorem 2 whose general ideas we recall below. Suppose that G_i is colored with $\omega(G_i)$ colors. If $\omega(G_{i+1}) > \omega(G_i)$, then assign a new color to v_{i+1}, else we can choose in $O(n)$ time the two colors α and β which induce a K_3-free graph in $N(v_{i+1})$. Now, choose any other color, say γ. We can then apply Tucker's algorithm for K_4-free Berge graphs to color the graph induced by $\{v_{i+1}\} \cup S_\alpha \cup S_\beta \cup S_\gamma$, whose running time is $O(n^3)$. Finally, we obtain an $\omega(G_{i+1})$-coloring of G_{i+1}, by keeping unchanged the other colors.

As we have pointed out in the beginning of this section, the 3-chromatic exchange tool was used for optimally coloring several classes of graphs and for proving, for some of them, the validity of SPGC. In fact, Theorem 1 shows that $\mathcal{G}_4$ graphs belong to a larger class of graphs, defined by S. Gravier in [6], so called $\mathcal{GH}$. In [6], S. Gravier also proved that $\mathcal{GH}$ graphs satisfy the SPGC. However, it is not known how to recognize $\mathcal{GH}$ graphs in polynomial time. The definition of $\mathcal{G}_4$ graphs immediately entails a polynomial-time recognition algorithm.

4 Conclusion

The aim of this paper was to show that the technique used by Tucker to find a special vertex in a $K_4 - e$-free Berge graph can be generalized. We wonder whether, by using this technique, it can be shown that :

Conjecture I *For $p > 2$, a $(K_p - e)$-free Berge graph has a vertex belonging to at most $p - 2$ cliques of size at least $p - 2$.*

Clearly, Conjecture I holds for $p = 3$ and 4 [7]. Notice that if Conjecture I holds for $p = 5$ then, by using the 3-chromatic exchange, we would obtain a polynomial-time algorithm to color, with $\omega(G)$ colors, any $(K_5 - e)$-free Berge graph G.

References

[1] H. Aït Haddadène and S. Gravier. On weakly diamond-free Berge graphs. *Discrete Mathematics*, 159:237–240, 1996.

[2] H. Aït Haddadène, S. Gravier and F. Maffray. An algorithm for coloring perfect graphs. *Discrete Mathematics*, 183:1–16, 1998.

[3] H. Aït Haddadène and F. Maffray. Coloring perfect degenerate graphs. *Discrete Mathematics*, 163:211–215, 1997.

[4] C. Berge. Les problèmes de coloration en théorie des graphes. *Publ. Inst. Stat. Univ. Paris*, 9:123–160, 1960.

[5] C. Berge. Farbung von graphen, deren ungerade kreise starr sind. *Wiss. z. Martin-Luther-Univ., Halle-Wittenberg Math-Natur. Reihe*, pages 114–115, 1961.

[6] S. Gravier. On Tucker vertices of graphs. 1998. To appear in *Discrete Mathematics.*

[7] A. Tucker. Coloring perfect $(K_4 - e)$-free graphs. *Journal of Combinatorial Theory, Series B*, 43:313–318, 1987.

[8] A. Tucker. A reduction procedure for coloring perfect K_4-free graphs. *Journal of Combinatorial Theory, Series B*, 43:151–172, 1987.