Transitive Orientations with Maximum Sets of Sources and Sinks*

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Abstract

Given a transitive orientation $\vec{G}$ of a comparability graph G, a vertex of G is a **source** (**sink**) if it has indegree (outdegree) zero in $\vec{G}$, respectively. A **source set** of G is a subset of vertices formed by sources of some transitive orientation of G. A pair of subsets $S, T \subseteq V(G)$ is a **source-sink pair** of G when the vertices of S and T are sources and sinks, respectively, of some transitive orientation of G. We describe algorithms for computing the cardinality of a maximum source set and of a maximum source-sink pair of a comparability graph. In addition, we describe algorithms for finding the corresponding transitive orientations. The algorithms are applications of modular decomposition and are all of linear-time complexity.

KEYWORDS: algorithms, comparability graphs, modular decomposition, source sets, transitive orientation.

1 Introduction

Let G denote a simple non-trivial connected undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Write $n = |V(G)|$ and $m = |E(G)|$. Let $\vec{G}$ represent an acyclic orientation of G. We say that $\vec{G}$ is **transitive** when $(v,w),(w,z) \in E(\vec{G})$ implies $(v,z) \in E(\vec{G})$, for all $v,w,z \in V(G)$. By definition, G is a **comparability graph** when it admits a

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transitive orientation. If v has indegree or outdegree zero in $\vec{G}$ then v is a **source** or a **sink**, respectively. A **source (sink) of** G is a vertex which is a source (sink) of some transitive orientation of G. A **source (sink) set** of G is a subset of vertices which are simultaneously sources (sinks) in some transitive orientation of the graph. A pair of subsets $S, T \subseteq V(G)$ is a **source-sink pair** of G when the vertices of S and T are sources and sinks, respectively, of some transitive orientation of G. A source set S is **exact** when G admits a transitive orientation $\vec{G}$ having the vertices of S as the only sources of $\vec{G}$. Analogously, we define exact source-sink pair.

The **max source number** of G, $n_s(G)$, is the maximum cardinality of a source set of G. The **max source-sink number** of G, $n_{st}(G)$, is the maximum cardinality of the union $S \cup T$ of a source-sink pair S, T of G. In this case, S, T is a **maximizing** source-sink pair of G. Similarly, we define the **min source number** and **min source-sink number** of G. We refer to source number and source-sink number with the meaning of max source number and max source-sink number of G, respectively.

A **subgraph** of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the **subgraph induced by** X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X. A **clique** K is a set of vertices pairwise adjacent in G and an **independent set** I is a set of vertices pairwise nonadjacent in G. The corresponding graphs $G[K]$ and $G[I]$, induced by a clique or by an independent set, are called **complete** and **empty** graph, respectively.

Gimbel [4] and Olariu [14] have formulated characterizations for a vertex of a comparability graph to be a source in some transitive orientation. In particular, [4] and [14] provide characterizations by forbidden induced subgraphs. On the other hand, [11] and [16] describe properties relating sources and sinks to the maximal cliques of the graph. The problem of finding a transitive orientation of a graph, if such exists, with prescribed sources and sinks can be solved using a method described by Möhring [12]. It generalizes Golumbic's transitive orientation algorithm [5]. Given any partial orientation of an undirected graph, [12] extends it to a transitive orientation, whenever possible. The complexity is that of comparability graph recognition.

Recently, the authors have described a characterization for a subset of vertices of a comparability graph G to be formed by sources in some transitive orientation of G. This characterization leads to another polynomial algorithm for finding transitive orientations with prescribed sources and sinks [3]. The complexity of this algorithm is also that of comparability graph recognition.

The computation of the source number was first asked in [3] and described in [1]. In the present paper, we prove recurrence equations for computing the source number and the source-sink number of a comparability graph. The equations lead to simple algorithms for computing these parameters. In addition, we also describe algorithms for finding the corresponding transitive orientations which maximize these parameters. The algorithms are of linear-time complexity. The methods are applications of the linear-time modular decomposition and transitive orientation algorithms by McConnell and Spinrad [9, 10].

The problem of finding a modular decomposition of a graph has been considered by several authors. See for instance [2, 5, 8, 10, 12, 13, 17]. Linear-time algorithms for this problem are due to Cournier and Habib [2] and to McConnell and Spinrad [8]. The **transitive orientation problem** is that of assigning a direction to each edge of a graph so that the resulting digraph is transitive. Previous algorithms required $O(n^2)$ [18], $O(\Delta m)$ [5, 7, 15], and $O(n+m\log n)$ time [8], where Δ is the maximum degree of any vertex in the graph. Recently, McConnell and Spinrad have formulated a linear-time algorithm for modular decomposition that can be used for obtaining a linear-time algorithm for the transitive orientation problem [9]. The complexity of comparability graph recognition remains $O(n^{2.39})$ [17].

In Section 2 we review the relationship between modular decomposition and transitive orientation. Section 3 describes the proposed equations for computing the source number, the source-sink number and a maximizing source-sink pair of a comparability graph. The corresponding algorithms and transitive orientations are given in Section 4.

2 Background

A **module** of a graph G is a subset M of vertices of $V(G)$ such that each vertex in $V(G)\setminus M$ either is adjacent to all vertices of M, or is adjacent to no vertex in M. The subsets formed by single vertices of G and the set $V(G)$ are **trivial** modules. We consider three types of modules. The module M is **parallel (P)** if $G[M]$ is disconnected; M is **serial (L)** if $\overline{G}[M]$ is disconnected; M is **neighbourhood (N)** if both $G[M]$ and $\overline{G}[M]$ are connected. Similarly, say that G[M] is parallel, serial or neighbourhood when M is respectively so. A graph is **prime** if it only contains trivial modules, and **degenerate** if every subset of its vertices is a module. A subset X of $V(G)$ induces a **maximal prime subgraph** if $G[X]$ is a prime subgraph, but no other proper superset X' of X induces a prime subgraph.

Theorem 1 *[2]: Let G be a graph with at least two vertices. Then exactly one of the following holds:*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *There is $X \subseteq V(G)$ and a unique partition Q of $V(G)$, such that:*

- *$|X| > 3$,*
- *$G[X]$ is a maximal prime subgraph of G,*
- *for every $Y \in Q$, Y is a module of G and $|Y \cap X| = 1$.*

From this theorem we know that every graph is parallel, serial or neighbourhood. Further, in each case shows how the graph admits a corresponding unique partition into special maximal modules. By recursive applications of Theorem 1, the **modular decomposition tree** $T(G)$ associated with a given graph G can be defined. Each vertex v of $T(G)$ is a module M of G, while the children of M are submodules M_i of $G[M]$ forming a partition of M. A module M of G corresponds to a node of $T(G)$ precisely when for any other module M' of G, modules M and M' are either disjoint or one of them is contained in the other. Such a module M is called a **strong module**. The modular decomposition tree for a given graph can be constructed in linear time [2, 8].

The **representative graph** $R[M]$ of a module M in the modular decomposition tree $T(G)$ is the graph induced in G by one vertex from each child M_i of M in $T(G)$. The representative graph shows whether there are edges in $G[M]$ between submodules of M. Note that for a P-module, an L-module and an N-module, the representative graph is respectively empty, complete and prime.

If the modular decomposition is available, then transitively orienting G reduces to the problem of transitively orienting each representative graph in the modular decomposition of G:

Theorem 2 *[18]: Let G be a comparability graph, $T(G)$ its modular decomposition tree, M a module of $T(G)$ with children $M_1, \ldots, M_k$, $k > 0$. Let $\vec{R}[M]$ be a transitive orientation* *of $R[M]$. Then there exists a transitive orientation $\vec{G}[M]$ of $G[M]$ such that, given $v_p \in M_i$ and $v_q \in M_j$, with $i \neq j$,*

$$(v_p, v_q) \in E(\vec{G}[M]) \Longleftrightarrow (M_i, M_j) \in E(\vec{R}[M]). \quad (1)$$

Conversely, given a transitive orientation $\vec{G}[M]$, there exists a transitive orientation $\vec{R}[M]$ satisfying statement 1.

A **linear extension** of a transitive orientation of G is a total ordering of the vertices such that whenever (u, v) is an undirected edge and v is a successor of u in the total ordering, then the orientation of (u, v) is from u to v.

McConnell and Spinrad [9] obtained a transitive orientation of G as follows. For each degenerate module M of $T(G)$, order the vertices of $G[M]$ arbitrarily and for each prime module M of $T(G)$, order its vertices according to a linear extension of a transitive orientation of $R[M]$. The leaf ordering of $T(G)$ corresponds to a linear extension of a transitive orientation of G.

In order to obtain a linear extension of a prime comparability graph, McConnell and Spinrad [9] start with a partition $\mathcal{L} = \{\{v\}, V(G) \setminus v\}$ of $V(G)$ and refine $\mathcal{L}$ using a pivot vertex u that splits partition classes that do not contain u into two subclasses, one with the vertices that are adjacent to u and the other with the vertices that are not adjacent to u until all classes in $\mathcal{L}$ are singleton vertices. They use the above described procedure twice. First, to identify a sink x in a transitive orientation of G, and second, using $\mathcal{L} = \{\{x\}, V(G) \setminus x\}$, to obtain a linear extension of a transitive orientation of G.

We shall use also theorems 3 and 4 below.

Theorem 3 *[6]: Let G be a non empty comparability prime graph. Then G has exactly two transitive orientations.*

Let $S, T \subset V(G)$. Denote by $G^{(S,T)}$ the graph constructed in the following way. Add a vertex u to G that is adjacent to exactly the vertices of $S \cup T$. Subdivide each edge incident to u in the following way. If the edge is incident to a vertex of S, subdivide it with one new vertex. If the edge is incident to a vertex of T, subdivide it with two new vertices.

Theorem 4 *[3]: Let G be a comparability graph with at least two vertices and $S, T \subset V(G)$. The following are equivalent:*

(i) *S, T is a source-sink pair;*

(ii) *$G^{(S,T)}$ is a comparability graph and the restriction to G of any transitive orientation of $G^{(S,T)}$ has S, T as source-sink pair.*

3 The Equations

In this section, we prove recurrence equations for computing the source number, $n_s(G)$, the source-sink number, $n_{st}(G)$, and a maximizing source-sink pair of a comparability graph G. The following theorem shows how to use the modular decomposition tree $T(G)$ to compute the parameters $n_s(G)$, $n_{st}(G)$ and the sets $f_1(G)$, $f_2(G)$ which form a maximizing source-sink pair of G.

Theorem 5 *Let G be a comparability graph, $T(G)$ its modular decomposition tree and M a node of $T(G)$. If $|M| = 1$ with $V(G[M]) = \{v\}$, then $n_s(G[M]) = n_{st}(G[M]) = 1$ and $f_1(G[M]) = f_2(G[M]) = \{v\}$. When $|M| > 1$, let $M_1, \ldots, M_k$, be the children of M. Then*

$$n_s(G[M]) = \begin{cases} \sum_{i=1}^{k} n_s(G[M_i]), & \textit{if } M \textit{ is } P; \\ \\ max_{i=1}^{k} n_s(G[M_i]), & \textit{if } M \textit{ is } L; \\ \\ max\{\sum_{M_i \in S} n_s(G[M_i]), \sum_{M_i \in T} n_s(G[M_i])\}, & \textit{if } M \textit{ is } N \textit{ and } S,T \textit{ are} \\ \quad \textit{the exact source sets of the two transitive orientations of } R[M]. \end{cases}$$

$$n_{st}(G[M]) = \begin{cases} \sum_{i=1}^{k} n_{st}(G[M_i]), & \textit{if } M \textit{ is } P; \\ \\ n_s(G[M_p]) + n_s(G[M_q]), & \textit{if } M \textit{ is } L \textit{ and } M_p,\ M_q,\ p \neq q, \textit{ are the two} \\ \quad \textit{children of } M \textit{ having highest source number}; \\ \\ \sum_{M_i \in (S \cup T)} n_s(G[M_i]), & \textit{if } M \textit{ is } N \textit{ and } S,T \textit{ is the exact source-sink} \\ \quad \textit{pair of a transitive orientation of } R[M]. \end{cases}$$

$$f_1(G[M]), f_2(G[M]) = \begin{cases} \bigcup_{i=1}^{k} f_1(G[M_i]),\ \bigcup_{i=1}^{k} f_2(G[M_i]),\ \textit{if } M \textit{ is } P; \\[2ex] f_1(G[M_p]),\ f_1(G[M_q]),\ \textit{if } M \textit{ is } L \textit{ and } M_p,\ M_q,\ p \neq q, \textit{ are the} \\ \textit{two children of } M \textit{ having highest source number}; \\[2ex] \bigcup_{M_i \in S} f_1(G[M_i]),\ \bigcup_{M_i \in T} f_1(G[M_i]),\ \textit{if } M \textit{ is } N \textit{ and } S, T \textit{ is the} \\ \textit{exact source-sink pair of a transitive orientation of } R[M]. \end{cases}$$

Proof: Let G be a comparability graph, $T(G)$ its modular decomposition tree and M a node of $T(G)$. The proof is by induction on $|M|$. If $|M| = 1$ with $V(G[M]) = \{v\}$, then clearly $n_s(G[M]) = n_{st}(G[M]) = 1$, and its source-sink pair is $f_1(G[M]) = f_2(G[M]) = \{v\}$. When $|M| > 1$, let $M_1, \ldots, M_k$ be the children of M with respective source and source-sink pair numbers $n_s(G[M_i])$, $n_{st}(G[M_i])$, and let $f_1(G[M_i])$, $f_2(G[M_i])$ be a maximizing source-sink pair of $G[M_i]$, for each i, $1 \leq i \leq k$. Note that $k > 1$. Consider the three possibilities according to M being P, L or N.

If M is P, then $R[M]$ is an empty graph with k vertices. By induction, we have $n_s(G[M]) = \sum_{1 \leq i \leq k} n_s(G[M_i])$ and $n_{st}(G[M]) = \sum_{1 \leq i \leq k} n_{st}(G[M_i])$. Note that every edge of $G[M]$ is an edge of $G[M_i]$, for some i. Therefore, a maximizing source-sink pair $f_1(G[M])$, $f_2(G[M])$ of $G[M]$ is obtained by setting: $f_1(G[M]) = \bigcup_{i=1}^{k} f_1(G[M_i])$ and $f_2(G[M]) = \bigcup_{i=1}^{k} f_2(G[M_i])$.

If M is L, then $R[M]$ is a clique. Hence, a transitive orientation $\vec{R}[M]$ corresponds to a permutation of the set $\{1, 2, \ldots, k\}$ and such orientation has only one source and only one sink. Let M_p and M_q, $p \neq q$, be the two children of M having highest and second highest source number, respectively. Choose $\vec{R}[M]$ having M_p as source and M_q as sink. A total order of the children of M in $T(G)$ such that M_p is the first child in the order and M_q is the last one yields a special transitive orientation of $\vec{R}[M]$. Now Theorem 2 implies that $n_s(G[M]) = n_s(G[M_p])$ and $n_{st}(G[M]) = n_s(G[M_p]) + n_s(G[M_q])$. Therefore, a maximizing source-sink pair $f_1(G[M])$, $f_2(G[M])$ of $G[M]$ is obtained by setting: $f_1(G[M]) = f_1(G[M_p])$ and $f_2(G[M]) = f_1(G[M_q])$.

If M is N, then $R[M]$ is a prime comparability graph. By Theorem 3, $R[M]$ has precisely two transitive orientations, one being the reverse of the other one. Hence, if S, T is the exact source-sink pair of a transitive orientation of $R[M]$, then T, S is the exact source-sink pair of the other transitive orientation of $R[M]$. Let $X = S \cup T$. Now Theorem 2 implies that $n_s(G[M]) = \max\{\sum_{M_i \in S} n_s(G[M_i]), \sum_{M_i \in T} n_s(G[M_i])\}$ and $n_{st}(G[t]) = \sum_{M_i \in X} n_s(G[M_i])$. To find a maximizing source-sink pair of $G[M]$, without loss of generality, assume that $|S| \geq |T|$ and set $f_1(G[M]) = \bigcup_{M_i \in S} f_1(G[M_i])$ and $f_2(G[M]) = \bigcup_{M_i \in T} f_1(G[M_i])$. $\square$

Note that, in general, the maximality of a source-sink pair S, T does not necessarily imply that S or T is a maximum source set. However the transitive orientation given by the algorithm described in the proof of Theorem 5, maximizes both the source-sink pair and the source set of a comparability graph G.

Corollary 1 *There exists a transitive orientation with maximum source-sink pair S, T, such that S is a maximum source set.*

4 The algorithms

We conclude by describing algorithms for computing the source number $n_s(G)$, the source-sink number $n_{st}(G)$ and a maximizing source-sink pair, S, T, of a comparability graph G.

The algorithms run from the leaves to the root of the modular decomposition tree $T(G)$. They make, according to Theorem 5, the appropriate assignments for the leaves and corresponding nodes P, L or N, for each node M of $T(G)$. The assignments on the root provide the required answers for the input graph.

For the complexity analysis, we recall that a transitive orientation of a comparability graph G and the modular decomposition tree of G can be computed in $\mathrm{O}(n+m)$ time. Hence the algorithms can be performed in linear time.

Now, we describe an algorithm that constructs in linear time a transitive orientation with max source-sink number and max source number in a comparability graph G.

Given a comparability graph G, use the algorithm that finds a maximizing source-sink pair, S, T. Construct the graph $G^{(S,T)}$ as described in the Section 2. Apply to $G^{(S,T)}$ the algorithm by McConnell and Spinrad to obtain an orientation $\vec{G}^{(S,T)}$. By Theorem 4, the orientation $\vec{G}^{(S,T)}$ is transitive, since the pair of sets S, T is a source-sink pair of G. Hence the restriction of $\vec{G}^{(S,T)}$ to G is a transitive orientation of G having S, T as a source-sink pair. By Corollary 1, this orientation also maximizes the source number of G.

Finally, note that the proposed algorithms also compute in linear time the weighted versions of the maximum source and source-sink set problems. In this case, we obtain the weighted parameters $n_s(G)$, $n_{st}(G)$, corresponding to a graph G with non-negative weights in its vertices as input.

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