Total-Chromatic Number and Chromatic Index of Dually Chordal Graphs

Celina M. H. de Figueiredo* João Meidanis† Célia Picinin de Mello†

Abstract

A graph is dually chordal if it is the clique graph of a chordal graph. Alternatively, a graph is dually chordal if it admits a maximum neighbourhood order. This class generalizes known subclasses of chordal graphs such as doubly chordal graphs, strongly chordal graphs and interval graphs. We prove that Vizing's total-colour conjecture holds for dually chordal graphs. We describe a new heuristic that yields an exact total-colouring algorithm for even maximum degree dually chordal graphs and an exact edge-colouring algorithm for odd maximum degree dually chordal graphs.

Key words. graph algorithms, chordal graphs, total colouring, edge colouring, clique graphs, maximum neighbourhood ordering.

AMS subject classification. 05C85, 05C15, 68R10, 90C27

1 Introduction

We consider the problem of total colouring and edge colouring dually chordal graphs. A *total colouring* of a graph is a colouring of its vertices and edges such that no adjacent vertices, no adjacent edges, and no incident vertices and edges obtain the same colour. *Edge colouring* is a partial case of total colouring when only the edges are coloured. The minimum number of colours needed is called the *total-chromatic number* and the *chromatic index*, respectively. Vizing's total-colour conjecture says that every graph admits a total colouring with $\Delta + 2$ colours and Vizing's celebrated theorem says that the chromatic index

*Universidade Federal do Rio de Janeiro, Instituto de Matemática, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil. `celina@cos.ufrj.br`. Partially supported by CNPq, grant 30 1160/91.0.

†Universidade Estadual de Campinas, Instituto de Computação, Caixa Postal 6176, 13081-970 Campinas, SP, Brazil. `{meidanis,celia}@dcc.unicamp.br`. Partially supported by FAPESP and CNPq.

is at most $\Delta + 1$, where Δ is the maximum degree [16]. A constructive proof of Vizing's theorem appeared in [13].

Graphs whose chromatic index equals Δ are said to be *Class* 1; graphs whose chromatic index equals $\Delta+1$ are said to be *Class* 2. It is well known that the set of graphs in Class 1 is NP-complete [12]. In [7, 8], we considered classes of graphs defined by special *vertex perfect elimination orders*. Our method provided for subclasses of chordal graphs such as doubly chordal graphs, strongly chordal graphs, interval graphs and indifference graphs, a simple constructive proof of Vizing's theorem.

Vizing's total-colour conjecture has been established for several classes of graphs and a recent survey book has been devoted to the subject [17]. The total-colour conjecture was proved recently for the class of split graphs, a subclass of chordal graphs [5]. Graphs whose total-chromatic number equals $\Delta + 1$ are said to be *Type* 1; graphs whose total-chromatic number equals $\Delta + 2$ are said to be *Type* 2.

In [7], we considered edge colouring doubly chordal graphs. The goal of the present note is to extend those results in two ways: we now consider both total colourings and edge colourings and we now consider dually chordal graphs, a larger class of graphs. Dually chordal graphs have attracted much attention lately [15, 4, 11, 2, 3], being the class of clique graphs of chordal graphs and also a natural generalization of strongly chordal graphs. Dually chordal graphs are defined by a special vertex elimination order: the maximum neighbourhood order whose algorithmic application has been studied to domination, packing and distance problems [2, 3]. The present note considers the algorithmic application of maximum neighbourhood order to graph-colouring problems. We prove that the total-colour conjecture holds for dually chordal graphs. Given a general dually chordal graph, our method constructs corresponding $\Delta+2$-total and $\Delta+1$-edge colourings. The presented heuristic actually yields an exact total-colouring algorithm for even maximum degree dually chordal graphs and an exact edge-colouring algorithm for odd maximum degree dually chordal graphs.

In Section 2 we define the classes of graphs considered in this note. In Section 3 we define the pull back graph homomorphism and obtain a sufficient local condition for a class of graphs to satisfy Vizing's total-colour conjecture. In Section 4 we apply our pull back method to the class of dually chordal graphs. Our conclusions in Section 5 discuss two natural extensions of our results to chordal graphs and to neighbourhood-Helly graphs.

2 Definitions and Notations

In this paper, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex and edge sets of G. A *clique* is a set of vertices pairwise adjacent in G. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph induced by* X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices which are adjacent to v. The *degree* of a vertex v is $\deg(v) = |Adj(v)|$. A vertex u is *universal* if $\deg(u) = |V(G)| - 1$. We denote by $N(v)$ the *neighbourhood* of v, that is, $N(v) = Adj(v) \cup \{v\}$. We denote by $N^2(v)$ the family of sets $\{N(u) : u \in N(v)\}$. The *distance* between two vertices u and v is the minimum length (i.e., number of edges) of a path between u and v. Given a graph G we denote by G^2 the graph having $V(G^2) = V(G)$ and satisfying $xy \in E(G^2)$ if and only if x and y are distinct and their distance in G is at most 2.

We follow the terminology of [1]. A vertex v is *simple* if $N^2(v)$ is linearly ordered by set inclusion. A vertex $u \in N(v)$ is a *maximum neighbour of* v if and only if for all $w \in N(v)$, $N(w) \subseteq N(u)$ holds. A *maximum neighbourhood elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that there is a maximum neighbour u_i of v_i in $G[v_1, v_2, \ldots, v_i]$.

A vertex v is *simplicial* if $N(v)$ is complete. A *perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *chordal* if it admits a perfect elimination order.

A *simple perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that v_i is simple in $G[v_1, v_2, \ldots, v_i]$. A vertex is *doubly simplicial* if it is simplicial and has a maximum neighbour. A *doubly perfect elimination order* of a graph G is a linear order on its vertex set $v_1, v_2, \ldots, v_n$ such that v_i is doubly simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *strongly chordal* if it admits a simple perfect elimination order. A graph is *doubly chordal* if it admits a doubly perfect elimination order [14]. Note that, by definition, every strongly chordal graph is doubly chordal.

A graph is *dually chordal* if it admits a maximum neighbourhood elimination order. Duality to chordal graphs is justified by the following characterization: a graph G has a maximum neighbourhood order if and only if its clique hypergraph $\mathcal{C}(G)$ forms a hypertree [4, 15, 11]. Recognition of dually chordal graphs can be done in $O(n^2m)$ time [15] and in $O(m)$ time [3]. Maximum neighbourhood orders are algorithmically useful, especially for domination-like problems and distance problems [2]. For a dually chordal graph a maximum neighbourhood order can be computed in linear time [3]. Note that unlike the chordal graphs, dually chordal graphs are not perfect: every graph containing a universal vertex is dually chordal. A graph is doubly chordal if and only if it is chordal and dually chordal. A graph is strongly chordal if and only if all induced subgraphs are dually chordal [1].

An *assignment of colours* to the vertices of G is a function $\lambda: V(G) \to S$, where S is a set of *colours*. A *conflict* in an assignment of colours is the existence of two adjacent vertices with the same colour. A *vertex colouring* of a graph is an assignment of colours such that there are no conflicts. The *chromatic number* of a graph G is the minimum number of colours used among all vertex colourings of G and is denoted by $\chi(G)$. Analogously we define *edge colouring*: an assignment of colours to the edges of G with no conflicts; and *total colouring*: an assignment of colours to the vertices and edges of G with no conflicts. The corresponding parameters are the *chromatic index*, $\chi'(G)$; and the *total-chromatic number*, $\chi''(G)$.

The *greedy algorithm* for vertex colouring considers the vertices of a graph according to a linear order and assigns to the current vertex the smallest available colour that does not create conflicts. A *perfect order* is a linear order on the vertex set of a graph such that the greedy algorithm vertex colours optimally all its induced subgraphs [6]. Every chordal graph admits a perfect order because every perfect elimination order is a perfect order.

In this note, we use a maximum neighbourhood order of a dually chordal graph G to vertex colour greedily G^2 with $\Delta(G) + 1$ colours. This optimal vertex colouring of G^2 is used in turn to give a simple constructive proof of Vizing's total-colour conjecture and of Vizing's theorem for the class of dually chordal graphs.

3 Pull back of complete graphs

A *pull back* from G to G' is a function $f: V(G) \to V(G')$, such that:

- f is a homomorphism, i.e., if $xy \in E(G)$, then $f(x)f(y) \in E(G')$;
- f is injective when restricted to $N(x)$, for all $x \in V(G)$.

The main use of pull backs is to transfer colourings:

Theorem 1 *If f is a pull back from G to G' and τ' is a total colouring of G', then the colour assignment τ defined by*

$$\tau(x) = \tau'(f(x))$$

$$\tau(xy) = \tau'(f(x)f(y))$$

is a total colouring of G.

Proof: Each vertex and each edge of G has a colour defined by τ because $f: V(G) \to V(G')$ is a homomorphism.

Moreover, the colour assignment τ has no conflicts: suppose $\tau(xy) = \tau(xz)$. Then, $\tau'(f(x)f(y)) = \tau'(f(x)f(z))$. Since τ' is a total colouring of G', $f(x)f(y) = f(x)f(z)$, which implies $f(y) = f(z)$. Since f is injective on $N(x)$, we have $y = z$, as required. Now suppose we have two distinct adjacent vertices x and y. Because f is a homomorphism, we have that $f(x)$ and $f(y)$ are two distinct adjacent vertices of G'. Hence, $\tau(x) = \tau'(f(x)) \neq \tau'(f(y)) = \tau(y)$. Finally, suppose we have a vertex x and an edge xy. Because f is a homomorphism, we have that $f(x)f(y)$ is an edge of G'. Hence, $\tau(x) = \tau'(f(x)) \neq \tau'(f(x)f(y)) = \tau(xy)$. ■

Corollary 1 *[7] If f is a pull back from G to G' and κ' is an edge colouring of G', then the colour assignment κ defined by*

$$\kappa(xy) = \kappa'(f(x)f(y))$$

is an edge colouring of G.

Proof: Analogous to the proof of Theorem 1. ■

A total colouring τ for G is said to be a *pull back* from a total colouring τ' for G' if τ is defined from τ' as in Theorem 1. Analogously, we define an edge colouring κ for G to be a *pull back* from an edge colouring κ' for G'.

For the convenience of the reader, we recall next the proof of a result from [7]: if G^2 admits a vertex colouring with ℓ colours, then there is a pull back from G to K_ℓ, where K_ℓ is the complete graph on ℓ vertices.

Theorem 2 *[7] There is a pull back from G to K_ℓ if and only if $\chi(G^2) \leq \ell$.*

Proof: For the "if" part, let $\lambda: V(G^2) \to S$ be a vertex colouring of G^2 with $|S| = \ell$. Consider a bijection $g: S \to V(K_\ell)$. We shall show that the composition $f = g \circ \lambda$ is a pull back from G to K_ℓ (recall that $V(G) = V(G^2)$). Since K_ℓ is a complete graph, the first condition of a pull back is satisfied by f. Now, notice that λ is also a vertex colouring of G with ℓ colours, and that all vertices which have distance in G at most two, have distinct colours. Hence, f is injective when restricted to $N(v)$, for all $v \in V(G)$.

For the "only if" part, let f be a pull back from G to K_ℓ. Then f can be viewed as a colour assignment with colour set $S = V(K_\ell)$, using at most ℓ colours. If $uv \in E(G)$, then $f(u)f(v) \in E(K_\ell)$, that is, $f(u) \neq f(v)$. If u and v are at distance 2 in G, then there is $w \in V(G)$ with $u, v \in N(w)$. Since f is injective in $N(w)$, again $f(u) \neq f(v)$. Hence, there are no conflicts and f is a vertex colouring of G^2 using at most ℓ colours. By definition this means $\chi(G^2) \leq \ell$. ■

Corollary 2 *If $\chi(G^2) \leq \ell$, then*

- $\chi''(G) \leq \ell$, *if ℓ is odd;*
- $\chi''(G) \leq \ell + 1$, *if ℓ is even.*

Proof: The total-chromatic numbers of complete graphs are well-known [9]: K_ℓ is Type 2, if ℓ is even and Type 1, if ℓ is odd. Now the corollary follows from Theorem 1 and Theorem 2. ■

Corollary 3 *If $\chi(G^2) = \Delta(G) + 1$, then*

- $\chi''(G) \leq \Delta(G) + 2$ *(i.e., Vizing's total-colour conjecture holds);*
- *If $\Delta(G)$ is even, then G is Type 1;*
- *If $\Delta(G)$ is odd, then G is Class 1.*

Proof: Any vertex colouring of G^2 needs $\Delta(G) + 1$ colours. Now take $\ell = \Delta(G) + 1$ in Corollary 2. ■

4 Dually chordal graphs

In this section we prove that Vizing's total-colour conjecture holds for the class of dually chordal graphs. This will follow from the result below:

Theorem 3 *If G is dually chordal, then $\chi(G^2) = \Delta(G) + 1$.*

Proof: Let $v_1, \dots, v_n$ be a maximum neighbourhood elimination order of G. Let $G_i = G[v_1, \dots, v_i]$ be the subgraph induced by $\{v_1, \dots, v_i\}$ and $N_i(v)$ be the neighbourhood of v in G_i. Let u_i be a maximum neighbour of v_i in G_i. By definition, for all $w \in N_i(v_i)$, $N_i(w) \subseteq N_i(u_i)$. Hence, there are at most $\Delta(G) + 1$ vertices in the family of sets $N_i^2(v_i) = \{N_i(w) : w \in N_i(v_i)\}$.

Thus, given a maximum neighbourhood order of G, the greedy algorithm uses at most $\Delta(G) + 1$ colours to colour the vertices of G^2. ■

It follows from Theorem 2 and Theorem 3 that we can pull back total colourings and edge colourings from complete graphs to colour dually chordal graphs.

Corollary 4 *Let G be a dually chordal graph. Then*

- $\chi''(G) \leq \Delta(G) + 2$ *(i.e., Vizing's total-colour conjecture holds);*
- *If $\Delta(G)$ is even, then G is Type 1;*
- *If $\Delta(G)$ is odd, then G is Class 1.*

In particular, these conditions hold if G is doubly chordal, strongly chordal or interval graph. ■

5 Conclusions

Note that Theorem 3 above actually shows that every maximum neighbourhood order of a graph G is a perfect elimination order of G^2, i.e., every dually chordal graph G has G^2 chordal [4]. We now discuss the extension of Theorem 3 to two classes graphs: neighbourhood-Helly graphs and chordal graphs. The extension of Theorem 3 to neighbourhood-Helly graphs was suggested to us by an unknown referee.

A graph is *neighbourhood-Helly* when the set $\{N(v) : v \in V\}$ satisfies the Helly property. A characterization of dually chordal graphs says that G is dually chordal if and only if G is neighbourhood-Helly and G^2 is chordal [1].

Consider the class of graphs G satisfying the following two properties:

1. the cardinality of a maximum clique of G^2 equals the chromatic number of G^2;

2. for every maximum clique C of G^2 there exists a vertex v of G such that $N_G(v) = C$.

These two properties are enough to imply that $\chi(G^2) = \Delta + 1$. Note that Property 1 is satisfied by graphs G having G^2 perfect. Property 2 is satisfied by all neighbourhood-Helly graphs. Thus we have the following extension of Theorem 3.

Corollary 5 *Vizing's total-colour conjecture holds for neighbourhood-Helly graphs G such that G^2 is perfect.* ■

Note that neighbourhood-Helly graphs are not known to be characterized by elimination orders. In order to get an optimal vertex colouring for the perfect graph G^2 we may use Grötschel, Lovász and Schrijver's vertex-colouring algorithm for perfect graphs [10]. Feodor Dragan pointed out that the validity of the Strong Perfect Graph Conjecture for neighbourhood-Helly graphs might provide an elimination order characterization for this class.

Consider now the class of chordal graphs, and the chordal graph G depicted in Figure 1. This graph has diameter 2; or more precisely $G^2 = K_7$. Thus, we need 7 colours for any vertex colouring of G^2, that is $\chi(G^2) = 7$. On the other hand, $\Delta(G) = 5$. Therefore, Theorem 3 does not generalize to arbitrary chordal graphs.

Nevertheless we were unable to find any evidence that Corollary 4 does not hold for chordal graphs. Note that although our pull back method does not apply to the graph in Figure 1, this odd maximum degree chordal graph is Class 1. We conjecture that all chordal graphs with odd maximum degree are in fact Class 1. A more general question, which we leave open, is to determine the largest graph class for which all its odd maximum degree graphs are Class 1 and for which all its even maximum degree graphs are Type 1.

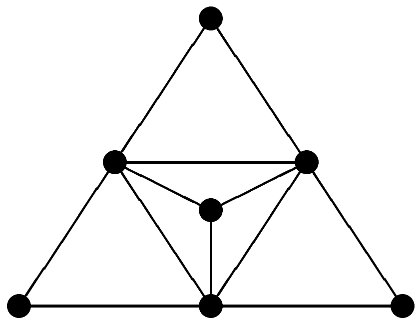

Figure 1: A chordal graph with $\chi(G^2) > \Delta(G) + 1$.

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