Optimal node-degree bounds for the complexity of nonplanarity parameters*

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Abstract

We prove that both the NP-completeness of the NONPLANAR DELETION decision problem and the Max SNP-hardness of the NONPLANAR DELETION problem remain true even for cubic graphs. We prove that the class of graphs with splitting number less than or equal to a fixed k is minor closed, which implies the existence of a corresponding polynomial-time recognition algorithm.

1 Introduction.

A natural question in the study of the complexity of a graph-theoretical decision problem is to determine the best possible bounds on the node degrees for which the problem remains NP-complete. Yannakakis [11] considered the complexity of edge-deletion decision problems and obtained corresponding best possible node-degree bounds for the NP-completeness of the edge-deletion bipartite problem and of the edge-deletion comparability graph problem.

The edge-deletion planar graph problem is also known as NONPLANAR DELETION: given a graph $G = (V, E)$, produce a smallest subset $L \subseteq E$ such that $(V, E \backslash L)$ is planar. The NONPLANAR DELETION decision problem was shown to be NP-complete by Liu and Geldmacher [8] and independently by Yannakakis in the fundamental paper cited above [11]. Both NP-completeness proofs left the corresponding best possible node-degree bounds unanswered. The *skewness* of a graph G is the smallest integer $\kappa(G)$ such that a planar graph can be obtained from G by removing $\kappa(G)$ edges.

More recently, Călinescu et al. [1] showed that NONPLANAR DELETION is Max SNP-hard, which implies that there is a constant $\varepsilon > 0$ such that the existence of a polynomial approximation algorithm with performance ratio at least $1 - \varepsilon$ implies that P = NP.

The goal of the present paper is to show that both the NP-completeness of the NONPLANAR DELETION decision problem and the Max SNP-hardness of the NONPLANAR DELETION problem remain true even for cubic graphs. The proofs cited above [11, 8, 1] did not use graphs with maximum node degree 3. Moreover, our results are optimum with respect to the allowed maximum node degree of the instances for the establishment of the mentioned complexity, because a graph with maximum degree 2 is a collection of paths and circuits that define a planar graph. Our optimum node-degree argument implies a surprising additional fact: the complexity results obtained for NONPLANAR DELETION hold even when restricted to graphs with no subdivision of K_5.

2 Splitting Number.

We considered in [4] the complexity of computing the splitting number, a graph parameter that is used as a measure of nonplanarity in many applications, such as graph drawing. The *splitting number* of a graph G is the smallest integer $\sigma(G)$ such that a planar graph can be obtained from G by $\sigma(G)$ vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 and P_2 and adds to $G \setminus v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a set of k splittings, then we say that H is the *resulting graph* of this set of k splittings in G. Note that the resulting graph H can be obtained either by splitting only vertices of G, or by splitting vertices of G and vertices created by former splittings.

Two aspects of the study of splitting numbers have been considered recently by Eades and Mendonça [2, 3]: they established the NP-completeness of a related problem (ELIGIBLE SPLIT SET) and they successfully used splitting numbers in layout algorithm design. The splitting number has been computed for complete graphs [5] and for complete bipartite graphs [6]. For a recent survey on splitting numbers, see [7].

3 Splitting Number and Nonplanar Deletion.

Research on splitting number can be also justified by its interdependency with other nonplanarity parameters. It

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is not difficult to prove that if G is a cubic graph, then $\kappa(G) = \sigma(G)$.

Recently [4], we have established the complexity of the SPLITTING NUMBER decision problem by constructing a reduction from 3-SAT. We proved that the SPLITTING NUMBER decision problem is NP-complete when restricted to cubic graphs. Consequently, we established that the NONPLANAR DELETION decision problem remains NP-complete when restricted to cubic graphs.

In the present paper, in order to establish that SPLITTING NUMBER and consequently that NONPLANAR DELETION are Max SNP-hard even for cubic graphs, we use the concept of *L-reductions* [9], a special kind of reduction that preserves approximability. To achieve the optimum node-degree bound with respect to Max SNP-hardness, we have strengthened our initial NP-completeness proof [4] by considering this time the MAX SNP–complete problem $3\text{SAT}_{\bar{3}}$ [9], a restricted version of 3-SAT, where each variable appears at most three times in the set of clauses.

We obtain the MAX SNP–hardness for SPLITTING NUMBER and NONPLANAR DELECTION by proving two results. Firstly, we show that if $I = (U, C)$ is an instance for $3\text{SAT}_{\bar{3}}$ with $|U| = n$ and $|C| = m$, then $\lceil \frac{n}{3} \rceil \leq m \leq 3n$ and $Opt_{3\text{SAT}_{\bar{3}}}(I) \geq \lceil \frac{n}{3} \rceil$, where $Opt_{3\text{SAT}_{\bar{3}}}(I)$ denotes the optimum value of $3\text{SAT}_{\bar{3}}$ for I. Secondly, we show that if G is the special instance for SPLITTING NUMBER constructed in [4] from I, then the optimum value of SPLITTING NUMBER for G is equal to $4n + 5Opt_{3\text{SAT}_{\bar{3}}}(I) + 6(m - Opt_{3\text{SAT}_{\bar{3}}}(I))$. These two statements yield the requirements for an L–reduction from $3\text{SAT}_{\bar{3}}$ to SPLITTING NUMBER. Since our instance is a cubic graph G, we have $\sigma(G) = \kappa(G)$ and thus the optimum values of SPLITTING NUMBER and of NONPLANAR DELECTION for G are the same. Therefore we have established the MAX SNP–hardness of both SPLITTING NUMBER and NONPLANAR DELECTION.

It is easily seen that, for a fixed value of k, the NONPLANAR DELETION decision problem is polynomial. On the other hand, the fact that the number of all possible splittings for a vertex in a graph G is of order $\Omega(2^{|V(G)|})$ seems to indicate that the SPLITTING NUMBER decision problem, even for a fixed value of k, is not a polynomial problem. However, we prove that the class of graphs with splitting number less than or equal to a fixed k is *minor closed*, which implies [10] the existence of a corresponding polynomial-time recognition algorithm.

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