Recognition of quasi-Meyniel graphs[*]

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Abstract

We present a polynomial-time algorithm for recognizing quasi-Meyniel graphs. Hertz defined quasi-Meyniel graphs and showed how this auxiliary generalization of Meyniel graphs could be used to solve efficiently both the maximum clique problem and the minimum coloring problem for Meyniel graphs through a sequence of even pair contractions.

1 Introduction

Two nonadjacent vertices x and y in a graph G form an *even pair* if every induced path between them has an even number of edges. For a given pair $\{x, y\}$ in a graph G, we denote by G_{xy} the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely those vertices of $G \setminus \{x, y\}$ which are adjacent to at least one of x or y in G. We say that G_{xy} is obtained by *contracting* on $\{x, y\}$. If $\{x, y\}$ is an even pair in G, then a largest clique in G_{xy} has the same size as a largest clique in G and the chromatic number of G is equal to the chromatic number of G_{xy}. Consider a sequence of graphs $G_0, \ldots, G_j$, such that for $i \leq j-1$, G_{i+1} is obtained from G_i by contracting an even pair $\{x_i, y_i\}$ and such that G_j is a clique. As G_j is a clique, it is trivial to obtain an optimal coloring and find a largest clique within it. We can then work backwards to find a largest clique and an optimal coloring of G_0. This simple procedure has been used to develop fast algorithms for finding a largest clique and an optimal coloring in many classes of perfect graphs [2].

A cycle is *even* if it contains an even number of vertices, and it is *odd* otherwise. A *hole* is a chordless cycle with at least four vertices. We denote by P_4 a chordless path on four vertices. A *cap* is a cycle of length greater than 4, with a single chord that forms a triangle with two edges of the cycle.

A graph is *Meyniel* if every odd cycle of length at least 5 has two or more chords. Clearly, a graph is Meyniel if and only if it contains no odd hole and no cap. Meyniel [4] proved that Meyniel graphs are perfect and subsequently [5] showed that every Meyniel graph is either a clique or contains an even pair. On the other hand, there are Meyniel graphs which are not a clique and yet contain no even pair whose contraction yields a Meyniel graph. One such graph can be obtained by substituting every vertex in a hole of length 6 by a pair of adjacent vertices. In order to optimize Meyniel graphs through a sequence of even pair contractions, Hertz [3], sidestepped this problem by defining a suitable larger class of graphs.

A graph G is *quasi-Meyniel* if it contains no odd hole and for some $x \in V(G)$, the chord of every cap in G has x as an endvertex. Such a vertex x is called a *tip* of G.

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Note that every Meyniel graph G is quasi-Meyniel, with every vertex of G being a tip. If G is quasi-Meyniel, but not Meyniel, then G has at most two tips and if x is a tip of G, then $G \setminus \{x\}$ is Meyniel.

Hertz proved that every quasi-Meyniel graph which is not a clique contains an even pair whose contraction yields a quasi-Meyniel graph.

Lemma 1 *[3] Let x be a tip of a quasi-Meyniel graph G which is not adjacent to all of $G \setminus \{x\}$, and let y be a nonneighbour of x maximizing $|N(x) \cap N(y)|$. Then x and y form an even pair in G, G_{xy} is quasi-Meyniel, and xy is a tip of G_{xy}.*

The polynomial-time recognition of Meyniel graphs was established by Burlet and Fonlupt [1] when they defined the amalgam decomposition and proved that the amalgam of two Meyniel graphs is a Meyniel graph and conversely, any Meyniel graph can be amalgam-decomposed in polynomial time into basic Meyniel graphs which in turn can be recognized in polynomial time. In this paper we present a polynomial-time recognition algorithm for the class of quasi-Meyniel graphs. Our algorithm follows from a decomposition of quasi-Meyniel graphs by star cutsets.

2 Decomposition

A graph G *contains* a graph H, if H is an induced subgraph of G. A graph is *H-free*, if it does not contain H.

A *wheel*, denoted by (H, x), is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H. Vertex x is the *center* of the wheel. A *sector* of the wheel is a subpath of H whose endvertices are neighbors of x and intermediate nodes are not. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. Two distinct sectors of a wheel are *adjacent* if they have a vertex in common. A *twin wheel* is a wheel with three sectors, two of which are short. A *starred wheel* is a wheel all of whose sectors are short. A *triangle-free wheel* is a wheel all of whose sectors are long.

Lemma 2 *Let G be a quasi-Meyniel graph with tip x. If (H, y) is a wheel in G with $y \neq x$, then it is a twin wheel, a starred wheel or a triangle-free wheel.*

Proof: Let (H, y), $y \neq x$, be a wheel in a quasi-Meyniel graph with tip x and let $y_1, y_2, \ldots y_n$ be the neighbors of y in H encountered in this order when traversing H clockwise. For $i = 1, \ldots, n$, let S_i be the sector of (H, y) with endvertices y_i and y_{i+1} (where $y_{n+1} = y_1$). Assume (H, y) is not a twin wheel, a starred wheel nor a triangle-free wheel. Then (H, y) contains at least one short sector and at least one long sector. In fact, for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, one of the sectors S_i, S_{i+1} is short and the other one is long and one of the sectors S_j, S_{j+1} is short and the other one is long. Since (H, y) is not a twin wheel, both vertex sets $V(S_i) \cup V(S_{i+1}) \cup \{y\}$ and $V(S_j) \cup V(S_{j+1}) \cup \{y\}$ induce a cap. But this contradicts the assumption that x is a tip of G, since $y \neq x$ and $y_{i+1} \neq y_{j+1}$. ■

Given a graph G and $S \subseteq V(G)$, $G \setminus S$ denotes a subgraph of G induced by the vertex set $V(G) \setminus S$. A node set $S \subseteq V(G)$ is a *cutset* of a connected graph G, if the graph $G \setminus S$ is disconnected. Let H_1 and H_2 be induced subgraphs of G. A cutset S of G *separates H_1 from H_2*, if H_1 and H_2 are contained in different components of $G \setminus S$.

Let A, B and S be disjoint vertex sets of a graph G, such that no vertex of A is adjacent to a vertex of B. A path $P = x_1, \ldots, x_n$ is a *direct connection from A to B in $G \setminus S$*, if $V(P) \subseteq V(G) \setminus S$, x_1 is adjacent to some vertex of A, x_n is adjacent to some vertex of B, and no proper subset of $V(P)$

induces a path with these properties. In other words, P is a chordless path, no intermediate vertex of P is adjacent to a vertex of $A \cup B$, and if $n > 1$ then x_1 is not adjacent to a vertex of B and x_n is not adjacent to a vertex of A. Note that if S is not a cutset separating A from B, then there exists a direct connection from A to B in $G \setminus S$.

Given a triangle $\{x_1, x_2, x_3\}$ and a vertex y adjacent to at most one vertex in $\{x_1, x_2, x_3\}$, a $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths, $P_1 = x_1, \ldots, y$, $P_2 = x_2, \ldots, y$ and $P_3 = x_3, \ldots, y$, having no common vertices other than y and such that the only adjacencies between the vertices of $P_1 \setminus \{y\}$, $P_2 \setminus \{y\}$ and $P_3 \setminus \{y\}$ are the edges of the triangle $\{x_1, x_2, x_3\}$. Note that for every $i, j \in \{1, 2, 3\}$, $i \neq j$, $V(P_i) \cup V(P_j)$ induces a hole. Since two of the paths must have equal parity, one of the three holes contained in $3PC(x_1x_2x_3, y)$ is odd. Therefore odd-hole-free graphs cannot contain a $3PC(x_1x_2x_3, y)$. This fact will be used in the next lemma.

Lemma 3 *Let G be a quasi-Meyniel graph with tip x. Suppose G contains a cap induced by a hole $H = x, x_1, \ldots, x_k, x$ and a vertex y adjacent to x and x_k. Then $S = (N(x) \cup \{x\}) \setminus \{y\}$ is a cutset separating y from $H \setminus S$.*

Proof: We prove that $S = (N(x) \cup \{x\}) \setminus \{y, x_1\}$ is a cutset separating y from $H \setminus S$, which clearly implies the lemma.

Suppose S is not a cutset separating y from $H \setminus S$ and let $P = p_1, \ldots, p_n$ be a direct connection from y to $H \setminus S$ in $G \setminus S$. Note that x does not have a neighbor in P but x_k possibly does. Let x_i be the vertex of H with lowest index adjacent to p_n. Note that $i < k$. Let H' be the hole induced by the vertex set $V(P) \cup \{y, x, x_1, \ldots, x_i\}$. We now consider the following two cases.

Case 1: x_k is adjacent to a vertex of P.

Then (H', x_k) is a wheel that contains at least one short sector (since x_k is adjacent to x and y) and at least one long sector (since x_k is not adjacent to x_1). Hence, by Lemma 2, (H', x_k) is a twin wheel. So the only neighbor of x_k in P is p_1. Let x_j be the neighbor of p_n in $H \setminus \{x_k\}$ with highest index. Note that possibly $i = j$. If $n \neq 1$ or $j \neq k - 1$, then the vertex set $V(P) \cup \{y, x_j, \ldots, x_k\}$ induces a cap, contradicting the assumption that x is a tip of G. Hence, $n = 1$ and $j = k - 1$. If $i = j$, then (H', x_k) is a wheel that contradicts Lemma 2. Hence, $i \neq j$, and so (H, p_1) is a wheel. By Lemma 2, (H, p_1) is a twin wheel. So the neighbors of p_1 in H are x_k, x_{k-1} and x_{k-2}. But then the vertex set $(V(H) \cup \{y, p_1\}) \setminus \{x_k\}$ induces a cap, contradicting the assumption that x is a tip of G.

Case 2: x_k is not adjacent to a vertex of P.

Let x_j be the neighbor of p_n in H with highest index. If $i = j$, then the vertex set $V(H) \cup V(P) \cup \{y\}$ induces a $3PC(xyx_k, x_i)$. Hence $i \neq j$. If x_ix_j is an edge, then the vertex set $V(P) \cup \{y, x_i, \ldots, x_k\}$ induces a cap, contradicting the assumption that x is a tip of G. Otherwise, the vertex set $V(P) \cup \{x, x_1, \ldots, x_i, x_j, \ldots, x_k\}$ induces a $3PC(xyx_k, p_n)$. ∎

A node set $S \subseteq V(G)$ is a *star cutset* of G if S is a cutset of G and for some $x \in S$, x is adjacent to all the vertices of $S \setminus \{x\}$. Vertex x is called the *center* of the star cutset S.

Let a node set S be a cutset of a graph G, and let $C_1, \ldots, C_n$ be the connected components of $G \setminus S$. The *blocks of decomposition* by S are graphs $G_1, \ldots, G_n$ where, for every $i \in \{1, \ldots, n\}$, G_i is the subgraph of G induced by the vertex set $V(C_i) \cup S$.

Lemma 4 *Suppose that a graph G and a vertex $x \in V(G)$ are such that $G \setminus \{x\}$ is Meyniel. Let S be a star cutset of G with center x, and let $G_1, \ldots, G_n$ be the blocks of decomposition by S. Then G is quasi-Meyniel with tip x if and only if, for every $i \in \{1, \ldots, n\}$, G_i is quasi-Meyniel with tip x.*

Proof: If G is quasi-Meyniel with tip x, then so are all the G_i's, since they are induced subgraphs of G. To prove the converse, assume that G contains an odd hole H or a cap H in which x is not and endvertex of a chord. Since $G \setminus \{x\}$ is Meyniel, H contains x. But then $H \setminus S$ is contained in some connected component C_i of $G \setminus S$, and so H is contained in G_i. Hence, G_i is not quasi-Meyniel with tip x. ■

3 Algorithm

We say that $S \subseteq V(G)$ is a *good star cutset with center x*, if for some nonadjacent neighbors y and z of x, $S = (N(x) \cup \{x\}) \setminus \{y\}$ and $G \setminus S$ contains distinct connected components C' and C'' such that $y \in C'$ and z has a neighbor in C''.

Recognition Algorithm

Input: A graph G and a vertex $x \in V(G)$.

Output: YES, if G is quasi-Meyniel with tip x, and NO otherwise.

Step 1: Check whether $G \setminus \{x\}$ is Meyniel. If it is, then continue. Otherwise stop and return NO.

Step 2: Let $\mathcal{L}_1 = G$ and $\mathcal{L}_2 = \emptyset$.

Step 3: If $\mathcal{L}_1 = \emptyset$, then go to Step 5, else continue.

Step 4: Remove a graph F from $\mathcal{L}_1$. Check whether there is a good star cutset S with center x in F. If there is such a cutset S, then decompose F by S and add the blocks of decomposition to $\mathcal{L}_1$. Otherwise, add F to $\mathcal{L}_2$. Go to Step 3.

Step 5: Check whether all the graphs in $\mathcal{L}_2$ are Meyniel. If they are, then return YES, and otherwise return NO.

Lemma 5 *The above algorithm correctly identifies whether G is quasi-Meyniel with tip x.*

Proof: If $G \setminus \{x\}$ is not Meyniel, then clearly G is not quasi-Meyniel with tip x, and the algorithm correctly terminates in Step 1. So assume that $G \setminus \{x\}$ is Meyniel. The correctness of the algorithm will follow from showing that G is quasi-Meyniel with tip x if and only if all the graphs in $\mathcal{L}_2$ are Meyniel. By repeated application of Lemma 4, G is quasi-Meyniel with tip x if and only if