Even and Odd Pairs in Comparability and in P_4-Comparability Graphs

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Abstract

We characterize even and odd pairs in comparability and in P_4-comparability graphs. The characterizations lead to simple algorithms for deciding whether a given pair of vertices forms an even or odd pair in these classes of graphs. The complexities of the proposed algorithms are $O(n+m)$ for comparability graphs and $O(n^2m)$ for P_4-comparability graphs. The former represents an improvement over a recent algorithm of complexity $O(nm)$.

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1 Introduction

Let G denote a simple non-trivial connected undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Write $n = |V(G)|$ and $m = |E(G)|$. A **chord** of a path P is an edge of G incident to two non-consecutive vertices of P. An **induced path** contains no chords. A path is **even** or **odd** according to the parity of its number of edges. Vertices v, w form an **even pair** when G contains no odd induced path between them. They constitute an **odd pair** when $(v, w) \notin E(G)$ and there is no even induced path between v and w.

Let $\vec{G}$ represent an acyclic orientation of G. For $v \in V(G)$, write $N^-_{\vec{G}}(v) = \{w|(w, v) \in E(\vec{G})\}$ and $N^+_{\vec{G}}(v) = \{w|(v, w) \in E(\vec{G})\}$. We say that $\vec{G}$ is **transitive** when $(v, w), (w, z) \in E(\vec{G})$ implies $(v, z) \in E(\vec{G})$, for all $v, w, z \in V(G)$. A **comparability** graph is one admitting a transitive orientation. Given a comparability graph, a transitive orientation can be constructed in linear time [9].

Denote by P_4 an induced path on four vertices. A graph G is **P_4-comparability** [6, 7] if it admits an acyclic orientation that is transitive when restricted to any P_4 of G. We call such an orientation, a **P_4-transitive orientation**. The Hajós graph, depicted in Figure 1, gives an example of a P_4-comparability graph that is not a comparability graph. Given a P_4-comparability graph, a P_4-transitive orientation can be constructed in $O(n^5)$ time [6]. More recently, such orientation has been constructed in $O(n^2m)$ time [11].

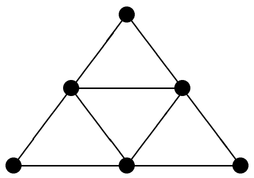

Figure 1: A P_4-comparability but not a comparability graph.

We formulate simple characterizations for a pair of vertices in a P_4-

comparability graph to form an even or odd pair of the graph. They lead to efficient algorithms for solving the **parity path problems** both in P_4-comparability graphs and in comparability graphs. That is, decide whether or not two given vertices form an even or an odd pair, respectively. The complexity of the algorithms is that of finding a P_4-transitive orientation or that of finding a transitive orientation, respectively. However, it reduces to linear time, when the corresponding orientations are given.

Even pairs have been first considered by Meyniel [10], in the context of perfect graphs. They have motivated the definition of some special classes of perfect graphs, as quasi-parity, strict quasi-parity and perfectly contractile graphs. The parity path problems are Co-NP-complete, in general [4]. However, special classes of graphs admit polynomial-time algorithms. They include chordal graphs [1], circular arc graphs [3] and planar perfect graphs having the two given vertices in a same face [8]. Recently, Arikati and Peled [2] presented an algorithm that solves in time $O(n^{21})$ the parity path problems for perfectly orientable graphs, a superclass of perfectly orderable graphs. The latter contains P_4-comparability graphs. However, the presently proposed algorithms for P_4-comparability graphs are substantially simpler and their complexity lower than that for perfectly orientable graphs.

The parity path problems were recently considered also in [12] for the following subclasses of perfect graphs: comparability, cocomparability and permutation graphs. The latter algorithms for comparability graphs require $O(nm)$ time. The algorithms for permutation and cocomparability graphs in [12] require $O(n+m)$ and $O(n^2m)$ time, respectively.

In the present paper we describe structural characterizations for even and odd pairs in comparability and P_4-comparability graphs. These characterizations lead to simple algorithms for deciding whether or not a given pair of vertices form an even or an odd pair, for graphs of these classes. The complexity of the proposed algorithms for comparability graphs is $O(n+m)$, provided a transitive orientation is given. The complexity of the algorithms described in [12] for the latter class remains $O(nm)$, even if such an orientation is given. We recall that a transitive orientation of a comparability graph can be obtained also in $O(n+m)$ time. The proposed algorithms for P_4-comparability graphs require linear time too, when a P_4-transitive orientation is given. We recall that a P_4-transitive orientation of a P_4-comparability graph can be obtained in $O(n^5)$ or as shown recently in $O(n^2m)$ time.

Besides their application in the study of perfect graphs, even and odd pairs are also of interest in the pure context of comparability graphs, as pairs of sources and sinks of a comparability graph G can be characterized in terms of even and odd pairs of G [5].

2 Characterizations

The following definition is needed. Let G be a connected graph, $S \subset V(G)$ and $v, w \in V(G) \setminus S$. We say set S **separates** v, w in G when v and w lie in distinct connected components of $G - S$.

Theorem 1 *Let G be a connected P_4-comparability graph, $\vec{G}$ any P_4-transitive orientation of it and $v, w \in V(G)$. Then v, w is an even pair if and only if*

$$N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \cup (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w)) \text{ and } N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cup (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$$

both separate v, w in G.

Proof: Let us assume first that v, w is an even pair of G. Then all induced paths of G between v and w are even. That is, all induced paths between v and w are of the form $v = x_1, ..., x_{2k+1} = w$, $k \geq 1$. Moreover, every general path P between v and w contains an even induced path between these two vertices formed by a subset of vertices of P. Let us consider first the case $k > 1$, i.e., induced paths with at least 4 edges between v and w. Since G is a P_4-comparability graph, the orientations of the edges in any induced path with at least 4 edges alternate. That is, all induced paths with at least 4 edges between v and w are of the form $v = x_1, ..., x_{2k+1} = w$, $k > 1$, where $(v, x_2) \in E(G)$ is oriented from v to x_2 if and only if $(w, x_{2k}) \in E(G)$ is oriented from w to x_{2k}. Hence $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w)$ meets all induced paths with at least 4 edges between v and w. So does $N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v)$. We are left with case $k = 1$, i.e., induced paths with precisely two edges between v and w. If the set $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w)$ does not meet a given induced path with precisely two edges between v and w, then set $(N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ does. Analogously if the set $N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v)$ does not meet a given induced path with precisely two edges between v and w, then set $(N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$ does. Therefore $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \cup (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ and $N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cup (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$ both separate v, w in G.

On the other hand, let us assume that $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \cup (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ and $N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cup (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$ both separate v, w in G. Because G is a P_4-comparability graph, all induced paths of G with at least 3 edges are alternating in $\vec{G}$. Partition the set of induced paths of G between v and w with at least 3 edges into four disjoint subsets E_1, E_2, O_1, O_2. The set $E_1 \cup E_2$ contains the even induced paths $v = x_1, ..., x_{2k+1} = w$, $k > 1$. Set E_1 is formed by the paths where $(v, x_2), (w, x_{2k}) \in E(\vec{G})$ while E_2 contains those satisfying $(x_2, v), (x_{2k}, w) \in E(\vec{G})$. The set $O_1 \cup O_2$ contains the odd induced paths $v = x_1, ..., x_{2k} = w$, $k > 1$. The paths belonging to O_1 satisfy $(v, x_2), (x_{2k-1}, w) \in E(\vec{G})$, while those of O_2 satisfy $(x_2, v), (w, x_{2k-1}) \in E(\vec{G})$. Set $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w)$ meets all paths of $E_1 \cup E_2 \cup O_2$, but none of O_1. Set $(N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ does not meet paths in O_1 either. Because $N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \cup (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ separates v, w in G it follows $O_1 = \emptyset$. Similarly, $N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cup (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$ separating v, w implies $O_2 = \emptyset$. Therefore v, w is an even pair. ■

Theorem 2 *Let G be a connected P_4-comparability graph, $\vec{G}$ any P_4-transitive orientation of it and $v, w \in V(G)$. Then v, w is an odd pair if and only if*

$$N^-_{\vec{G}}(v) \cup N^-_{\vec{G}}(w) \setminus (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w)) \text{ and } N^+_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \setminus (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$$

both separate v, w in G.

Proof: Similar to that of Theorem 1. ■

By noting that for comparability graphs both sets $(N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))$ and $(N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))$ are empty, whenever $(v, w) \notin E(G)$, we obtain in that particular case the following characterizations:

Corollary 1 *Let G be a connected comparability graph, $\vec{G}$ any transitive orientation of it and $v, w \in V(G)$. Then v, w is an even pair if and only if*

$$N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \textit{ and } N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v)$$

both separate v, w in G.

Corollary 2 *Let G be a connected comparability graph, $\vec{G}$ any transitive orientation of it and $v, w \in V(G)$. Then v, w is an odd pair if and only if*

$$N^-_{\vec{G}}(v) \cup N^-_{\vec{G}}(w) \textit{ and } N^+_{\vec{G}}(v) \cup N^+_{\vec{G}}(w)$$

both separate v, w in G.

3 Conclusions

The parity path problems for P_4-comparability graphs can be solved applying Theorems 1 and 2. Given a P_4-comparability graph G and $v, w \in V(G)$, find a P_4-transitive orientation $\vec{G}$ of it. Then v, w form an even pair of G precisely when they lie in distinct connected components of $G \setminus [N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w)]$ and $G \setminus [N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v) \cup N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w)]$. The odd pair problem is similar, except that the latter sets are $G \setminus [N^-_{\vec{G}}(v) \cup N^-_{\vec{G}}(w) \setminus (N^+_{\vec{G}}(v) \cap N^-_{\vec{G}}(w))]$ and $G \setminus [N^+_{\vec{G}}(v) \cup N^+_{\vec{G}}(w) \setminus (N^-_{\vec{G}}(v) \cap N^+_{\vec{G}}(w))]$, respectively.

Now consider the particular case when we have a comparability graph. In this case, v, w form an even pair of G precisely when they lie in distinct connected components of $G \setminus [N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(w)]$ and $G \setminus [N^-_{\vec{G}}(w) \cup N^+_{\vec{G}}(v)]$. The odd pair problem is similar, except that the latter are $G \setminus [N^-_{\vec{G}}(v) \cup N^-_{\vec{G}}(w)]$ and

$G\setminus[N^+_{\vec{G}}(v)\cup N^+_{\vec{G}}(w)]$, respectively. A transitive orientation of a comparability graph can be constructed in linear time [9]. The remaining operations of the proposed algorithms are simple and can also be performed within this bound. The given graph G is supposed to be a comparability graph. Otherwise the algorithms would require G to be recognized as such, in a previous step.

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