The splitting number of the 4–cube*

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Abstract. The splitting number of a graph G consists in the smallest positive integer $k \geq 0$, such that a planar graph can be obtained from G by k splitting operations, such operation replaces v by two nonadjacent vertices v_1 and v_2, and attaches the neighbors of v either to v_1 or to v_2. One of the most useful graphs in computer science is the n–cube. Dean and Richter devoted an article to proving that the minimum number of crossings in an optimum drawing of the 4–cube is 8, but no results about splitting number of nonplanar n–cubes are known. In this note we give a proof that the splitting number of the 4–cube is 4. In addition, we give the lower bound 2^{n-2} for the splitting number of the n–cube. Because it is known that the splitting number of the n–cube is $O(2^n)$, our result implies that the splitting number of the n-cube is $\Theta(2^n)$.

1 Introduction

Applications in Computer Science are frequently modeled with nonplanar graphs. Graph visualization and VLSI projects many times require strategies of layout techniques. Layout algorithms are limited to special classes of graphs. For instance, there is a wealth of layout algorithms for planar graphs; however, these algorithms are useless for nonplanar graphs. One approach to handling nonplanarity in layout algorithms is to consider another topological invariant of the graph, the splitting number. The splitting number is a graph invariant that is used as a measure of nonplanarity in many applications such as graph drawing. Research on topological properties of the n–cube is important for applications such as Parallel Processing. In this article we prove that the splitting number of the 4–cube is 4. As we shall see, this result implies that the splitting number of the n–cube is $\Theta(2^n)$.

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the

* Work partially supported by CNPq, CAPES, FAPERJ and FAPESP, Brazilian research agencies.

plane such that no edges cross. In what follows, all drawings are assumed to be simple.

A drawing of a graph G is *optimum* when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $\nu(G)$. The *skewness* $\kappa(G)$ of G is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *splitting number* $\sigma(G)$ of a graph G is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbours of v into two nonempty sets P_1 and P_2 and adds to $G \setminus v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbours of v_1 and P_2 is the set of neighbours of v_2. If a graph H is obtained from G by a set of k splittings, we say that H is the *resulting graph* of this set of k splittings in G. We note that the resulting graph H can be obtained either by splitting only vertices of G, or by splitting vertices of G and vertices created by former splittings.

Two aspects of the study of splitting numbers have been considered recently by Eades and Mendonça [5, 6, 20]: they established the NP-completeness of a related problem — ELIGIBLE SPLIT SET —, and they successfully used splitting numbers in layout algorithm design.

Research on the splitting number of graphs can be also justified by the interdependency of the following three nonplanarity parameters of a graph: crossing number, skewness and splitting number. Consider an optimum drawing for G with $\nu(G)$ crossings. If in this drawing one of the edges in each crossing is removed from G, then there exists a set with at most $\nu(G)$ edges such that the removal of these edges yields a planar graph. This means that the crossing number $\nu(G)$ is greater than or equal to the skewness $\kappa(G)$. On the other hand, there are $\kappa(G)$ edges whose removal yields a planar resulting graph. Suppose that $e = (u, u_1)$ is one of these $\kappa(G)$ edges and that the neighbourhood of u is given by $Adj(u) = \{u_1, u_2, \ldots, u_d\}$. Let the splitting of u into the vertices v_1 and v_2 be such that $Adj(v_1) = \{u_2, u_3 \ldots, u_d\}$ and $Adj(v_2) = \{u_1\}$. Note that this splitting removes the crossing that occurred in $e = (u, u_1)$. Hence there is a set of splittings of size at most $\kappa(G)$, which yields a planar graph from G. Hence, $\kappa(G) \geq \sigma(G)$. Therefore, the study of the splitting number for a given graph helps to obtain lower bounds both for its skewness and its crossing number.

Very little is known about skewness, crossing numbers or splitting numbers for specific classes of graphs. The corresponding decision problems are all NP-complete [18, 11, 10]. For fixed k, CROSSING NUMBER turns out to be polynomial [11]. We show in Lemma 1 that the number of all possible splittings for a vertex in a graph G is $\Omega(2^{|V(G)|})$, which suggests that SPLITTING NUMBER is not a polynomial problem, even for fixed k.

The difficulty of finding the values for these invariants can justify articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [1, 2].

The knowledge of one of these invariants for the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, $\nu(C_3 \times C_3) = 3$ was proved in [12] and used later in [21] to establish that $\nu(C_3 \times C_n) = n$.

The splitting number has been computed for complete graphs [13] and for complete bipartite graphs [14]. For a recent survey on splitting numbers, see [17].

Let Q_n denote the n–*cube* graph. The vertices of Q_n are all n–tuples of 0's and 1's, of which there are $|V(Q_n)| = 2^n$. Two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if $x_i \neq y_i$, for exactly one index i.

Much has been done about properties of nonplanarity for the class of n–cubes. Eggleton and Guy [7] conjectured that $\nu(Q_n) \leq 4^n \frac{5}{32} - 2^{n-2}\lceil \frac{n^2+1}{2} \rceil$. Madej [19] established an upper bound for the crossing number of the n–cube: $\nu(Q_n) \leq 4^n \frac{1}{6} - 2^{n-3}n^2 + 2^{n-4}3 + (-2)^n \frac{1}{48}$. Sýkora and Vrťo [22] used Madej's upper bound to prove that $\nu(Q_n) = \Theta(4^n)$. Faria and Figueiredo [8,9] constructed drawings for the 6-,7- and 8-cubes with the same upper bound of crossings predicted by Eggleton and Guy. Cimikowski [3] showed that $\kappa(Q_n) = 2^n(n-2) - 2^{n-1}n + 4$.

Note that $\sigma(Q_n) = 0$, for $n = 1,\ 2,\ 3$, but $\sigma(Q_n) > 0$, for $n \geq 4$. Figure 1 shows drawings for the 1-, 2-, 3- and 4-cubes. Recently, Dean and Richter [4] devoted an entire article to proving that $\nu(Q_4) = 8$. Their proof consists of two main steps. Firstly, they show that in any optimum drawing of Q_4 there exists a C_4 with at least 4 crossings. Secondly, they show that the removal of the edges of a C_4 in Q_4 leaves a subdivision of $C_3 \times C_4$. Using that $\nu(C_3 \times C_4) = 4$, they establish that $\nu(Q_4) = 8$.

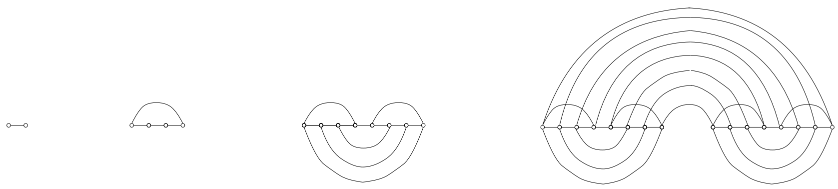

Fig. 1. Optimum drawings for Q_1, Q_2, Q_3 and Q_4.

In this article, we prove that the splitting number of the 4–cube is 4. We prove in Lemma 2 that the removal of a C_4 from Q_4 leaves the same graph up to isomorphism strengthening a result of Dean and Richter [4]. The strategy of the main proof is to show that if it were possible to obtain a planar graph from Q_4 with three splittings, then it would be needed to do them in three different vertices of Q_4 with each pair of them not in the same C_4. Adding to this statement the property that for every triple of vertices in Q_4 there exists a C_4 containing two of them, we have the need of four splittings, as required.

We notice that, for $n > 4$, there are 2^{n-4} vertex-disjoint subgraphs of Q_n isomorphic to Q_4. Hence the splitting number of Q_4 gives the following lower bound for the splitting number of the n–cube, for $n \geq 4$: $\sigma(Q_n) \geq 4 \cdot 2^{n-4} = 2^{n-2}$.

The skewness of the n–cube is known [3] to be $\kappa(Q_n) = 2^n(n-2) - 2^{n-1}n + 4$. As noted above, skewness and splitting number are related as follows: $\kappa(Q_n) \geq \sigma(Q_n)$. Therefore, our lower bound implies that the splitting number of the n-cube is in fact $\Theta(2^n)$.

2 The Result

In this section we prove that $\sigma(Q_4) = 4$.

Figure 2 exhibits a set of four splittings that obtains a planar resulting graph from Q_4. This proves that $\sigma(Q_4) \leq 4$.

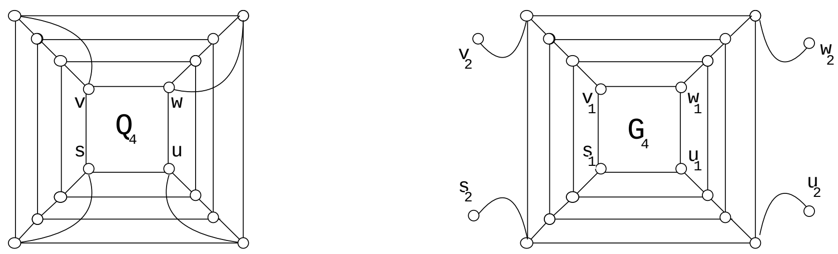

Fig. 2. $\sigma(Q_4) \leq 4$.

Kuratowski [15] characterized the class of planar graphs by saying that "a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph".

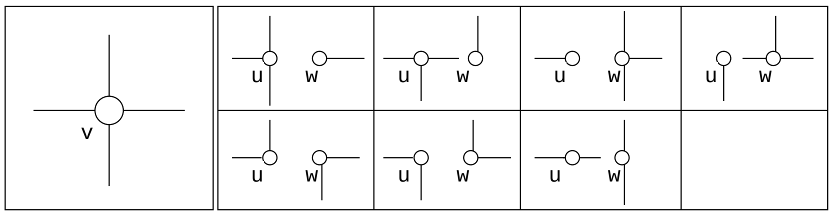

Fig. 3. All possible splittings in a vertex v of degree 4.

We recall that Q_4 is a 4–regular graph. A vertex v of degree 4 can be split into vertices v_1 and v_2 in seven different ways as shown in Fig. 3. We can generalize this observation with the following lemma.

Lemma 1. *If v is a vertex of degree d in G, then there are exactly $2^{d-1} - 1$ possible different splittings that can be done in v.*

Proof. Let $Adj(v) = \{u_1, u_2, \ldots, u_d\}$ be the set of vertices adjacent to v. Let H be the resulting graph obtained from G by splitting v into vertices v_1 and v_2.

We shall show that this splitting can be done in $2^{d-1} - 1$ different ways, by considering the possibilities for partitioning $Adj(v)$ into two sets.

For each $u_i \in Adj(v)$ we have to decide whether this vertex is adjacent to v_1 or not. We have 2^d possibilities for this decision. Let P_1, P_2 be a partition of $Adj(v)$. The assignment of the set P_1 to be the neighborhood of v_1 and of P_2 to be the neighborhood of v_2 gives the same graph as the assignment of the set P_1 to be the neighborhood of v_2 and of P_2 to be the neighborhood of v_1. Thus we must divide 2^d by 2 in order to obtain non isomorphic graphs. Finally, as the assignment that has the empty set is not allowed, we must subtract 1 from 2^{d-1}, which gives the result. □

An *automorphism* α of G is a bijective function $\alpha : V(G) \to V(G)$, such that $(u, v) \in E(G)$ if and only if $(\alpha(u), \alpha(v)) \in E(G)$.

Given a graph G and a subgraph S of G, we say that G is *S–transitive* if for each pair F, H of subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$.

Lemma 2 proves that the n–cube Q_n, for $n \geq 2$ is C_4-transitive, that is, a C_4 can be chosen with no loss of generality among all the subgraphs C_4 of Q_n.

Lemma 2. *If Q_n, for $n \geq 2$ is considered without the labels of its vertices, then any C_4 can be selected in Q_n with no loss of generality between all the subgraphs C_4 of Q_n.*

Proof. Given S and W two C_4's of Q_n, we shall exhibit an automorphism of Q_n carrying S to W.

Because an automorphism is a bijective function, it has inverse and it admits composition with another automorphism. Thus given T, a fixed C_4 of Q_n, it is enough to define for a each C_4 of Q_n an automorphism α of Q_n carrying this C_4 to T.

For we first show a key property needed for the definition of α: in the four binary n–tuples of the vertices of a fixed C_4 of Q_n there are precisely $(n-2)$ fixed digits.

Consider Q_n, with $n \geq 2$ and a C_4 induced by vertices v_1, v_2, v_3, v_4, where $(v_i, v_{(i+1) \bmod 4}) \in E(Q_n)$. Let $v_1 = (a_1, a_2, \ldots, a_n)$ be the n–tuple of v_1.

As v_1 is adjacent to v_2, by the definition of the n–cube, there is $k \in \{1, 2, \ldots n\}$ such that $v_2 = (a_1, a_2, \ldots, a_{k-1}, \bar{a}_k, a_{k+1}, \ldots, a_n)$, where $\bar{a}_k$ denote the binary complement of a_k, i.e., $\bar{a}_k = 0$ if and only if $a_k = 1$.

As $v_1 \neq v_3$ and $(v_2, v_3) \in E(Q_n)$ there is $j \neq k$ such that $v_3 = (a_1, a_2, \ldots, a_{j-1}, \bar{a}_j, a_{j+1}, \ldots, a_{k-1}, \bar{a}_k, a_{k+1}, \ldots, a_n)$. We assume $j < k$ with no loss of generality.

As $v_4 \neq v_2$ and $(v_1, v_4) \in E(Q_n)$, we have that the n–tuple of v_4 must be given by $v_4 = (a_1, a_2, \ldots, a_{j-1}, \bar{a}_j, a_{j+1}, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)$.

Hence, the n–tuples of the four vertices of a C_4 of Q_n have $(n-2)$ fixed digits.

In this way, we can define S a fixed generic C_4 of Q_n by writing $S = (I, s_{|I|+1}, II, s_{|I,II|+2}, III) = (s_1, s_2, \ldots, s_n)$, where $s_{|I|+1}$ and $s_{|I,II|+2}$ assume values in $\{0,1\}$, and (I, II, III) is the fixed $(n-2)$–tuple with respect to the four vertices defining S.

Let us fix T to be the C_4 of Q_n given by $T = (IV, t_{|IV|+1}, t_{|IV|+2})$, where IV is the $(n-2)$–tuple consisting of 0's only.

We define an automorphism $\alpha : Q_n \to Q_n$ carrying S to T by setting $\alpha(x_1, x_2, \ldots, x_{|I|-1}, x_{|I|}, x_{|I|+1}, \ldots, x_{|I,II|}, x_{|I,II|+1}, x_{|I,II|+2}, \ldots, x_n) = (y_1, y_2, \ldots, y_{|I|}, y_{|I|+2}, y_{|I|+3}, \ldots, y_{|I,II|+1}, y_{|I,II|+3}, y_{|I,II|+4}, \ldots, y_n, x_{|I|+1}, x_{|I,II|+2})$, where $y_i = x_i$ if and only if $s_i = 0$ and $i \notin \{|I|+1, |I,II|+2\}$.

Trivially, α carries S to T. For each n–tuple of Q_n the map α binary complements a set of digits defined by the fixed part of S. It follows that α is an automorphism. $\square$

Note that an argument similar to that used in the proof of Lemma 2 shows that the n–cube is vertex transitive.

Lemma 3. *For every set of three vertices in a Q_4, there is a C_4 containing two of them.*

Proof. Consider, with no loss of generality, the black vertex of Q_4 depicted in Fig. 4a. The vertices in Q_4 that are not in the same C_4 with respect to this vertex are the black vertices depicted in Fig. 4b. Since each pair of these vertices share a C_4, the result is obtained. $\square$

Fig. 4. For every set of three vertices in a Q_4 there is a C_4 containing two of them.

Next we show that a graph obtained from Q_4 by three splittings such that one of the splittings is not done in a vertex of Q_4 is nonplanar.

Lemma 4. *If G is obtained from Q_4 by two splittings, such that the first splits a vertex v of Q_4 into u and w, and the other splits u, then $\sigma(G) \geq 2$.*

Proof. We establish that $\sigma(G) \geq 2$ by considering the subgraph F of G defined by the drawing $D(F)$ in Fig. 5.

Because F is a subgraph of G, we have $\sigma(G) \geq \sigma(F)$. So it is enough to show that $\sigma(F) \geq 2$. For we show that splitting an arbitrary vertex of F yields a graph containing a subdivision of $K_{3,3}$.

In Fig. 5 we show also ten auxiliary copies of $D(F)$. We partition the vertex set of F into two sets: black vertices and striped vertices. We show first that the removal of any black vertex always produces a graph containing a subdivision of $K_{3,3}$. Although the removal of a striped vertex produces a planar graph, we show that the splitting of such a vertex produces a graph containing a subdivision of $K_{3,3}$.

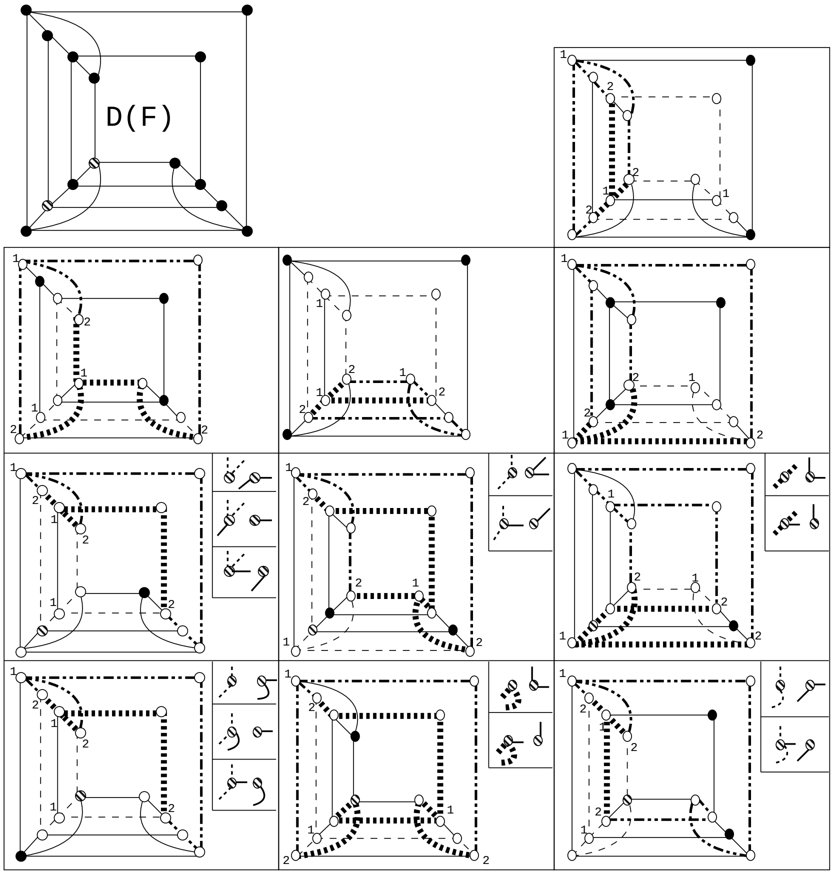

Fig. 5. Auxiliary graph F with splitting number at least 2 (for Lemma 4).

Consider first the top rightmost copy of $D(F)$. It contains two black vertices and a subgraph of F that is a subdivision of $K_{3,3}$ having partitions labeled respectively with 1 and 2. This means that the removal of any of those two black vertices yields a graph that still has a subdivision of $K_{3,3}$. An analogous argument shows that this is the case for any black vertex in the other nine copies of $D(F)$. As a black vertex can be removed without producing a planar graph, the splitting of a black vertex cannot produce a planar graph either.

The key property we use with respect to the splitting of a striped vertex v is that v is a vertex of degree four and so any splitting of v into vertices v_1 and v_2 is such that at least one of them has degree at least 2. We list three copies of $D(F)$ for each one of the two striped vertices with the corresponding three subdivisions of $K_{3,3}$. Each subdivision uses two edges incident to each striped vertex. For the convenience of the reader, we list beside each of the six last drawings all possible splittings for a striped vertex and the corresponding subdivision of $K_{3,3}$ in the resulting graph. □

Now we show that three splittings, done in three vertices of Q_4, two of these vertices sharing the same C_4 are not enough to obtain a planar resulting graph from Q_4.

Lemma 5. *If G is obtained from Q_4 by two splittings in the same C_4, then* $\sigma(G) \geq 2$.

Proof. We shall show that G contains the auxiliary graph F of Lemma 4 as subgraph, which implies $\sigma(G) \geq 2$.

The definition of G fixes two vertices u and v in the same C_4 of Q_4 with corresponding splittings. We define S_G as the graph obtained from Q_4 by removing v and by splitting u in the same way u is split to obtain G. Note that S_G is a subgraph of G. We show that F is a subgraph of S_G by considering all possibilities for S_G.

We consider in Fig. 6 two cases according to u and v being adjacent in Q_4 or not:

Case 1: vertices u and v are adjacent in Q_4. For the convenience of the reader, we label the three possibilities for S_G in order to show that these three graphs are isomorphic. We note that $S_G \setminus \{a, b\}$ is in turn isomorphic to F, as required.

Case 2: vertices u and v are not adjacent in Q_4. The seven possibilities for S_G are also shown in Fig. 6. In each case, it is easy to find F as subgraph.

This completes the proof of the lemma. □

Finally, we state and prove the main theorem.

Theorem 6. *The splitting number of Q_4 is* 4.

Proof. Figure 2 shows it is enough to establish $\sigma(Q_4) \geq 4$. It follows from Lemma 3, Lemma 4 and Lemma 5 that there is no set of three splittings that obtains from Q_4 a planar resulting graph. Thus, the splitting number of Q_4 is at least 4, which implies the equality $\sigma(Q_4) = 4$. □

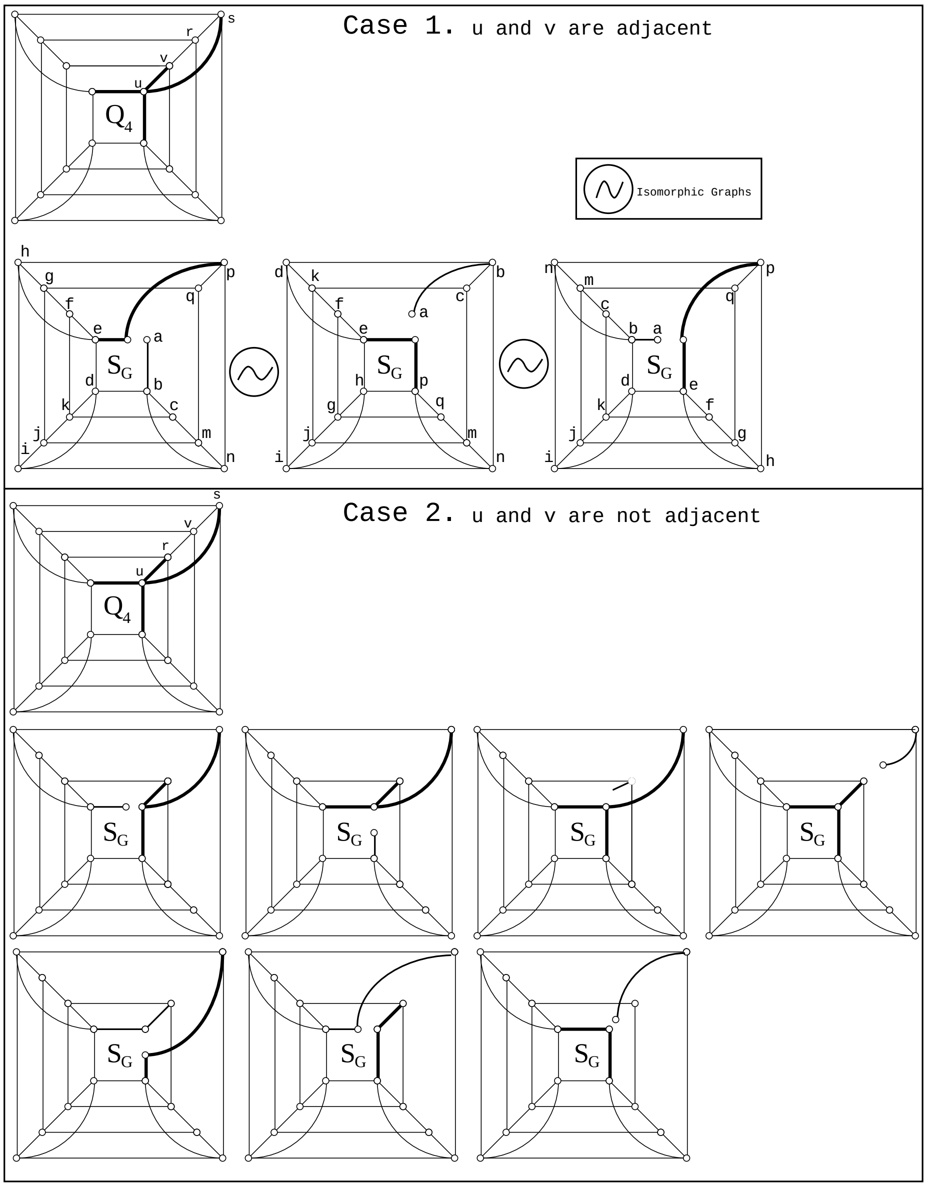

Fig. 6. The possibilities for the subgraph S_G of G, where u and v are the vertices in Q_4 that are split to obtain G (for Lemma 5).

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