On Eggleton and Guy conjectured upper bound for the crossing number of the n–cube

Luerbio Faria* Celina Miraglia Herrera de Figueiredo†

Abstract. The crossing number $\nu(G)$ of a graph G is the smallest integer such that there is a drawing for G with $\nu(G)$ crossings of edges. Let Q_n denote the n–dimensional cube. Eggleton and Guy conjectured in 1970 that $\nu(Q_n) \leq 4^n \frac{5}{32} - 2^{n-2}\lfloor \frac{n^2+1}{2} \rfloor$. We exhibit a drawing for $n = 6$ with the same value of Eggleton and Guy's conjectured upper bound. We construct a family of drawings for the n–cubes, $n \geq 7$, with number of crossings $\frac{165}{1024}4^n - \frac{2n^2-11n+34}{2}2^{n-2}$, establishing a new upper bound for $\nu(Q_n)$. Our family of drawings confirms Eggleton and Guy's conjectured upper bound when $n = 7$ and 8. In addition, our upper bound improves the upper bound $\nu(Q_n) \leq 4^n\frac{1}{6} - 2^{n-3}n^2 - 2^{n-4}3 + (-2)^n\frac{1}{48}$ due to Madej.
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1 Introduction

A *simple drawing* $D(G)$ of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. In what follows, all drawings are assumed to be simple. A drawing of a graph G is *optimum* when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $\nu(G)$. The algorithmic problem of computing the crossing number of a graph has been shown to be NP-complete [5].

Let Q_n denote the n–dimensional cube. The vertices of Q_n are all n–tuples of 0's and 1's, of which there are $|V(Q_n)| = 2^n$. Two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are adjacent if and only if $x_i \neq y_i$, for exactly one index i.

Note that $\nu(Q_n) = 0$, for $n = 1$, 2, 3, but $\nu(Q_n) > 0$, for $n \geq 4$. Figure 1 shows drawings for Q_1, Q_2, Q_3 and Q_4. Recently, Dean and Richter [1] devoted an entire article to proving that $\nu(Q_4) = 8$. Their proof consists of two main steps. Firstly, they show that in any optimum drawing of Q_4 there exists a C_4 with at least four crossings. Secondly, they show that the removal of the edges of a C_4 in Q_4 leaves a subdivision of $C_3 \times C_4$. Using that $\nu(C_3 \times C_4) = 4$ was known [8], they establish that $\nu(Q_4) = 8$.

*Faculdade de Formação de Professores/UERJ and COPPE/UFRJ, Caixa Postal 68511, 21945-970 Rio de Janeiro, RJ, Brazil. E-mail: `luerbio@cos.ufrj.br`. Partially supported by CAPES.

†Instituto de Matemática and COPPE/UFRJ, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil. E-mail: `celina@cos.ufrj.br`. Partially supported by CNPq.

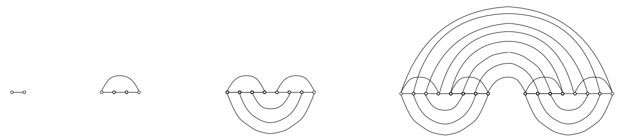

Figure 1: Optimum drawings for Q_1, Q_2, Q_3 and Q_4.

Much has been done about properties of nonplanarity for the class of n–cubes. Eggleton and Guy [2] announced that $\nu(Q_n) \leq 4^n\frac{5}{32} - 2^{n-2}\lfloor\frac{n^2+1}{2}\rfloor$. But a gap was found in the description of the construction [6], so this upper bound still remains a conjecture. Later, Erdös and Guy [3] conjectured equality in the above relation.

Some tight upper and lower bounds have been recently established for $\nu(Q_n)$. Madej [7] established an upper bound for the crossing number of the n–cube: $\nu(Q_n) \leq 4^n\frac{1}{6} - 2^{n-3}n^2 - 2^{n-4}3 + (-2)^n\frac{1}{48}$ and the lower bound: $\nu(Q_n) = \Omega(2^n n^{lgn})$. Madej [7] also exhibited a drawing for the 5–cube with the number of crossings confirming the conjectured upper bound of Eggleton and Guy. Subsequently, Sýkora and Vrťo [9] used Madej's upper bound to prove that $\nu(Q_n) = \Theta(4^n)$. Therefore, the exact value for $\nu(Q_n)$ is known for $n \leq 4$ only and the conjectured upper bound is verified for $n \leq 5$.

In this note, we confirm Eggleton and Guy's conjectured upper bound for $n = 6, 7, 8$ which represents so far the best drawings of these n–cubes. These drawings were described in the first author's master thesis [4]. We exhibit in Section 2.1 a drawing for $n = 6$ with the same value of Eggleton and Guy's conjectured upper bound. We extend in Section 2.2 the construction given in [4] and obtain a family of drawings for Q_n, $n \geq 7$, with number of crossings $\frac{165}{1024}4^n - \frac{2n^2-11n+34}{2}2^{n-2}$, establishing a new upper bound for $\nu(Q_n)$. The proposed family of drawings confirms Eggleton and Guy's conjectured upper bound when $n = 7$ and 8. Note that while our leading term constant is $\frac{165}{1024}$, Madej's leading term constant is $\frac{1}{6} > \frac{170}{1024}$, whereas Eggleton and Guy conjectured leading term constant is $\frac{5}{32} = \frac{160}{1024}$. Therefore, our result improves Madej's established upper bound, being closer to the expected value conjectured by Eggleton and Guy.

2 The proposed drawings for the n–cubes, $n \geq 6$

Let $D(G)$ be a drawing for G and v a vertex of G. Let v_∞ be a new vertex placed in the infinite region of $D(G)$. Let c be a curve representing the edge $e = (v, v_\infty)$ linking v to v_∞. We call *exterior distance of v with respect to $D(G)$ and to c* the number of crossings produced between c and the edges of $D(G)$. We call *exterior distance of v with respect to $D(G)$* the minimum number of crossings produced between a curve linking v to v_∞ and the edges of $D(G)$.

The drawings we are about to describe have all the following common constructive strategy: first of all, we consider just one eighth part of the drawing of the n–cube. The remaining seven eighth parts are similar to the first and differ only with respect to reflections, as defined below (see Figure 2). Subsequently, we define how to link the superior four eighth parts and the inferior four eighth parts obtaining two drawings for Q_{n-1}. The total number of crossings in the proposed drawing is eight times the number of crossings in the corresponding one eighth copy plus eight times the sum of the exterior distances of its vertices.

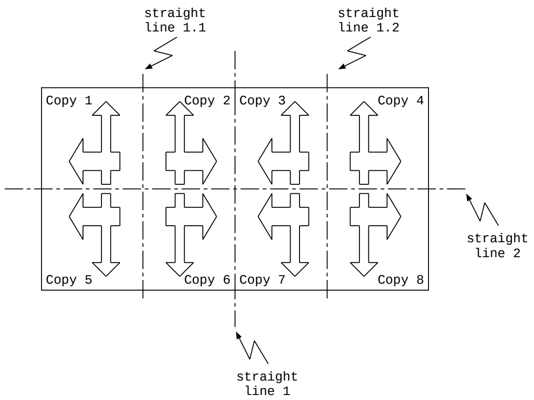

Figure 2: Reflections of the copy 1 square.

Our constructive method will be applied to Figure 3 in order to obtain the proposed drawing for Q_6 and to Figure 5 in order to obtain the proposed drawing for Q_n, $n \geq 7$.

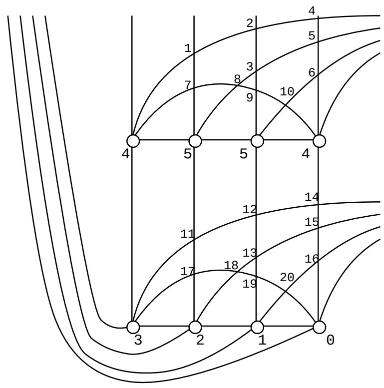

Figure 3: One eighth of the 6–cube.

2.1 The basic construction and the case $n = 6$

Consider the diagram in Figure 2 obtained by seven reflections of copy 1 square as follows. Copy 2 square is obtained from copy 1 square by reflection with respect to straight line 1.1. Copy 3 is identical to copy 1. Copy 4 is identical to copy 2. Copy 5 is obtained from copy 1 square by reflection with respect to straight line 2. We define the remaining three copies analogously.

Next consider Figures 3 and 5 where we depict copy 1 square corresponding to the construction of the desired drawing of Q_6 or of Q_n, $n \geq 7$, respectively.

In Figures 3 and 5, edges that take horizontal directions link vertices of copy 1 square to vertices of copy 2 square that are symmetric with respect to straight line 1.1. Edges that take vertical directions link vertices of copy 1 square to vertices of copy 4 square that are

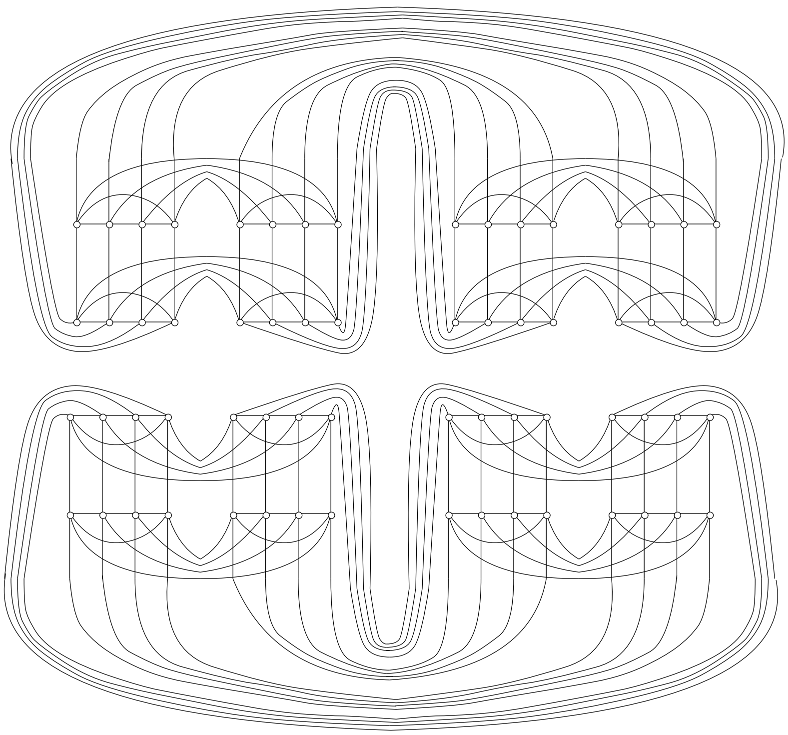

Figure 4: Complete drawing of the 6–cube (edges linking parts superior and inferior are omitted).

symmetric with respect to straight line 1. Note that copy 3 square is linked to copy 4 square and to copy 2 square in an analogous way by using straight line 1.2 and straight line 1, respectively. See Figure 4 for a diagram of the proposed drawing of Q_6.

We consider in Figure 3 exterior distances attached to the vertices to represent the minimum number of crossings that an edge linking vertices of copy 1 square to the symmetric vertices with respect to straight line 2 of copy 5 square produces with edges of copy 1 square. The sum of exterior distances in Figure 5 are evaluated separately. The total number of crossings in each one of the two obtained drawings, respectively for Q_6 and for Q_n, $n \geq 7$ is eight times the number of crossings in the corresponding copy plus eight times the sum of the exterior distances of its vertices.

The drawing of copy 1 square depicted in Figure 3 corresponding to one eighth of Q_6 has labels indicating 20 crossings and satisfies that the sum of exterior distances is 24. The Eggleton and Guy conjectured upper bound for $n = 6$ has value $(4^6 \times 5)/32 - \lfloor (6^2 + 1)/2 \rfloor 2^{6-2} = 352 = 44 \times 8$. To illustrate the complete method, we depict in Figure 4 the case $n = 6$, where we omit the edges between the top and the bottom parts symmetric with respect to straight line 2 to simplify the drawing.

2.2 The proposed family of drawings

We define a drawing for Q_n, $n \geq 7$, with $\nu(Q_n) \leq \frac{165}{1024}4^n - \frac{2n^2-11n+34}{2}2^{n-2}$ crossings as follows.

Consider in Figure 5 one eighth part of the proposed drawing of an n-cube.

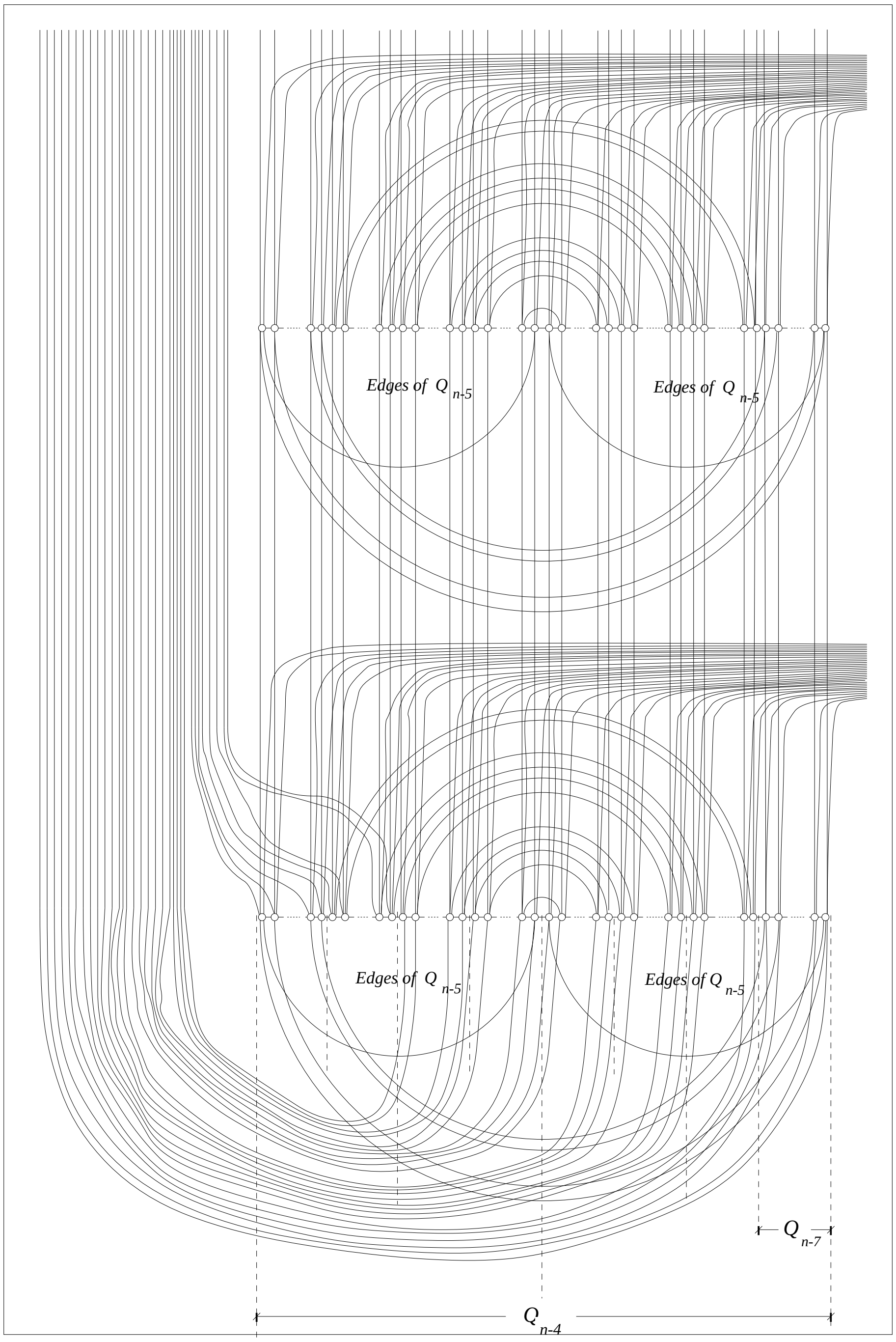

Figure 5: Drawing for the copy 1 square corresponding to a n–cube, with $n \geq 7$.

The vertices in Figure 5 induce a Q_{n-3} and are depicted in two horizontal lines. The vertices in the superior horizontal line of Figure 5 induce a Q_{n-4} and the vertices in the inferior horizontal line of Figure 5 also induce a Q_{n-4}. Consider first the superior Q_{n-4}. The vertices in the left half induce a Q_{n-5} and the vertices in the right half also induce a Q_{n-5}. The edges corresponding to the left Q_{n-5} are placed below the superior horizontal line. The edges corresponding to the right Q_{n-5} are placed below the superior horizontal line. Now consider the inferior Q_{n-4}. The edges corresponding both to the left Q_{n-5} and to the right Q_{n-5} are placed below the inferior horizontal line.

Consider the superior Q_{n-4}. There are 2^{n-5} edges joining the vertices of the left Q_{n-5} to the vertices of the right Q_{n-5}. The 2^{n-7} edges joining the leftmost vertices of the left Q_{n-5} to the rightmost vertices of the right Q_{n-5} are placed below the superior horizontal line.

The remaining $2^{n-5} - 2^{n-7} = 2^{n-7}(2^2 - 1) = 3 \times 2^{n-7}$ edges joining vertices of the left Q_{n-5} to the vertices of the right Q_{n-5} are placed above the superior horizontal line. The analogous placement of edges is done with respect to the inferior Q_{n-4}.

Edges joining vertices of the superior Q_{n-4} to the vertices of the inferior Q_{n-4} are placed vertically as shown in Figure 5.

Edges joining each vertex of the two Q_{n-4}'s of copy 1 square to vertices of the corresponding Q_{n-4}'s in copy 2 square are placed horizontally to the right as shown in Figure 5. Edges joining each vertex of the superior Q_{n-4} of copy 1 square to the vertices of the corresponding Q_{n-4} in copy 4 square are placed vertically to the top as shown in Figure 5. Edges joining the $\frac{3}{4}2^{n-4}$ rightmost vertices of the inferior Q_{n-4} to the vertices of the corresponding Q_{n-4} in copy 4 square are placed as shown in Figure 5: these edges go first below the inferior horizontal line and then go upwards and are placed in the leftmost region of the drawing. Edges joining the $\frac{1}{4}2^{n-4}$ leftmost vertices of the inferior Q_{n-4} to the vertices of the corresponding Q_{n-4} in copy 4 square are placed as shown in Figure 5: these edges go first above the inferior horizontal line and then, together with the $\frac{3}{4}2^{n-4}$ edges described above, also go upwards and are also placed in the leftmost region of the drawing.

The eight copy squares are connected as in Section 2.1. We describe edges joining vertices of copy squares 1, 2, 3 and 4 to the corresponding vertices in copy squares 5, 6, 7 and 8 by using exterior distances. This concludes the definition of our drawing of Q_n, $n \geq 7$.

We proceed to count the number of crossings in our proposed drawing of Q_n, $n \geq 7$. We follow the same method used in Section 2.1. First, we count the number of crossings in copy 1 square. Second, we sum the exterior distances in copy 1 square. Then, we sum these two values and we multiply this total by 8 in order to obtain the claimed number of crossings in our proposed drawing.

Consider Figure 5, where we depict a drawing for one eighth part of Q_n corresponding to square copy 1. Note that the corresponding drawings for Q_7 and Q_8 can be obtained as particular cases from the drawing in Figure 5. The case of Q_6 is dealt with separately in Figure 3 because the definition of the drawing of Q_n in Figure 5 requires a subgraph corresponding to Q_{n-7}, which is not defined for Q_n, $n \leq 6$.

- Computation of the number of crossings for the proposed drawing of Q_n, $n \geq 7$.

  Now we compute the number of crossings in one eighth part of Q_n depicted in Figure 5. For, we exhibit in Figure 6(a) a plane drawing with some edges of Figure 5. Figure 6(b) is a copy of Figure 6(a) with some edges of Figure 5 that are not in Figure 6(a). These new edges induce a certain number of crossings in Figure 6(b). We count the additional number of crossings in Figure 6(b) and proceed to Figure 6(c). Figure 6(c) is a copy of Figure 6(b) with some edges of Figure 5 that are not in Figure 6(b). We count this additional number of crossings in Figure 6(c). Analogously, in Figures 6(d), 6(e) and 6(f), we consider additional sets of edges and we count the corresponding additional crossings. Since Figure 6(f) is a copy of Figure 5, we have decomposed our computation of the number of crossings into six steps.

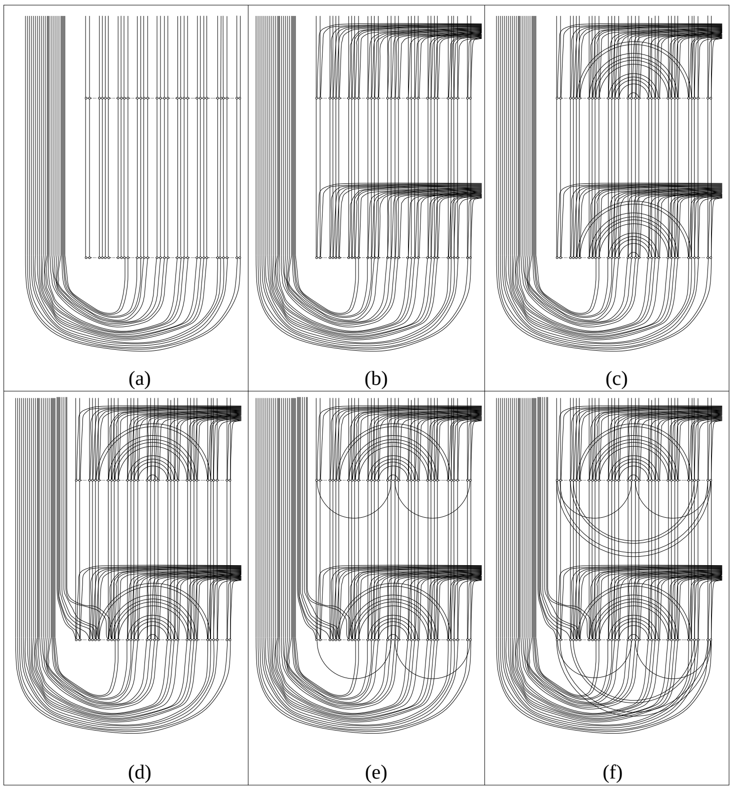

Figure 6: Counting process of crossings.

– Computation of the number of additional crossings in Figure 6(b).

  Consider first the superior horizontal line. We have $(2^{n-4}-1)+(2^{n-4}-2)+(2^{n-4}-3)+\ldots+(1)=\sum_{i=1}^{2^{n-4}-1} i$ corresponding crossings. This number is the sum of the first $(2^{n-4}-1)$ natural numbers. Thus we have $2^{n-5}(2^{n-4}-1)$ additional crossings above the superior horizontal line. Analogously, we have $2^{n-5}(2^{n-4}-1)$ crossings between the superior horizontal line and the inferior horizontal line. Therefore, there are $2^{n-4}(2^{n-4}-1)=4^{n-4}-2^{n-4}=64\times 4^{n-7}-16\times 2^{n-8}$ additional crossings in Figure 6(b).

– Computation of the number of additional crossings in Figure 6(c).

  For each vertex of the superior Q_{n-4} and for each vertex of the inferior Q_{n-4} there are two incident edges going upwards in Figure 6(b). Consider first the crossings above the superior horizontal line. There are $3 \times 2^{n-7}$ edges linking the vertices of the left Q_{n-5} to the vertices of the right Q_{n-5} in the superior horizontal line. Consider e the innermost edge with crossings among these $3 \times 2^{n-7}$ edges. There are two vertices between the endpoints of this edge e. Since for each one of these two vertices there are two edges going upwards in Figure 6(b), we have in e the number of 2×2 crossings. The innermost edge with crossings among the remaining edges has four vertices between its endpoints, so we have 4×2 additional crossings. This counting process stops in the outermost edge among the $3 \times 2^{n-7}$ edges. That edge has $2 \times (3 \times 2^{n-7} - 1)$ vertices between its endpoints, which means that it has $(2 \times (3 \times 2^{n-7} - 1)) \times 2$ additional crossings. Thus, we have $4 \sum_{i=1}^{3\times 2^{n-7}-1} i$ crossings for the superior part and $4 \sum_{i=1}^{3\times 2^{n-7}-1} i$ crossings for the inferior part. Therefore, that are $3 \times 2^{n-5}(3 \times 2^{n-7} - 1) = 9 \times 4^{n-6} - 3 \times 2^{n-5} = 36 \times 4^{n-7} - 24 \times 2^{n-8}$ additional crossings in Figure 6(c).

– Computation of the number of additional crossings in Figure 6(d).

  We denote the function of the number of additional crossings in Figure 6(d) by $\psi(n)$. Consider the leftmost 2^{n-6} vertices of the inferior Q_{n-4} that induce a Q_{n-6} in Figure 5. Consider this set of vertices in the drawing of Figure 6(d). Consider first the Q_{n-7} in the left half of this Q_{n-6}. For each vertex in this Q_{n-7}, there are two edges going upwards in the drawing of Figure 6(c). Now consider the Q_{n-7} in the right half of this Q_{n-6}. For each vertex in this Q_{n-7} there are three edges going upwards in the drawing of Figure 6(c). Therefore, in the drawing of Figure 6(d) we count 2 for $2^{n-6} - 1$ vertices of the leftmost Q_{n-6} and we count 1 more crossing for $2^{n-7} - 1$ vertices of its right half Q_{n-7}, that obtains the following number of additional crossings:

$$\psi(n) = 2 \sum_{i=1}^{2^{n-6}-1} i + \sum_{i=1}^{2^{n-7}-1} i =$$

$$= 2^{n-6}(2^{n-6} - 1) + 2^{n-8}(2^{n-7} - 1) = \frac{4^n}{2^{12}} + \frac{4^n}{2^{15}} - 2^{n-6} - 2^{n-8}.$$

  That yields:

$$\psi(n) = \frac{9}{2} 4^{n-7} - 5 \times 2^{n-8}.$$

– Computation of the number of additional crossings in Figure 6(e).

  Consider Figure 7. We present in a horizontal straight line a drawing for Q_n, $n \geq 1$, such that all edges of Q_n are below this straight line. The drawing for Q_n is obtained from two drawings for Q_{n-1} by appropriate adding of 2^{n-1} edges. The correctness that this drawing is in fact a drawing for Q_n is due to that Q_n can be defined as $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$, for $n \geq 2$. We want to compute the number of crossings produced in this drawing of Q_n by 2^n vertical new edges

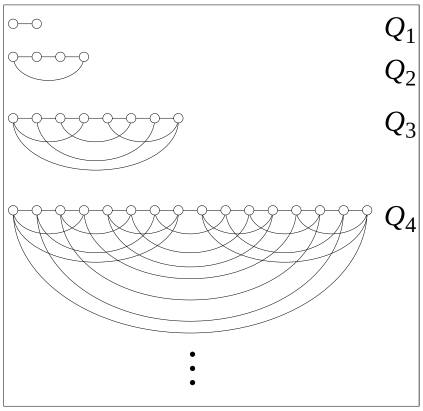

Figure 7: Auxiliary drawing for Q_n with the edges below a horizontal straight line.

each one incident to each one of the vertices of Q_n. In other words, we want to compute the sum of exterior distances of the vertices of Q_n with respect to the infinite region below the horizontal straight line of this drawing for Q_n. We denote this number by $\phi(n)$. See Figure 8. This means that $\phi(n)$ is the number of crossings that 2^n edges going downwards in Figure 8(a) share with the edges of Q_n.

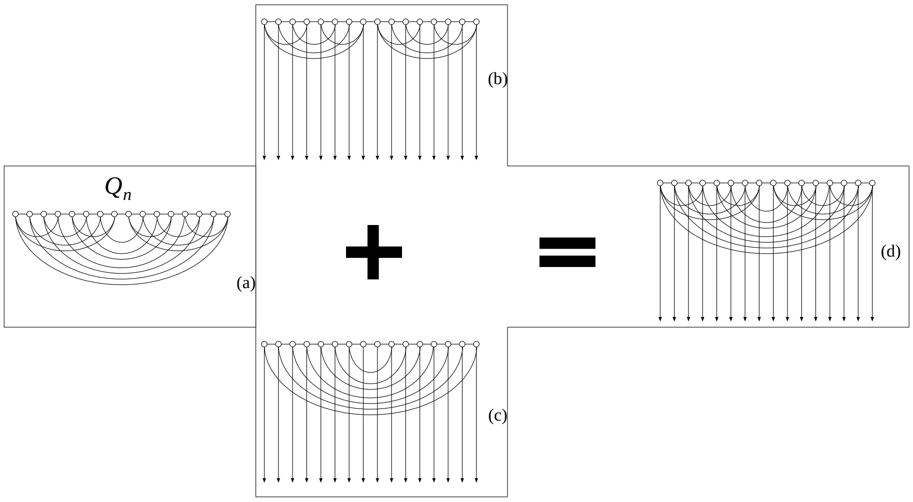

Figure 8: Exterior distances for $\phi(n)$.

We compute the value of $\phi(n)$ in two parts. In the first part, as shown in Figure 8(b), we compute the number of crossings of the vertical edges with the edges of each one of the Q_{n-1}'s on each one of the halves right and left of Q_n, that is $2\phi(n-1)$. In the second part, as shown in Figure 8(c), we compute the number of crossings of the vertical edges with the 2^{n-1} edges linking the vertices of the Q_{n-1} on the right half of Q_n to the vertices of the Q_{n-1} on the left half of Q_n. The outermost of these 2^{n-1} edges linking the vertices of the two Q_{n-1}'s has $(2^{n-1}-1)2$ crossings, corresponding to the $(2^{n-1}-1)2$ vertices between its two endpoints. Analogously, from the exterior to the interior we have, respectively, $(2^{n-1}-2)2, (2^{n-1}-3)2, \ldots, 2\times 2, 1\times 2$ and 0×2 crossings for the innermost of these 2^{n-1} edges. Hence, the value of $\phi(n)$ satisfies the recurrence:

$$\begin{cases} \phi(1) = 0 \\ \phi(n) = 2\phi(n-1) + 2\sum_{i=1}^{2^{n-1}-1} i, n \geq 1, \end{cases}$$

giving that $\phi(n) = \frac{4^n}{2} - 2^{n-1}(n+1)$.

Now we want to compute the number of crossings in the drawing of Q_n in Figure 7. This number of crossings is denoted by $\xi(n)$. We shall use $\phi(n)$ to compute the value of $\xi(n)$.

By definition of the drawing in Figure 7, the number of crossings in this drawing of Q_n is equal to twice the number of crossings in the drawing of Q_{n-1} plus twice the number of crossings of the edges of Q_{n-1} with the 2^{n-1} edges linking the corresponding vertices to the two Q_{n-1}'s. This means that $\xi(n) = 2\xi(n-1) + 2\phi(n-1)$.

Solving the recurrence for $\xi(n)$:

$$\begin{cases} \xi(2) = 0 \\ \xi(n) = 2\xi(n-1) + 2\phi(n-1), n \geq 2, \end{cases}$$

gives that $\xi(n) = \frac{4^n}{2} - 2^{n-2}(n^2+n+2)$.

Figure 6(e) is obtained from Figure 6(d) by adding the corresponding edges to four Q_{n-5}'s. First of all, we consider the number of crossings in the pairs of edges of each Q_{n-5}, that, by definition, is $\xi(n-5)$. It remains to compute the number of crossings of the edges of the four Q_{n-5}'s with the edges of Figure 6(a). These crossings are, by definition, $\phi(n-5)$ for the left half Q_{n-5} of the superior Q_{n-4}, $\phi(n-5)$ for the right half Q_{n-5} of the superior Q_{n-4}, $\phi(n-5)$ for the right half Q_{n-5} of the inferior Q_{n-4} and $\frac{1}{2}\phi(n-5)$ for the left half Q_{n-5} of the inferior Q_{n-4}. So the number of additional crossings in Figure 6(e) is $4\xi(n-5) + \frac{7}{2}\phi(n-5)$. Thus, we have the following number of crossings:

$$4\xi(n-5) + \frac{7}{2}\phi(n-5) =$$

$$= 15 \times 4^{n-6} - 2^{n-7}(4n^2 - 29n + 60).$$

Therefore, there are $60 \times 4^{n-7} - 2^{n-8}(8n^2 - 58n + 120)$ additional crossings in Figure 6(e).

– Computation of the number of additional crossings in Figure 6(f).

In this case, the additional number of crossings is computed in two parts. First, we compute the number of crossings of the additional edges in Figure 6(f) with the edges of Figure 6(a). Second, we compute the number of crossings of the additional edges in Figure 6(f) with the edges of the four Q_{n-5}'s placed in Figure 6(e).

We start by computing the first part. We consider first the crossings with the 2^{n-7} edges placed in Figure 6(f), in the region between the superior and inferior horizontal lines. Consider the outermost edge of the 2^{n-7} edges. There are $(2^{n-4}-2)$ crossings in this edge corresponding to the $(2^{n-4}-2)$ vertical edges between the two endpoints of this edge. Analogously, the next edge has $(2^{n-4}-2\times 2)$ crossings. Hence, we have: $(2^{n-4}-2)+(2^{n-4}-2\times 2)+(2^{n-4}-2\times 3)+\ldots+(2^{n-4}-2\times 2^{n-7}) = 2^{n-4}2^{n-7}-2\times 2^{n-8}\times(2^{n-7}+1) = 2^{n-4}2^{n-7}-2^{n-7}\times(2^{n-7}+1)$ crossings in this region. Consider now the outermost edge of the 2^{n-7} edges below the inferior horizontal line. There are $(\frac{3}{4}2^{n-4}-1)$ crossings in this edge corresponding to the $(\frac{3}{4}2^{n-4}-1)$ vertical edges between the two endpoints of this edge. Analogously, there are $(\frac{3}{4}2^{n-4}-1)+(\frac{3}{4}2^{n-4}-2)+(\frac{3}{4}2^{n-4}-3)+\ldots+(\frac{3}{4}2^{n-4}-2^{n-7}) = \frac{3}{4}2^{n-4}2^{n-7}-2^{n-8}(2^{n-7}+1)$ corresponding crossings. Therefore, in first part we have

$$\frac{7}{4}2^{n-4}2^{n-7}-3\times 2^{n-8}(2^{n-7}+1) = \frac{25}{2}4^{n-7}-3\times 2^{n-8},$$

additional crossings.

Let us compute the second part. Observe first that the additional crossings produced between the additional edges of Figure 6(f) with the edges of the four Q_{n-5}'s are exactly four times the sum of exterior distances of the 2^{n-7} leftmost vertices with respect to the infinite region below the the horizontal straight line of the drawing of Q_{n-5}, defined in Figure 7. We denote this sum by $\phi'(n)$. We derive $\phi'(n)$ in three parts. First part corresponds to the sum of exterior distances in the leftmost Q_{n-7} that is equal to $\phi(n-7)$. Second part corresponds to the 2^{n-7} edges linking those 2^{n-7} vertices to the vertices of the corresponding Q_{n-7} to produce the Q_{n-6} on the left half of the Q_{n-5}. For these edges we have to sum to the exterior distances: $2^{n-7}-1$ for the outermost edge, $2^{n-7}-2$ for the next edge and respectively, $2^{n-7}-3$, $2^{n-7}-4$, $\ldots$, $2^{n-7}-(2^{n-7}-1)$ for the remaining edges. Third part consists in the $2^{(n-7)}$ edges linking the 2^{n-7} vertices of the leftmost Q_{n-7} to the vertices of the corresponding rightmost Q_{n-7} of the Q_{n-5}. For these edges we have to sum the exterior distances: $2^{n-7}-1$ for the outermost edge, $2^{n-7}-2$ for the next edge, and respectively, $2^{n-7}-3$, $2^{n-7}-4$, $\ldots$, $2^{n-7}-(2^{n-7}-1)$ for the remaining edges.

In this way, $\phi'(n)$ is computed as follows:

$$\phi'(n) = \phi(n-7) + 2\sum_{i=1}^{2^{n-7}-1} i$$

which gives the value:

$$\phi'(n) = \frac{3}{2}4^{n-7} - 2^{n-8}(n-4).$$

Thus, the corresponding number of crossings to the second part is: $4\times\phi'(n) = \frac{3}{2}4^{n-6}-2^{n-6}(n-4) = \frac{12}{2}4^{n-7}-2^{n-8}(4n-16)$.

Therefore, in Figure 6(f) there are $\frac{37}{2}4^{n-7} - 2^{n-8}(4n-13)$ additional crossings.

Therefore, we have that the sum of the crossings in the drawing of one eighth for Q_n in Figure 5 is given by: $183 \times 4^{n-7} - 2^{n-8}(8n^2 - 54n + 152)$.

Now we compute the sum of the exterior distances in Figure 5.

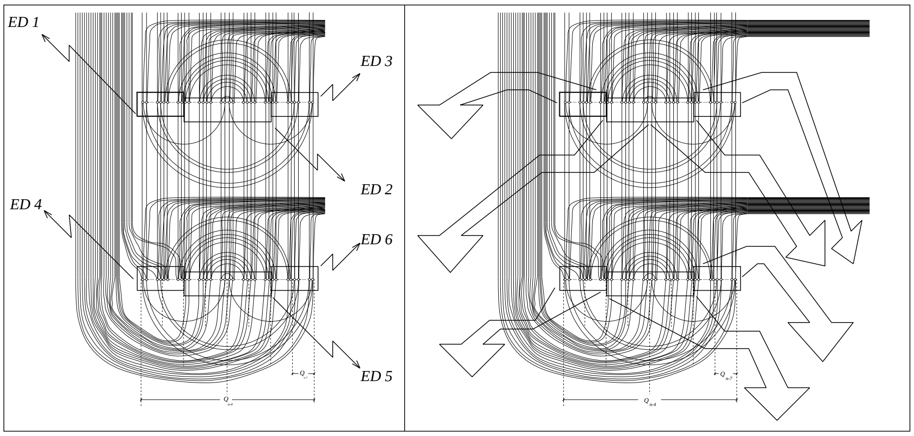

Figure 9: Counting regions of exterior distances.

- Computation of the sum of the exterior distances in Figure 5.

  We partition the vertices of Figure 5 into six groups of vertices, which are the vertices in the regions $ED1, ED2, ED3, ED4, ED5$ and $ED6$ defined by Figure 9(a). The exterior distances are derived with respect to curves following the flow chart diagram corresponding to each region in Figure 9(b).

  In this way, we derive the sum of exterior distances of the vertices in each one of the regions $ED1, ED2, ED3, ED4, ED5$ and $ED6$.

  - Regions $ED1$ and $ED3$.

    By definition of $\psi(n)$ and because we have 2^{n-6} vertices in each one of the regions $ED1$ and $ED3$, the sum of exterior distances of the vertices in each of these regions is:

    $$\psi(n) + 2^{n-4}2^{n-6} = \frac{9}{2}4^{n-7} - 5 \times 2^{n-8} + 4^{n-5} = \frac{41}{2}4^{n-7} - 5 \times 2^{n-8}.$$

  - Region $ED2$.

    Because of the additional edges in Figure 6(e), we have $\phi(n-5)$ crossings.

    Because of the edges linking the vertices of copy 1 square to the vertices of copies 2 and 4 squares and because we have 2^{n-5} vertices in region $ED2$, we have additional $2^{n-4}2^{n-5}$ crossings.

    Because of the additional 2^{n-7} edges in Figure 6(f) between the superior and inferior horizontal lines and because we have 2^{n-5} vertices in region $ED2$, we have $2^{n-7}2^{n-5}$ crossings.

Let us analyze the crossings with the 2^{n-4} vertical edges linking the vertices of the superior Q_{n-4} to the vertices of the inferior Q_{n-4}. Consider the leftmost vertex of $ED2$. This vertex is responsible for 2^{n-6} crossings with the vertical edges. The first vertex in $ED2$ on the right hand side of this vertex is responsible by $2^{n-6}+1$ crossings with the vertical edges. It is easy too see that the i–th vertex, $1 \leq i \leq 2^{n-6}-1$, of $ED2$ on the right hand side of the leftmost vertex of $ED2$ contributes with $2^{n-6}+i$ crossings with the vertical edges. So, the rightmost vertex of the left half of $ED2$ is responsible for $2^{n-5}-1$ crossings with the vertical edges. Following the direction of the flow chart diagram of $ED2$ an analogous sum of exterior distances is obtained for the vertices in the right half of $ED2$. That gives, $2\sum_{i=2^{n-6}}^{2^{n-5}-1} i = 2^{n-6}(3\times 2^{n-6}-1)$ additional crossings. Thus we have the following number of crossings in region $ED2$:

$$\phi(n-5) + 2^{n-4}2^{n-5} + 2^{n-7}2^{n-5} + 2^{n-6}(3\times 2^{n-6}-1) =$$

$$= \frac{1}{2}4^{n-5} - 2^{n-6}(n-4) + \frac{1}{2}4^{n-4} + 4^{n-6} + 3\times 4^{n-6} - 2^{n-6} =$$

$$= (8+32+16)4^{n-7} - 2^{n-8}(4n-12) = 56\times 4^{n-7} - 2^{n-8}(4n-12).$$

– Region $ED4$.

Because of additional edges in Figure 6(e), we have in region $ED4$ the number of $\frac{1}{2}\phi(n-5)$ of crossings.

Because of the additional 2^{n-7} edges in Figure 6(f), we have the sum $\sum_{i=1}^{2^{n-7}-1} i = 2^{n-8}(2^{n-7}-1)$ of crossings for the 2^{n-7} vertices in the left half of $ED4$ and $2^{n-7}2^{n-7}$ crossings, for the 2^{n-7} vertices in the right half of $ED4$.

Because of the $\frac{3}{4}2^{n-4}$ edges leaving the bottom part of the inferior horizontal line and linking the vertices of the inferior Q_{n-4} with the corresponding vertices of copy 4 square, and because there are 2^{n-6} vertices in region $ED4$, we have $\frac{3}{4}2^{n-4}2^{n-6}$ crossings. This gives the following number of crossings in region $ED4$:

$$\frac{1}{2}\phi(n-5) + 2^{n-7}2^{n-7} + 2^{n-8}(2^{n-7}-1) + 3\times 2^{n-6}2^{n-6} =$$

$$= 4^{n-6} - 2^{n-7}(n-4) + 4^{n-7} + \frac{1}{2}4^{n-7} - 2^{n-8} + 3\times 4^{n-6} =$$

$$= (4+1+\frac{1}{2}+12)4^{n-7} - 2^{n-8}(2n-8+1) = \frac{35}{2}4^{n-7} - 2^{n-8}(2n-7).$$

– Region $ED5$.

Because of the additional edges in Figure 6(e), we have $\phi(n-5)$ additional crossings.

Let us consider the $\frac{3}{4}2^{n-4}$ rightmost edges linking the vertices of the inferior Q_{n-4} to the corresponding vertices in copy 4 square. Consider the rightmost vertex of $ED5$. The flow chart direction of $ED5$ shows that this vertex contributes with 2^{n-6} crossings with the 2^{n-6} rightmost edges of the $\frac{3}{4}2^{n-4}$ edges. The first vertex of $ED5$ on the left hand side of this vertex contributes with $2^{n-6}+1$ crossings with the $2^{n-6}+1$ rightmost edges among the $\frac{3}{4}2^{n-4}$ edges. It is easy to see that the i–th vertex, $1 \le i \le 2^{n-5}-1$, of $ED5$ on the left hand side of the rightmost vertex of $ED5$ contributes with $2^{n-6}+i$ crossings with $2^{n-6}+i$ rightmost edges among the $\frac{3}{4}2^{n-4}$ edges. So, we have the sum $\sum_{i=2^{n-6}}^{2^{n-6}+2^{n-5}-1} i = 2^{n-6}(2^{n-4}-1)$ of additional crossings.

Because of the additional edges in Figure 6(f) and because we have 2^{n-5} vertices in region $ED2$, we have $2^{n-7}2^{n-5}$ crossings. That gives the following number of crossings for region $ED5$:

$$\phi(n-5) + 2^{n-7}2^{n-5} + 2^{n-6}(2^{n-4}-1) =$$

$$= \frac{1}{2}4^{n-5} - 2^{n-6}(n-4) + 4^{n-6} + 4^{n-5} - 2^{n-6} =$$

$$= (8+4+16)4^{n-7} - 2^{n-8}(4n-16+4) = 28 \times 4^{n-7} - 2^{n-8}(4n-12).$$

– Region $ED6$.

By definition of $\psi(n)$, the sum of exterior distances of the vertices in this region is $\psi(n) = \frac{9}{2}4^{n-7} - 5 \times 2^{n-8}$.

That obtains $147 \times 4^{n-7} - (10n-16)2^{n-8}$ for the sum of the exterior distances in Figure 5. Therefore, the number of crossings in Figure 5 added to the exterior distances is:

$$330 \times 4^{n-7} - (8n^2 - 44n + 136)2^{n-8}.$$

In this way, we obtain the claimed upper bound for the crossing number of the n–cube:

$$\nu(Q_n) \le 8(330 \times 4^{n-7} - (8n^2 - 44n + 136)2^{n-8}) =$$

$$= \frac{165}{1024}4^n - \frac{2n^2 - 11n + 34}{2}2^{n-2}.$$

■

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