Even pairs

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Abstract

Two non-adjacent vertices in a graph form an even pair if every chordless path between them has an even number of edges. The salient fact about even pairs is that contracting an even pair in a graph G yields a graph that has the same clique number and chromatic number as G. It follows that (a) the contraction of two vertices that form an even pair in a perfect graph produces a new perfect graph, and (b) no minimal imperfect graph can contain an even pair. Fact (a) can be exploited to devise simple combinatorial algorithms for coloring many perfect graphs. Fact (b) can be used to define large classes of perfect graphs. We will review the wealth of results that have appeared on these topics and discuss various related concepts such as odd pairs. This is an updated version of [H. Everett, C.M.H. de Figueiredo, C. Linhares-Sales, F. Maffray, O. Porto, B.A. Reed, Path Parity and Perfection, *Discrete Mathematics*, 165-166 (1997), 223-242.]

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1 Introduction

The main object under consideration here is the following.

Definition: Even pair [62] *Two non-adjacent vertices x, y in a graph G form an even pair if every induced path between them has an even number of edges.*

Definition: Contraction *For two vertices x, y in a graph G, we denote by G/xy the graph obtained by deleting x and y and adding a new vertex xy adjacent to precisely those vertices of $G - x - y$ which were adjacent to at least one of x, y in G. We say that G/xy is obtained by contracting on the pair $\{x, y\}$.*

For a graph G, we let as usual $\chi(G)$ denote the chromatic number of G, $\omega(G)$ denote the size of a largest clique in G, and $\alpha(G)$ denote the size of a largest stable set in G. Also $\overline{G}$ denotes the complement of G. The problems of computing $\omega(G)$ and $\chi(G)$ are in general NP-complete [26]. The following easy facts motivate our interest in even pairs and the contraction operation.

Lemma 1 *Let G be any graph that contains an even pair $\{x, y\}$. Then,*

1.1. $\omega(G/xy) = \omega(G)$;

1.2. $\chi(G) = \chi(G/xy)$.

Proof To prove Fact 1.1, let C be any clique of G. Clearly C cannot contain both x and y. If C contains none of them, then C is also a clique in G/xy. If C contains precisely one of x, y, then replacing this vertex by xy yields a clique of G/xy that has the same size as C. Thus $\omega(G/xy) \geq \omega(G)$.

Conversely, let C be a clique in G/xy. If xy is not in C then C is also a clique in G. If xy is in C then every vertex in $C - xy$ is adjacent to at least one of x or y in G. Recall that $\{x, y\}$ is an even pair. Thus there cannot be two vertices a and b in $C - xy$ such that a is adjacent to x and not to y but b is adjacent to y and not to x. It follows that one of x or y is adjacent to all of $C - xy$. So, there is a clique of G with the same size as C. Thus $\omega(G) \geq \omega(G/xy)$.

To prove Fact 1.2, consider any coloring of G/xy. There corresponds a coloring of G, using the same number of colors, where x and y receive the same color (the color which was given to xy), all other colors remain unchanged. Thus $\chi(G) \leq \chi(G/xy)$.

Conversely, consider any coloring c of G. If x and y receive the same color, there corresponds a coloring of G/xy with the same number of colors. Now assume x and y receive different colors in c, say x has color 1 and y has color 2. Let H be the bipartite subgraph induced by the vertices of colors 1 and 2. The vertices x, y must be in different components of H, for otherwise there would exist an odd chordless path between them (with vertices alternately colored 1 and 2), contradicting that they form an even pair. Thus we can obtain a new coloring by swapping colors 1 and 2 in the component of H containing x, without affecting y. In this new coloring, x and y have the same color. It follows that there is a corresponding coloring of G/xy with the same number of colors. Thus $\chi(G/xy) \leq \chi(G)$. □

The proof of Fact 1.1 yields a simple procedure which, given a largest clique in G/xy, finds a largest clique in G. Similarly the proof of Fact 1.2 yields a simple procedure which, given a k-coloring of G/xy, yields a k-coloring of G. As we shall see, these procedures can be used to develop fast algorithms for finding a largest clique and an optimal coloring in certain kinds of graphs. To illustrate the technique, we consider the sequence of graphs $G_0, \ldots, G_j$ depicted in Figure 1. For $i \leq j-1$, G_{i+1} is obtained from G_i by contracting the even pair $\{x_i, y_i\}$. As G_j is a clique, it is trivial to obtain an optimal coloring and find a largest clique within it. We can then work backwards to find a largest clique and an optimal coloring of G_0. We note that the size of the largest clique in G_0 is equal to its chromatic number. We will often iteratively contract even pairs to obtain optimal colorings which use the same number of colors as there are vertices in a largest clique.

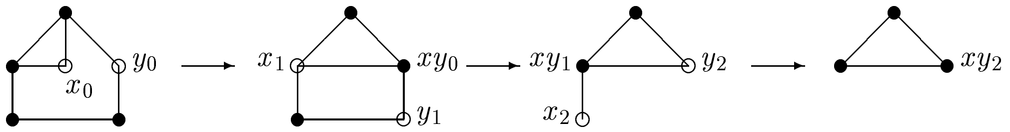

Figure 1: A sequence of even pair contractions

Bertschi proposed the following definitions.

Definition: Even-contractile graph [6] *A graph G is even-contractile if there is a sequence $G_0 = G, G_1, \ldots, G_j$ such that G_j is a clique, and, for $i \leq j-1$, G_{i+1} is obtained from G_i via the contraction of an even pair of G_i.*

Facts 1.1 and 1.2 imply that if H is even contractile then $\chi(H) = \omega(H)$. Even-contractile graphs form an interesting class of graphs, but little is known about them. Perhaps they are too difficult to characterize as they do not form a hereditary class. Bertschi [6] proposed to study the class of graphs all of whose induced subgraphs are even contractile.

Definition: Perfectly contractile graph [6] *A graph is perfectly contractile if each of its induced subgraphs is even contractile.*

This definition has been very fruitful, and much of this paper will be devoted to discussing perfectly contractile graphs. Facts 1.1 and 1.2 imply that *every perfectly contractile graph is perfect.* It turns out that many classical types of perfect graphs are perfectly contractile. As we shall see in the next section, this is not the only link between even pairs and perfect graphs.

Stuff that may be deleted from the book chapter version:

Following Berge [3] a graph G is called *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. A graph is then *minimal imperfect* if it is not perfect but all its proper induced subgraphs are perfect. The chordless cycle on $k \geq 5$ vertices is denoted by C_k and called a *hole*; its complement is called an antihole. It is easy to see that for $k \geq 2$, the odd hole C_{2k+1} and the odd antihole $\overline{C_{2k+1}}$ are minimal imperfect graphs; hence no perfect graph can contain them as induced subgraphs. No other minimal imperfect graphs have, to date, been found. When he first introduced perfect graphs, Berge made the following conjecture (see [3]):

Strong Perfect Graph Conjecture (SPGC) *A graph is perfect if and only if it contains no C_{2k+1} or $\overline{C}_{2k+1}$, for $k \geq 2$.*

A graph that contains no C_{2k+1} or $\overline{C}_{2k+1}$, for $k \geq 2$, will be called a *Berge graph.* Proving that all Berge graphs are perfect is equivalent to prove the Strong Perfect Graph Conjecture. In 1972, Lovász proved the following theorem [58]:

Lovász's Theorem [58] *A graph G is perfect if, and only if, we have $\omega(H)\alpha(H) \geq |V(H)|$ for every induced subgraph H of G.*

An immediate corollary is the following (which was known earlier as the Weak Perfect Graph Conjecture):

Perfect Graph Theorem [58] *A graph is perfect if and only if its complement is perfect.*

Grötschel, Lovász and Schrijver [29] have shown that the problem of computing $\omega(G)$ and $\chi(G)$ can be solved in polynomial time for perfect graphs. Their algorithm is based on the ellipsoid method and is quite complicated and unpractical. Research in perfect graph theory has centered around the SPGC, finding fast combinatorial algorithms for optimizing on perfect graphs, and developing a polynomial-time recognition algorithm for the class of perfect graphs.

Let us recall some standard definitions and notation. The *length* of a path is its number of edges. A path is *odd* if its length is odd and *even* otherwise. We denote by P_k a chordless path on k vertices. A *triangle* is a clique on three vertices.

We say that x is a *neighbour* of y, or that x *sees* y, if xy is an edge of G. The *neighbourhood* of x is the set $N(x)$ of all neighbours of x. If xy is not an edge of G then we say that x *misses* y.

The subgraph of a graph G induced by a subset S of vertices is denoted by $G[S]$; but we often confound a set of vertices with the induced subgraph on that set. When discussing complexity we often use n for the number of vertices in a graph and m for the number of edges.

An *arc* is an edge with an order on its endpoints. A *directed graph* consists of a set of vertices and a set of arcs. An *orientation* of a graph G is a directed graph obtained by choosing for each edge e of G one of the two possible arcs corresponding to e.

We close this section with some comments on even pairs in general graphs.

Bienstock [8] proved that it is NP-complete to decide whether a graph admits an odd hole containing a specified vertex. Moreover:

Theorem 1 ([8]) *It is co-NP-complete to decide whether a graph admits an even pair.*

Dutton and Brigham [18] and Hertz [37] defined a graph-coloring heuristic based on the contraction of some non-adjacent vertices. Dutton and Brigham choose at each step a pair $\{x, y\}$ that has the largest number of common neighbours among all non-adjacent pairs in G; Hertz fixes a vertex x and (while x has a non-neighbour) picks among all non-neighbours of x a vertex y that maximizes the number of common neighbours with x. In either variant, $\{x, y\}$ might not be an even pair, but one can still contract x and y and iterate the procedure until a clique K is obtained; then one can get a $|K|$-coloring of G by walking backwards along the contraction sequence. See [18, 37] for comments on the performance of this heuristic.

2 Even pairs and perfect graphs

Even pairs are intimately related to perfect graphs as the following consequences of Lemma 1 show:

Lemma 2 ([24, 51]) *Let $\{x, y\}$ be an even pair in a perfect graph G. Then G/xy is perfect.*

Proof Consider any induced subgraph H of G/xy. If xy is not a vertex of H, then H is also an induced subgraph of G and so $\chi(H) = \omega(H)$. If xy is a vertex of H, let H' be the induced subgraph of G with vertex set $V(H)-xy+x+y$. Since G is perfect, $\chi(H') = \omega(H')$. Facts 1.1 and 1.2 imply $\chi(H) = \omega(H)$. Thus, G/xy is perfect. □

Note that the converse of Lemma 2 is not true, as the first even pair contraction in Figure 1 shows.

¿From this result one can derive the major fact linking perfect graphs and even pairs:

The Even Pair Lemma [7, 62] *No minimal imperfect graph contains an even pair.*

Proof Assume that the Even Pair Lemma is false and consider vertices x, y that form an even pair in a minimal imperfect graph G. Let H be any proper induced subgraph of G/xy. If H is also a subgraph of G, then H is perfect. Else, there is a proper induced subgraph F of G such that $H = F/xy$. We know that F is perfect thus, by Fact 2, so is H. So, every proper induced subgraph of G/xy is perfect. Now, $\chi(G/xy) = \chi(G) > \omega(G) = \omega(G/xy)$ and so G/xy is minimal imperfect. By Lovász's Theorem we know that $|V(G)|-1 = \alpha(G)\omega(G)$ and $|V(G/xy)| - 1 = \alpha(G/xy)\omega(G/xy)$. Thus $1 = (|V(G)| - 1) - (|V(G/xy)| - 1) = \omega(G)(\alpha(G) - \alpha(G/xy))$. This implies that $\omega(G) = 1$, a contradiction. □

The Even Pair Lemma implies that any graph G such that every induced subgraph of G (or its complement) has an even pair must be a perfect graph. Thus it is of interest to study such graphs, and we will review results in this direction in Section 4.

One might naïvely hope to validate the Strong Perfect Graph Conjecture by proving that every Berge graph has an even pair. Unfortunately this approach is doomed to failure as there exist Berge graphs that are not cliques and yet have no even pair, see e.g. Figure 2. Some more complex examples can be found in [72, 73].

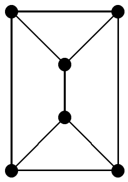

Figure 2: A Berge graph with no even pair

Not surprisingly, since we believe that all Berge graphs are perfect, it turns out that if we contract an even pair in a Berge graph then we again obtain a Berge graph.

Lemma 3 *Let $\{x, y\}$ be an even pair in a graph G. Then:*

3.1. If G contains no odd hole then G/xy contains no odd hole;

3.2. If G contains no antihole then G/xy contains no antihole different from $\overline{C}_6$.

3.3. If G contains no odd antihole then G/xy contains no odd antihole;

3.4. If G is a Berge graph then G/xy is a Berge graph.

Proof For the proof of 3.1, suppose that G/xy contains an odd hole H. If H does not contain xy, then H is also an odd hole in G. If H contains xy, then one of $H - xy + x$ or $H - xy + y$ is an odd hole in G as otherwise, H corresponds to an induced odd path between x and y in G.

For the proof of 3.2, first note that C_5 is isomorphic to $\overline{C}_5$. So, by Fact 3.1, we need only show that G/xy contains no antihole of length seven or greater. Assume the contrary and let A be an antihole of length seven or greater in G/xy. Obviously, xy is a vertex of A. We enumerate the vertices of A as $a_0 = xy, a_1, \ldots, a_k$ in such a way that a_i misses a_j if and only if $j = i \pm 1 \bmod k+1$. Now obviously, in G, both x and y miss both a_1 and a_k. Furthermore, no vertex of $A - a_0 - a_1 - a_k$ can miss both x and y in G. So, we can split $A - a_0 - a_1 - a_k$ up into three sets X, Y and Z such that every vertex in X sees x but not y, every vertex in Y sees y but not x, and every vertex in Z sees both x and y. There is no edge from a vertex in X to a vertex in Y as otherwise there would be an induced path of length 3 from x to y in G. But now, it is easy to verify that there exists some i and j with $1 \leq i < j \leq k$ such that either $\{x, a_i, a_{i+1}, \ldots, a_j\}$ or $\{y, a_i, a_{i+1}, \ldots, a_j\}$ induces an antihole in G.

The proof of 3.3 is similar to Fact 3.2; details are left to the reader. Fact 3.4 is immediate from Facts 3.1 and 3.3. □

3 Perfectly Contractile Graphs

We discuss here the proofs that various classes of graphs are perfectly contractile. The classes discussed include weakly triangulated graphs, Meyniel graphs and perfectly orderable graphs as well as subclasses of these classes such as triangulated graphs, comparability graphs, parity graphs and clique separable graphs. We also exhibit what we believe is a list of all the minimal non-perfectly contractile graphs.

Proving that a class of graphs contains only perfectly contractile graphs not only means that every graph in the class is perfect, it also suggests a natural combinatorial algorithm which will find optimal colorings and largest cliques for graphs in the class. We simply need an efficient procedure which, given a graph G in the class, finds a sequence of even pair contractions which transforms G into a clique.

Most of the classes in which we are interested are hereditary, i.e., every induced subgraph of a graph in the class is also in the class. Thus, we can apply the following simple lemma.

Lemma 4 *If A is a hereditary class of graphs and every graph in A is either a clique or contains an even pair whose contraction yields a graph in A then every graph in A is perfectly contractile.*

Proof Since A is hereditary, we need only show that every graph in A is even contractile. Assume the contrary and let G be a smallest graph in A which is not even contractile. Clearly, G is not a clique so by assumption there is an even pair $\{x, y\}$ in G such that G/xy is in A. Now, by the minimality of G, there exists a sequence of graphs $H_0 = G/xy, H_1, \ldots, H_k$ such that H_k is a clique and for i between 1 and k, H_i is obtained from H_{i-1} by contracting an even pair. Now, the sequence $G_0 = G, G_1 = H_0, G_2 = H_1, \ldots, G_{k+1} = H_k$ shows that G is even contractile, a contradiction. □

In what follows we say a graph is *non-complete* if it is not a clique.

In order to prove that a hereditary class $\mathcal{C}$ of graphs contains only perfectly contractile graphs, it is not sufficient to prove that every non-complete graph in $\mathcal{C}$ has an even pair. One must show that every graph in $\mathcal{C}$ is the beginning of a sequence of even-pair contractions that leads to a clique. It is important to note that the intermediate graphs of the sequence need not be in the class $\mathcal{C}$. Thus two situations can occur. Given a graph G of $\mathcal{C}$ we may be so lucky as to find an even pair $\{x, y\}$ of G such that G/xy is in $\mathcal{C}$; in this case an induction argument as in Lemma 4 will suffice to prove that G is perfectly contractile. Such a situation is illustrated below by the class of weakly triangulated graphs. On the other hand, $\mathcal{C}$ could contain a graph G such that all possible even-pair contractions from G produce graphs that are not in $\mathcal{C}$. This more complicated case is illustrated by the class of Meyniel graphs.

3.1 Weakly triangulated graphs

A graph is *weakly triangulated* if it contains no hole (C_k, ≥ 5) and no antihole ($\overline{C}_k$, ≥ 5). Hayward [30] proved that every weakly triangulated graph is perfect. The key lemma in his proof was:

Lemma 5 ([30]) *If C is a minimal cutset in a weakly triangulated graph and $\overline{C}$ is connected, then every component of $G - C$ contains a vertex adjacent to all of C.*

With the help of this lemma, Hoàng and Maffray [43] proved that every weakly triangulated graph contains an even pair. However, it is not true that contracting an even pair in a weakly triangulated graph always yields a new weakly triangulated graph. For instance, contracting the endpoints of a P_{2k+1} ($k \geq 3$) produces a C_{2k}. In order to be able to apply Lemma 4 to weakly triangulated graphs, one can define a special kind of even pair whose contraction cannot create holes or antiholes: two vertices x and y in a graph G form a 2-*pair* if every chordless path between them has length two. It is not hard to prove, particularly given Facts 3.1 and 3.2, the following fact.

Lemma 6 ([32]) *Contracting a 2-pair in a weakly triangulated graph yields a weakly triangulated graph.*

Hayward, Hoàng and Maffray [32] proved:

Theorem 2 ([32]) *Every weakly triangulated graph which is not a clique contains a 2-pair.*

Combining Theorem 2, Lemma 6 and Lemma 4 we obtain:

Corollary 1 *Every weakly triangulated graph is perfectly contractile.*

Let us now show how one can take advantage of the even-pair contraction sequences to optimally color weakly triangulated graphs. We note that two non-adjacent vertices x, y in a graph G form a 2-pair if and only if x and y are in different connected components of the graph $G - N(x) \cap N(y)$. We can test this condition for a given pair $\{x, y\}$ in $O(m + n)$ time. Thus, we can find a 2-pair in a graph (if it contains any) in $O(n^2 m)$ time. It follows that for any weakly triangulated graph G we can find, in $O(n^3 m)$ time, a sequence of 2-pair contractions that reduce G to a clique. As described in the introduction, we can use such a sequence to find an optimal coloring and largest clique in G. In fact, given the sequence of contractions, this can be done in $O(nm)$ time. In [32], Hayward, Hoàng and Maffray formally described $O(n^3m)$ algorithms which solve the maximum clique and minimum coloring problems on weakly triangulated graphs in this manner. As the complement of a weakly triangulated graph is weakly triangulated their algorithms can also be used to solve the minimum clique cover and maximum stable set problems on this class. They also developed algorithms of a similar flavour to solve the *weighted* versions of these four optimization problems in $O(n^4m)$ time. Later, Arikati and Pandu Rangan [2] have developed an $O(nm)$ algorithm to find a 2-pair (if any) in a graph. Using their algorithm yields a corresponding speedup in the optimization algorithms.

Spinrad and Sritharan [76] proved that if G is any graph that has a 2-pair $\{x, y\}$, then G is weakly triangulated if and only if G/xy is weakly triangulated. This astute use of 2-pairs yields an $O(mn^2)$ recognition algorithm for this class. They also improved on the complexity of the weighted version of the four optimization problems.

3.2 Meyniel graphs

A *Meyniel graph* is any graph in which every odd cycle of length at least five has at least two chords. Thus, a graph is Meyniel if and only if it contains no odd hole C_{2k+1} ($k \geq 2$) and no D_{2k+1} ($k \geq 2$), where D_p denotes a cycle of length p with exactly one chord which forms a triangle with two consecutive edges of the cycle. Meyniel [61] proved that these graphs are perfect, whence their name. Later [62], Meyniel showed that every such graph is either a clique or contains an even pair. However, there are non-complete Meyniel graphs which do not contain an even pair whose contraction yields a Meyniel graph. One such graph can be obtained by substituting every vertex in a C_6 by a pair of adjacent vertices. (This example was found by Sarkossian, and independently by Hougardy; a larger example was found by Bertschi and appears in his thesis [5].) Thus, we cannot hope to apply Lemma 4 directly to prove that Meyniel graphs are perfectly contractile. Hertz [36], sidestepped this problem by defining a slightly larger class of graphs to which he could apply the lemma. Specifically, a graph G is *quasi-Meyniel* if (i) it contains no C_{2k+1}, and (ii) for some vertex x of G, every edge which is the chord of some D_{2k+1} of G has x as an endpoint. Given a quasi-Meyniel graph G we call *tip* of G any vertex x that is endpoint of every chord of a D_{2k+1}. (Every vertex of a Meyniel graph G is a tip of G.)

Note that every Meyniel graph is quasi-Meyniel. In addition, if G is quasi-Meyniel but not Meyniel, then G has at most two tips and if x is a tip of G, then $G - x$ is Meyniel.

Hertz proved that every non-complete quasi-Meyniel graph contains an even pair whose contraction yields a quasi-Meyniel graph. Thus, by Lemma 4, quasi-Meyniel graphs are perfectly contractile.

Actually Hertz proved the following lemma.

Lemma 7 ([36]) *Let x be a tip of a quasi-Meyniel graph G which is not adjacent to all of*

$G - x$, and let y be a non-neighbour of x maximizing $|N(x) \cap N(y)|$. Then x and y form an even pair in G, G/xy is quasi-Meyniel, and xy is a tip of G/xy.

Proof Let G, x and y be as in the statement of the lemma. We show first that x and y form an even pair in G. If not, there is an odd induced path P from x to y in G. We enumerate the vertices along P as $p_0 = x, p_1, \ldots, p_{2k+1} = y$. Note that p_1 sees p_2 but not y. Thus by our choice of y there is a vertex z which sees x and y but not p_2. Now, z sees both endpoints of P but not all of it. Since $z + P$ is not bipartite, it contains an induced odd cycle which must be a triangle since G is quasi-Meyniel. In fact, we can think of P as being the concatenation of a set of subpaths each of which forms an induced cycle with z. Each of these paths is either even or an edge. Since at least one but not all of these subpaths are edges, there is an edge subpath and a non-edge subpath which have a common endpoint. But then these two paths together with z induce a D_{2k+1}. Furthermore, the chord of this D_{2k+1} has as endpoints z and an interior vertex of P. This contradicts our choice of x. Thus, x and y are indeed an even pair.

We now show that G/xy is quasi-Meyniel with tip xy. By Fact 3.1, G/xy contains no odd hole so we need only show that it contains no D_{2k+1} which does not have xy as an endpoint for its chord. Assume G/xy does have such a subgraph D. Let d be the unique vertex of D adjacent to both endpoints of the chord. If $xy = d$ then either one of $D - xy + y$ or $D - xy + x$ is an odd cycle with one chord in G contradicting the fact that x is a tip, or there is a path of length three from x to y in G through the chord of D contradicting the fact that x and y form an even pair. So, we can assume that xy is not d. Now, $D - d$ is an induced cycle C of G/xy. If neither $D - xy + y$ nor $D - xy + x$ is an odd cycle in G contradicting the fact that x is a tip then C corresponds to a path P from x to y in G. Let w be the vertex on this path which has a common neighbour with x on P. Now, by our choice of y, there is a vertex z which sees both x and y but not w. Using arguments similar to those in the above paragraph we can show that $P + d + z$ contains an odd cycle which contradicts the fact that G is quasi-Meyniel with tip x. We omit the details. $\square$

Corollary 2 ([36]) *Every non-complete quasi-Meyniel graph contains an even pair whose contraction yields a quasi-Meyniel graph.*

Proof If G is Meyniel and not a clique then we can apply Lemma 7 to any vertex x of G with fewer than $n-1$ neighbours and hence find the desired even pair. If G is quasi-Meyniel but not Meyniel then any tip must have fewer than $n-1$ neighbours so again we can apply Lemma 7. $\square$

Coloring Meyniel graphs with even-pair contractions

Even-pair contractions can be used as an optimization tool in Meyniel and quasi-Meyniel graphs. Recall that Lemma 7 describes the even pair to be contracted at each step. Given a quasi-Meyniel graph G and a tip x of G which is not adjacent to all of $G - x$ we can find, in $O(m)$ time, a vertex y such that G/xy is a quasi-Meyniel graph and xy is one of its tips—we simply chose a non-neighbour y of x maximizing $|N(x) \cap N(y)|$. Actually, if x is adjacent to all of $G - x$ then G obviously must be Meyniel, so every vertex of G is a tip and if G is not a clique we can choose some tip z of G which is not adjacent to all of $G - z$. It follows that given a quasi-Meyniel graph G which is not a clique and a tip z of G we can find, in $O(m)$ time, an even pair $\{z, y\}$ whose contraction yields a quasi-Meyniel graph G/zy with tip zy. Recursively applying this procedure yields an $O(nm)$ algorithm which given a quasi-Meyniel graph G and a tip x of G provides a sequence of even contractions reducing G to a clique. Since we can quickly find an optimum coloring and maximum clique of G by working through this sequence backwards, this yields an $O(nm)$ algorithm for these two optimization problems given a quasi-Meyniel graph with a specified tip. Since every vertex of a Meyniel graph is a tip, this yields an $O(nm)$ algorithm for the two optimization problems on Meyniel graphs.

This algorithm for optimizing on Meyniel graphs was first developed by Hertz, who introduced the quasi-Meyniel graphs only in order to be able to treat them as they appear along the contraction sequences (and when one quasi-Meyniel graph which is not Meyniel appears in the sequence, we know one tip of that graph). Since then, De Figueiredo and Vušković [23] have devised a polynomial-time algorithm which decides if a given graph G is quasi-Meyniel and, if it is, returns a vertex x that is a tip of G. Thus Hertz's algorithm can be applied to any quasi-Meyniel graph from the start.

More recently, Roussel and Rusu [71] have given an $O(n^2)$ algorithm which colors Meyniel graphs without using even pairs (but "simulates" even-pair contractions); this algorithm is based on Lexicographic Breadth-First Search and greedy sequential coloring.

3.3 Perfectly orderable graphs

A *perfect order* on the vertices of a graph G is an order $<$ such that, for every induced subgraph H of G, the greedy sequential coloring algorithm applied on $(H, <)$ produces an optimal coloring of H. Equivalently (see [10]), an order is perfect if and only if G contains no path P_4 on four vertices w, x, y, z with edges wx, xy, yz such that $w < x$ and $z < y$. A graph is *perfectly orderable* if it admits a perfect order. If the graph is given together with a perfect order, we say that the graph is *perfectly ordered.* This class of graphs was introduced by Chvátal [10] who proved that all such graphs are perfect. Meyniel [62] later proved that every non-complete perfectly orderable graph has an even pair. Hertz and de

Werra [38] extended Meyniel's result to prove:

Lemma 8 ([38]) *Every non-complete perfectly orderable graph has an even pair whose contraction yields a new perfectly orderable graph.*

Proof Let G be a perfectly ordered graph, where $<$ is a perfect order on the vertices of G. Let x be the first vertex in this ordering with less than $n-1$ neighbours. Let y be the first vertex in the ordering not adjacent to x. Then $\{x, y\}$ is called a *minimal pair* with respect to $<$. It is easy to see that every perfectly orderable graph which is not a clique contains a minimal pair. Let us show that any such pair is an even pair whose contraction yields a new perfectly ordered graph.

So let $\{x, y\}$ be the minimal pair with respect to $<$. We show first that $\{x, y\}$ is an even pair. Suppose, by way of contradiction, that there is an odd induced path $p_0p_1 \cdots p_{2m}p_{2m+1}$ from $x = p_0$ to $y = p_{2m+1}$. Since p_1 misses y, it does not have $n-1$ neighbours and hence $x < p_1$. Since $<$ is a perfect order, it follows that $p_2 < p_3$ and more generally $p_{2j} < p_{2j+1}$, for any j with $1 \leq j \leq m$. Since y is the first vertex missed by x under the order, it must be that x sees p_{2m}, a contradiction. Now, we claim that G/xy is perfectly orderable. To see this we consider the order $<'$ defined as equal to $<$ on $G - x - y$ and such that for z in $G - x - y$, we have $z <' xy$ if and only if $z < x$. The routine verification of the fact that $<'$ is a perfect order is left to the interested reader. The following three facts are all that is needed: (i) $<$ is a perfect order; (ii) every vertex which comes before xy under $<'$ has $n-2$ neighbours in G/xy; and (iii) every vertex of G which misses x comes after y under $<$. $\square$

Corollary 3 *Perfectly orderable graphs are perfectly contractile.*

Developing optimization algorithms for the class of perfectly orderable graphs on the basis of even-pair contractions poses a problem. If we are given a perfectly *ordered* graph, then we can apply the above arguments. On the other hand, it is NP-complete to determine if a graph is perfectly orderable [64]. Thus, if we are given a perfectly orderable G but not a perfect order of G, we can probably not find such an order. Yet, it is still possible to determine which pairs of vertices form even pairs in polynomial time ([1], see the next paragraph). However, once we have contracted on an arbitrary even pair in G, we do not know and can probably not check whether the resulting graph is perfectly orderable. There may be a sophisticated way of finding quickly an even pair whose contraction yields another perfectly orderable graph, but this problem is open and seems hard.

Hoàng [41] developed algorithms that solve the optimization problems on perfectly orderable without using even-pair contractions.

Generalizations of perfectly orderable graphs

The algorithm for finding even pairs in perfectly orderable graphs actually works on a much larger class of graphs. A *perfect orientation* of a graph G is a choice of direction for each edge under which there is no P_4 $wxyz$ such that wx is directed towards x while yz is directed towards y. A graph is *perfectly orientable* if it has a perfect orientation. We note that every perfect order corresponds to a perfect orientation, we simply direct each edge so that if xy is directed towards x then $y \leq x$. Thus, every perfectly orderable graph is perfectly orientable. In fact a graph is perfectly orderable if and only if it permits a perfect orientation containing no directed cycles. Arikati and Peled [1] developed a polynomial-time algorithm which given two vertices in a perfectly orientable graph determines if they form an even pair.

Their algorithm takes advantage of two facts. The first is that it is easy to find a perfect orientation of a perfectly orientable graph (this is in contrast to finding perfect orders of perfectly orderable graphs which by Middendorf and Pfeiffer's result is NP-complete). Thus, they can first find a perfect orientation of G and then use this orientation to help check for odd induced paths between the two vertices. The second fact they use is that if P is a path between two vertices x, y in a perfectly orientable graph G, then, under any orientation, we can find vertices a, b of P such that the path P can be broken into a directed a-x path, a directed b-y path and a path from a to b in which every vertex is a source or a sink. This fact, which follows immediately from the definition, allows them to develop a dynamic-programming method to check for the existence of odd and even induced paths between all pairs of vertices, given a perfect orientation.

We note that perfectly orientable graphs are not perfectly contractile, because odd holes are perfectly orientable. Thus we cannot hope to apply Lemma 4 to perfectly orientable graphs. There is, however, a class of graphs between perfectly orientable and perfectly orderable which seems like a natural candidate for our optimization technique. If $wxyz$ is a P_4 then we say that the arc wx *directly forces* the arc yz. We say that wx *forces* yz if there is a sequence of arcs $a_1, a_2, \ldots, a_k$ with $a_1 = wx$ and $a_k = yz$ such that the arc a_i directly forces a_{i+1} for each $i = 1, \ldots, k-1$. A directed cycle in an orientation is *forced* if there is some arc xy of the orientation which forces every arc of the cycle. A *strong perfect orientation* is a perfect orientation which contains no forced cycle. A graph is *strongly perfectly orientable* if it permits a strong perfect orientation. We can easily check in polynomial time if a graph permits a strong perfect orientation. Clearly every perfectly orderable graph is strongly perfectly orientable. It is also trivial to show that every strongly perfectly orientable graph is Berge. Reed conjectured that every strongly perfectly orientable graph is perfect and actually perfectly contractile. In fact, he conjectured that in every non-complete strongly perfectly orientable graph there is an even pair whose contraction yields another strongly perfectly orientable graph. If this is true, then the algorithm for determining if two vertices form an even pair in a perfectly orientable graph can be combined with the recognition algorithm for strongly perfectly orientable graphs to obtain a polynomial-time algorithm for optimally coloring and finding a largest clique in strongly perfectly orientable graphs.

3.4 Other classes of perfectly contractile graphs

Other classical families of perfect graphs that are perfectly contractile include: the triangulated graphs [28], which are contained in all three of the classes mentioned above; the comparability graphs [28], which are perfectly orderable; the parity and i-triangulated graphs [9], which are Meyniel; and the clique-separable graphs [27]. For an alternative proof that these last three classes are perfectly contractile see [6].

3.5 Graphs that might be perfectly contractile

We hope that the reader's interest in perfectly contractile graphs has been piqued by the results discussed above. We mention here a few classical families of perfect graphs which may be perfectly contractile.

Strongly perfect graphs A graph G is *strongly perfect* [4] if every induced subgraph G' of G contains a stable set that meets all maximal cliques of G'. It is easy to see that a graph is perfect if and only if each of its induced subgraphs contains a stable set meeting all maximum cliques. Thus, strongly perfect graphs are perfect.

Alternately orientable graphs A graph in *alternately orientable* if it permits an orientation in which no induced cycle contains a directed path with two edges, i.e., each cycle "alternates". Alternately orientable graphs were shown to be perfect by Hoàng [39].

The class Bip* A *star cutset* in a graph G is a cutset C of G that contains a vertex x adjacent to all of $C - x$. A graph is in *Bip** [11] if each of its induced subgraphs either is bipartite or contains a star cutset. Chvátal [11] proved that no minimal imperfect graph contains a star cutset, from which it follows that the graphs in *Bip** are perfect. The following three questions are all open:

> *Are strongly perfect graphs perfectly contractile?*
>
> *Are alternately orientable graphs perfectly contractile?*
>
> *Are the graphs in Bip* perfectly contractile?*

3.6 Forbidden subgraphs in perfectly contractile graphs

It is easy to see that antiholes and odd holes are not perfectly contractile (indeed they have no even pair). In this direction it is interesting to consider the "refinements of $\overline{C}_6$", i.e., the graphs obtained from a $\overline{C}_6$ by subdividing the edges that do not lie in a triangle. For short any such graph is called a *stretcher*. In other words a stretcher consists in two disjoint triangles and three disjoint paths, each path having one endpoint in each triangle. Note that if the paths have different parities then the stretcher contains an odd hole. A stretcher is *odd* if the three paths have odd length, and *even* if the three paths are even. Two examples of odd stretchers are depicted in Figure 3. One can establish that odd stretchers are minimally not-perfectly contractile, indeed: *Any possible sequence of even-pair contractions starting from an odd stretcher leads to $\overline{C}_6$ itself* (this fact needs some checking; see [55] for a formal proof).

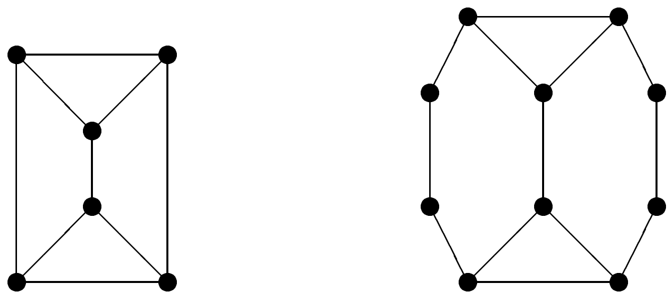

Figure 3: Two odd stretchers

In 1993, Everett and Reed made the following conjectures (publicly presented in [68]):

Conjecture 1 ([68]) *A graph is perfectly contractile if and only if it contains no antiholes, no odd holes and no odd stretchers.*

It is not known whether the characterization proposed in Conjecture 1 leads to a polynomial-time recognition algorithm for perfectly contractile graphs.

Reed proposed to say that G is a *Grenoble* graph if it contains no antiholes, no odd holes and no odd stretchers, and that G is an *Artemis* graph if it contains no antiholes, no odd holes and no stretchers. Conjecture 1 states that all Grenoble graphs are perfectly contractile. One can easily prove that stretchers and antiholes are neither alternately orientable nor strongly perfectly orientable. Furthermore, stretchers are not in *Bip** and odd stretchers are not strongly perfect. Thus, Conjecture 1 implies:

Conjecture 2 ([68]) *If G is alternately orientable, strongly perfect, strongly perfectly orientable, or in Bip* then G is perfectly contractile.*

The following weakening of Conjecture 1 would still imply that alternately orientable graphs, strongly perfect graphs, strongly perfectly orientable graphs, and the graphs in *Bip** are perfectly contractile.

Conjecture 3 ([68]) *Artemis graphs are perfectly contractile.*

We also believe:

Conjecture 4 ([68]) *Every non-complete perfectly contractile graph contains an even pair whose contraction leaves the graph perfectly contractile.*

and

Conjecture 5 ([68]) *There is a polynomial-time algorithm which given a non-complete perfectly contractile graph G finds an even pair $\{x, y\}$ of G such that G/xy is perfectly contractile.*

Since we believe that Conjecture 1 is true, we can make these last two conjectures more precise. We say that an even pair $\{x, y\}$ in a graph G is a *strong even pair* if there is no induced subgraph of G whose edge set can be partitioned into two vertex disjoint triangles and four vertex disjoint paths such that: two paths have an endpoint in each triangle, one has s as endpoint and an endpoint in a triangle, one has t as an endpoint and an endpoint in a triangle and one of s or t is not contained in the union of the two triangles (see Figure 4 for two examples of the forbidden configuration).

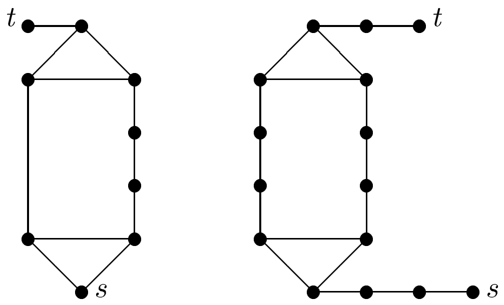

Figure 4: Forbidden configurations for strong even pairs

It is not hard to see that if $\{x, y\}$ is an even pair in a Grenoble graph then G/xy is a Grenoble graph if and only if $\{x, y\}$ is a strong even pair. Thus the following conjecture implies both Conjecture 1 and Conjecture 4.

Conjecture 6 ([68]) *Every non-complete Grenoble graph contains a strong even pair.*

Similarly, the following conjecture implies both Conjecture 1 and Conjecture 5.

Conjecture 7 ([68]) *There is a polynomial-time algorithm which, given a non-complete Grenoble graph G, returns a strong even pair of G.*

The results cited in this manuscript give evidence for the following conjecture, which can be compared with Theorem 1:

Conjecture 8 ([69]) *Even-pair testing is polynomial when restricted to the class of perfect graphs.*

A *homogeneous set* in a graph G is a subset S of vertices such that every vertex in $G-S$ sees either all or none of the vertices of S. The following question posed itself to some of the authors:

Conjecture 9 ([21]) *If every proper induced subgraph of G is perfectly contractile, and G has a homogeneous set, then G is perfectly contractile.*

We note that Conjecture 9 is implied by Conjecture 1. We provide support for some of these conjectures in Section 5 below.

4 Quasi-parity graphs

In 1987 Henry Meyniel [62] defined two classes of graphs that are of much interest.

Definition: Quasi-parity graph [62] *A graph G is a quasi-parity (QP) graph if, for every induced subgraph H of G on at least two vertices, either H or its complement $\overline{H}$ has an even pair.*

Definition: Strict quasi-parity graph [62] *A graph G is a strict quasi-parity (SQP) graph if, for every induced subgraph H of G, either H has an even pair or H is a clique.*

The Even Pair Lemma and the Perfect Graph Theorem imply that quasi-parity graphs are perfect. It is clear from the definitions that (a) every SQP graph is a QP graph, and (b) every perfectly contractile graph is an SQP graph. Hence, all the classes of perfectly contractile graphs mentioned in the preceding section (weakly triangulated graphs, Meyniel and quasi-Meyniel graphs, perfectly orderable graphs, and their subclasses) are SQP. The reverse of (a) and (b) above is not true: for example, the graph $\overline{C}_6$ is QP but not SQP, and any odd stretcher other than $\overline{C}_6$ itself is SQP but not perfectly contractile. Also note that some perfect graphs are not quasi-parity, see e.g. Figure 5 (the graph on the left is the smallest such graph).

Some other classes of perfect graphs have been shown to be QP or SQP, as follows.

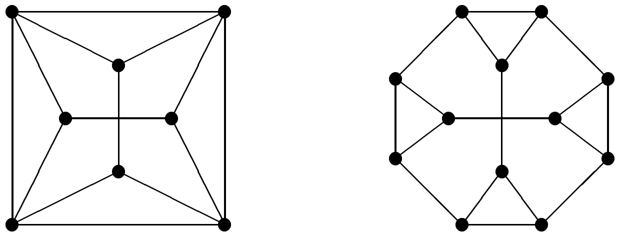

Figure 5: Perfect graphs that are not quasi-parity

Bull-free graphs A *bull* is a graph on five vertices a, b, c, d, e with edges ab, bc, cd, be, ce, see Figure 6. De Figueiredo, Maffray and Porto [21] proved that every bull-free perfect graph is a quasi-parity graph. As observed by Hougardy (private communication) the bull is "best of its type" in this theorem in the sense that if F is a graph such that every F-free perfect graph is QP, then F must necessarily be an induced subgraph of the bull. Some bull-free perfect graphs are not SQP, e.g., $\overline{C}_6$.

Slim graphs, slender graphs, skeletal graphs A graph is *slim* [34] if it can be obtained from a Meyniel graph by removing all the edges of one induced subgraph. Hertz [34] proved that slim graphs are perfect and conjectured that they are strict quasi-parity graphs; partial results in this direction appear in [45]. In addition, since it seems that slim graphs cannot contain any stretcher, in view of Conjecture 1 one may wonder whether all slim graphs are perfectly contractile.

A graph is *slender* [35] if it can be obtained from an i-triangulated graph by removing the edges of a matching. Hertz [35] proved that slender graphs are perfect. Not all slender graphs are strict quasi-parity, e.g., $\overline{C}_6$ is slender. One may wonder whether all slender graphs are quasi-parity graphs.

A graph is *skeletal* [33] if it can be obtained from a parity graph by removing the edges of stars whose centers are at least distance three apart in the parity graph. Hertz [33] proved that every skeletal graph is SQP. Since a skeletal graph cannot contain any stretcher or antihole, in view of Conjecture 3 it may be conjectured that every skeletal graph is perfectly contractile.

Opposition graphs A graph is an *opposition graph* [65] if it is possible to orient its edges in such a way that there is no directed circuit and every P_4 $abcd$ has its wings (the edges ab and cd) both pointing in or both pointing out. Hoàng and Maffray [44] proved that every opposition graph is an SQP graph. It is easy to see that an opposition graph cannot contain any stretcher or antihole; thus, in view of Conjecture 3, it can be conjectured that every opposition graph is perfectly contractile.

Wing-triangulated graphs Given a graph G, let $W(G)$ be the graph whose vertices are the edges of G and whose edges are the pairs of edges of G that are the wings of a P_4 (i.e., the edges ab, cd of a P_4 $abcd$). A graph G is then called *wing-triangulated* [50] if $W(G)$ is triangulated. Hougardy, Le and Wagler [50] proved that every wing-triangulated graph is SQP. Since a wing-triangulated graph cannot contain any odd stretcher or any antihole, in view of Conjecture 1 it may be conjectured that every wing-triangulated graph is perfectly contractile.

Graphs with no P_5 or K_5 Maffray and Preissmann [59] proved the Strong Perfect Graph Conjecture for graphs containing no P_5 or K_5, and suggested that the class of Berge graphs with no P_5 or K_5 is in QP. (This class is not in SQP since it contains $\overline{C}_6$; neither is its complement in SQP, since it contains the odd stretcher with path lengths $1, 3, 3$.)

4.1 Characterization of QP and SQP graphs

The question of determining in polynomial time whether a given graph is QP, or SQP, is still open. Since the classes QP and SQP are hereditary, they can be characterized by their minimal forbidden subgraphs. In this subsection we examine what is known on this problem. Let us call *minimally non quasi-parity* (MNQP) and *minimally non strict quasi-parity* (MNSQP), respectively, any graph which is not QP (resp. not SQP) but such that every proper induced subgraph is QP (resp. SQP). As pointed out above, antiholes and odd holes contain no even pair, and it is easy to check that they are MNSQP graphs. Likewise, odd holes and odd antiholes are MNQP (even antiholes are of course QP). Infinitely many other such graphs can be built, as follows. Recall that the *line-graph* of a graph G is the graph $L(G)$ whose vertices are the edges of G and whose edges are the pairs of incident edges in G. A classical corollary of Kőnig's matching theorem is that the line-graph of any bipartite graph is perfect. Hougardy studied even pairs in line-graphs of bipartite graphs.

Theorem 3 ([48]) *One can test in polynomial time if the line-graph of a bipartite graph has an even pair.*

Theorem 4 ([48]) *Let B be a 3-edge-connected simple bipartite graph. Then $L(B)$ and its complement have no even pair.*

We note that in Theorem 4 the condition of 3-edge-connectedness is not necessary for the line-graph to be even pair-free. As an example, consider the bipartite graph BW_{2k} ("bipartite wheel") obtained from an even cycle on $2k$ vertices by adding a new vertex adjacent to every second vertex on the cycle. It is not difficult to check that $L(BW_{2k})$ has no even pair; in fact $L(BW_{2k})$ is MNQP for $k \geq 3$ and MNSQP for $k \geq 2$.

Hougardy proposed an analogue of Conjecture 1 for the class of strict quasi-parity graphs.

Conjecture 10 ([46]) *Every minimal non-strict quasi-parity (MNSQP) graph is either an odd hole, an antihole, or the line-graph of some bipartite graph.*

We should point out that Conjecture 10 does not propose an explicit description of the structure of those line-graphs that are MNSQP. Indeed the above discussion and Theorem 4 show that there are infinitely many such graphs, and suggest that their collection may be too complex to be described simply. Linhares Sales and Maffray [54] devised a polynomial-time algorithm which decides if a given line-graph of bipartite is SQP; a variant of this algorithm also enables us to decide in polynomial time if a given line-graph of bipartite is MNSQP. Thus the recognition of such graphs is polynomially solvable. The method used in [54], however, does not give a simple description of the structure of the line-graphs of bipartite that are MNSQP.

Conjecture 10 has been confirmed for several classes of graphs, as we will see in the next section.

Two general results are known on MNSQP graphs. Assume G is a graph with a clique-cutset C, and call $B_1, \ldots, B_k$ ($k \geq 2$) the components of $G - C$.

Lemma 9 *A graph G with a clique-cutset C is strict quasi-parity if and only if all the induced subgraphs $G[B_i \cup C]$ are strict quasi-parity.*

Proof The only if part is trivial. To prove the if part, first suppose that $G[B_i \cup C]$ is not a complete graph, for some $i = 1, \ldots, k$. Then $G[B_i \cup C]$ possesses an even pair $\{x, y\}$. Then $\{x, y\}$ is also an even pair of G, because no chordless path between x and y can have interior vertices in another component of $G - C$, since C is a clique. In case $G[B_i \cup C]$ is a complete graph for all $i = 1, \ldots, k$, then any vertex from B_1 and any vertex from B_2 form an even pair of G. The proof is similar for every induced subgraph of G. $\Box$

Kézdy and Scobee proved:

Lemma 10 ([53]) *Every minimal counterexample to Hougardy's conjecture is* 3*-connected.*

Proof Let G be a minimal counterexample to Hougardy's conjecture, and suppose that G is not 3-connected. By Lemma 9, G is 2-connected. Let $\{u, v\}$ be a 2-cutset of G. The vertices u, v are not adjacent, again by Lemma 9. It is not difficult to establish that all chordless paths from u to v have the same parity (or else G would contain an odd hole); this parity must then be odd, since G has no even pair. Now, for each component B of $G - C$ build the graph $G_B = G[B \cup \{u, v\}] + uv$. It can be checked (see [53]) that each G_B is an even pair-free graph, which contradicts the minimality of G. $\Box$

It is tempting to formulate the analogue of Conjecture 10 for non quasi-parity graphs, although less evidence is available here:

Conjecture 11 *If G is a minimal non quasi-parity graph, then G or $\overline{G}$ is an odd hole or the line-graph of some bipartite graph.*

This conjecture was also formulated by Hougardy (private communication).

5 Recent Progress

This section surveys recent attempts to characterize perfectly contractile graphs and strict quasi-parity graphs, which consist in proofs of Conjecture 1 (or 3) and of Conjecture 10 for restricted classes of graphs. Recall that the Strong Perfect Graph Conjecture has been established for planar graphs [77], claw-free graphs [67], diamond-free graphs [79] and bull-free graphs [15]. The corresponding problem of recognizing the Berge graphs within each of these subclasses was solved respectively in [52], [16], [25] and [70]. Let us now explain how Conjectures 1 and 10 have been verified for planar graphs [57], claw-free graphs [54] and bull-free graphs [21]. In each case, the proof consists of a sequence of even-pair contractions that turns an input graph with none of the forbidden induced subgraphs into a clique. In addition, in each case, a decomposition theorem leads to a polynomial-time algorithm that recognizes the subclass of perfectly contractile graphs. Also for diamond-free graphs Conjectures 3 and 10 are now established.

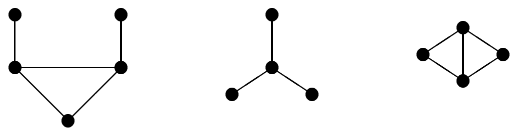

Figure 6: bull, claw, diamond

A little result turned out to be quite useful. When a graph G contains a clique-cutset C, and the components of $G - C$ are denoted $B_1, \ldots, B_k$ ($k \geq 2$), it is well-known that G is perfect if and only if all the induced subgraphs $G[B_i \cup C]$ are perfect [28]. The same holds for perfectly contractile graphs.

Lemma 11 *A graph G with a clique-cutset C is perfectly contractile if and only if all the induced subgraphs $G[B_i \cup C]$ are perfectly contractile.* $\square$

Indeed, each $G[B_i \cup C]$ can be even-contracted into a clique as any even pair of $G[B_i \cup C]$ is an even pair of G. We then get a graph G' in which any vertex of C sees all vertices of $G - C$ and each component of $G - C$ is a clique. Now G' is triangulated and so perfectly contractile.

5.1 Planar graphs

Note that any antihole with at least seven vertices is not planar. Thus for planar graphs Conjecture 1 reduces to the following statement:

Theorem 5 ([55]) *A planar graph with no odd hole and no odd stretcher is perfectly contractile.*

The proof consists of a polynomial algorithm which finds an $\omega(G)$-coloring for G through a sequence of even-pair contractions, as follows. Assume that the graph is drawn in the plane. Since G can be assumed to have no clique cutset by Lemma 11, every face of G is an even hole or a triangle. Then, every non-triangular face (if any) can be shown to contain an even pair at distance two along the face. The contraction of such a pair obviously keeps the graph planar; it also preserves the absence of odd holes and of odd stretchers. Finally, when every face of G is a triangle, an argument similar to that used in [52] implies that G is a comparability graph and so is perfectly contractile [38].

The corresponding recognition problem: "does a planar graph contain an odd hole or an odd stretcher?" can be solved by a revised version of Hsu's decomposition tree for planar perfect graph recognition [52].

For planar graphs Conjecture 10 reduces to the following statement:

Theorem 6 ([57]) *Every planar minimally non strict quasi-parity graph is either an odd hole or the line-graph of a bipartite graph.*

In [56] a polynomial-time algorithm is presented which recognizes if a given planar graph is strict quasi-parity; if it is not the algorithm exhibits a induced subgraph which is MNSQP.

5.2 Claw-free graphs

Chvátal and Sbihi [16] presented a polynomial-time algorithm for the recognition of claw-free Berge graphs. The theorem that supports this algorithm is that a claw-free graph is perfect if and only if either it has a clique-cutset, or it belongs to one of two classes of graphs called respectively "elementary" and "peculiar". These terms will be defined soon, but we can immediately observe that, with Lemma 11, the problem of characterizing claw-free perfectly contractile graphs reduces to the same problem on elementary graphs and peculiar graphs. Likewise, with Lemma 9, the problem of characterizing claw-free strict quasi-parity graphs reduces to elementary graphs and to peculiar graphs.

A graph G is *peculiar* if it can be constructed as follows: take a complete graph K whose set of vertices is split into six pairwise disjoint non-empty sets $A_1, B_1, A_2, B_2, A_3, B_3$; for each $i = 1, 2, 3$ remove at least one edge between A_i and $B_{i+1 \bmod 3}$; add pairwise disjoint non-empty cliques K_1, K_2, K_3 and, for each $i = 1, 2, 3$, make each vertex in K_i adjacent to all vertices in $K - (A_i \cup B_i)$. Chvátal [13] proved that all peculiar graphs containing no antiholes are perfectly orderable. Thus they are perfectly contractile by Lemma 8, and consequently strict quasi-parity. Chvátal's proof is such that it is possible to test in polynomial-time if a given peculiar graph contains an antihole and, if it does not, to build a perfect ordering. By the results in Section 3 we then find a sequence of even-pair contractions that turns G into a clique.

A graph is *elementary* if its edges can be colored with two colors in such a way that every chordless path on three vertices has its edges colored differently. This definition implies easily that elementary graphs can be recognized in polynomial time. Their structure was elucidated in the decomposition theorem of Maffray and Reed [60] which states that any elementary graph G is obtained from a line-graph of bipartite graph $L(B)$ (the "skeleton" of G) by replacing each vertex by a clique and each edge by a co-bipartite graph; most of these co-bipartite graphs are complete except for the edges of a matching of $L(B)$ which may be replaced by arbitrary (i.e., non-complete) co-bipartite graphs. This decomposition leads to study co-bipartite graphs and line-graphs of bipartite graphs. Chvátal [13] proved that all co-bipartite graphs with no antihole are perfectly orderable; his proof contains a polynomial-time algorithm for constructing a perfect ordering in any perfectly orderable co-bipartite graph. On the other hand, Linhares Sales and Maffray [54] that in any line-graph of bipartite graph $L(B)$ containing no odd stretcher and no clique-cutset there exists a vertex whose neighbourhood is a stable set of size two (a "nice" vertex). The contraction of the two neighbours of such a vertex and the elimination of this nice vertex (now pendant) gives again a line-graph of bipartite graph with no odd stretcher. From this result on the skeleton of G one can derive in G the existence of a set of even pairs "near" a nice vertex of $L(B)$ whose contraction reduces G to another elementary graph. In summary:

Theorem 7 ([54]) *Every claw-free graph that contains no odd hole, no antihole, and no odd stretcher is perfectly contractile.* □

In addition, the decomposition given in [60] is used in [54] to exhibit an odd stretcher in case the elementary graph is not perfectly contractile.

Using the decomposition from [60] it is also possible to prove:

Theorem 8 ([54]) *Every claw-free minimally non strict quasi-parity graph is either and odd hole, an antihole, or the line-graph of a bipartite graph.* □

The proof of this theorem contains an algorithm which, given any claw-free graph, determines if it is a strict quasi-parity graph and, if it is not, exhibits an induced subgraph which is either an odd hole, an antihole, or an MNSQP line-graph of bipartite.

5.3 Bull-free graphs

Bull-free graphs are interesting because they generalize P_4-free graphs and bipartite graphs. Chvátal and Sbihi [15] proved that all bull-free Berge graphs are perfect. De Figueiredo, Maffray and Porto [21] proved that every bull-free Berge graph either has a homogeneous set or is weakly triangulated or has a special structure called a *box partition* (this definition is too technical to develop here, and we refer the reader to [21] for the details). From this structure theorem one can derive:

Theorem 9 ([20, 21]) *Every bull-free Berge graph with no antihole either is weakly triangulated, or contains a homogeneous set which is not a clique, or is a comparability graph.* □

We note that any stretcher different from $\overline{C}_6$ contains a bull. Thus for bull-free graphs, both Conjectures 1 and 10 reduce to the following.

Corollary 4 ([21]) *Every bull-free Berge graph with no antihole is perfectly contractile.*

To prove Corollary 4 from Theorem 9, let G be any bull-free graph with no antihole. We first reduce to a clique of size $\omega(H)$ every non-complete homogeneous set H of G; indeed, by induction we can find a sequence of even-pair contractions that turns H into a clique of size $\omega(H)$. It is easy to see that any even pair of H is also an even pair of G, and that its contraction gives a Berge graph containing no antihole. Moreover, this reduced graph is also bull-free. Hence, Theorem 9 and the fact that both weakly triangulated graphs and perfectly orderable graphs are perfectly contractile yield the desired result.

The methods used in this proof also give rise to a polynomial-time algorithm which finds an $\omega(G)$-coloring for G through a sequence of even-pair contractions.

An argument from [20, 21, 22] states that in a bull-free graph G that has a box partition, there exists an antihole if and only if there exists a $\overline{C}_6$. This argument and the polynomial-time algorithm for bull-free Berge graph recognition [70] gives a polynomial-time recognition algorithm for bull-free perfectly contractile graphs.

Recently, Hayward [31] established that all bull-free weakly triangulated graphs are perfectly orderable. This result together with Theorem 9 imply now the following (which was a conjecture of Chvátal [12]):

Theorem 10 *Every bull-free Berge graph with no antihole is perfectly orderable.*

5.4 Diamond-free graphs

Tucker [79] proved that diamond-free graphs satisfy the Strong Perfect Graph Conjecture, in other words (since every antihole on at least seven vertices contains a diamond), that every diamond-free graph containing no odd hole is perfect. His proof is a polynomial-time algorithm that colors every diamond-free graph G containing no odd hole with $\omega(G)$ colors.

The recognition of diamond-free perfect graphs can be done in polynomial time [25], see also [14].

For diamond-free graphs, Conjecture 3 reduces to the following:

Theorem 11 ([75]) *Any diamond-free graph containing no odd hole and no stretcher is perfectly contractile.*

(The weaker fact that every diamond-free graph containing no odd hole and no stretcher has an even pair or is a clique was also established by Rusu in [74].) The key result in the proof of Theorem 11 [75] is that, in any diamond-free graph G containing no odd hole and no stretcher, there exists a vertex w whose neighbourhood is either a clique or a stable set. In case $N(w)$ is a clique, one can obtain that G is perfectly contractile by induction, using Lemma 11. In case $N(w)$ is a stable set, it is easy to see that any two neighbours of w form an even pair; however, some additional effort must be made to prove that there exists two neighbours of w whose contraction yields a graph that contains no diamond, no odd hole, and no stretcher (so that the contraction procedure can be continued). The proofs in [75] yield a polynomial-time algorithm which, given a diamond-free graph G containing no odd hole, produces a sequence of even-pair contractions leading from G to a clique of size $\omega(G)$. The corresponding recognition problem: "does a diamond-free graph contain an odd hole or a stretcher?" remains open.

Kézdy and Scobee [53] examined Conjecture 10 for diamond-free graphs. Their proof uses the decomposition which is at the basis of Fonlupt and Zemirline's recognition algorithm for diamond-free perfect graphs [25]. Fonlupt and Zemirline [25] proved that any diamond-free graph G that contains no odd hole must satisfy one of: (a) G is a bipartite graph, (b) G is the line-graph of a bipartite graph, (c) G has a clique cutset, (d) G has a cutset consisting of two non-adjacent vertices, (e) there exists a vertex z such that removing from G this vertex and all edges in $N(z)$ yields a disconnected graph. Now consider a diamond-free MNSQP graph G which would be a counterexample to Hougardy's Conjecture 10. Clearly G does not satisfy (a); by Lemmas 9 and 10, G does not satisfy (c) or (d). Kézdy and Scobee [53] further proved that G cannot satisfy (e). Hence G must satisfy (b), a contradiction. Thus:

Theorem 12 ([53]) *Conjecture 10 is true for diamond-free graphs.*

The corresponding algorithmic question remains open, namely: decide in polynomial time whether a diamond-free graph is SQP, and, in case it if not, exhibit an induced subgraph which is minimally SQP.

6 Odd pairs

Two non-adjacent vertices x, y of a graph form an *odd pair* if all induced paths between x and y have an odd number of edges. It is natural to try to formulate the analogue of the Even Pair Lemma for odd pairs.

The Odd Pair Conjecture [63, 69] *No minimal imperfect graph contains an odd pair.*

Let us say that an edge is *flat* if it does not lie in a triangle, i.e., its two endpoints have no common neighbour. If x, y are non-adjacent vertices in a graph G, denote by $G + xy$ the graph obtained from G by adding the edge xy. Clearly, $\{x, y\}$ is an odd pair in G if and only if, in the graph $G + xy$, the edge xy is flat and belongs to no odd hole in $G + xy$. Meyniel and Olariu [63] proved:

Lemma 12 ([63]) *The odd pair conjecture is true if and only if every minimal imperfect graph that contains a flat edge is an odd hole.*

For odd pairs one can obtain analogues of Lemmas 1 and 3, with $G + xy$ replacing G/xy, as follows.

Lemma 13 *Let $\{x, y\}$ be an odd pair in a graph G. Then:*

13.1. If $\omega(G) \geq 2$ then $\omega(G + xy) = \omega(G)$ and $\chi(G + xy) = \chi(G)$;

13.2. G contains an odd hole if and only if $G + xy$ contains an odd hole;

13.3. G contains an odd antihole if and only if $G + xy$ contains an odd antihole;

13.4. G is Berge if and only if $G + xy$ is Berge.

Proof To establish Fact 13.1 we need only prove $\chi(G + xy) \leq \chi(G)$. For this purpose let c be any coloring of G. If x, y have different colors, then c is a coloring of $G + xy$. If x, y have the same color (say color 1), let H be the bipartite subgraph induced by the vertices of colors 1 and 2. Then x and y are not in the same connected component of H, or else they would be joined by a path of even length. Hence, swapping colors 1 and 2 yields a coloring of G with the same number of colors and with x, y colored differently, which is also a coloring of $G + xy$. Facts 13.2–13.4 are easily checked. □

The results summarized in the next lemma make the relation between flat edges and odd pairs even more explicit.

Lemma 14 *If G is perfect and e is a flat edge of G, then $G - e$ is perfect* [80]).

If G is minimal imperfect, not an odd hole, and e is a flat edge of G, then $G - e$ is minimal imperfect [17]).

If G is a perfect graph and $\{x, y\}$ is an odd pair of G, then $G + xy$ is perfect [51]).

Hsu [51] proved that if x, y are two vertices in a graph G such that either (a) x, y are not adjacent and form an odd pair, or (b) x, y are adjacent and form a flat edge, then the graph obtained by removing x and y and adding all edges between $N(x)$ and $N(y)$ is perfect. It does not seem, however, that this would imply the Odd Pair Lemma.

Hougardy gave the analogues of Theorems 3 and 4 for odd pairs, namely:

Theorem 13 ([48]) *One can test in polynomial time if the line-graph of a bipartite graph has an odd pair.*

Theorem 14 ([48]) *Let B be a 3-edge-connected simple bipartite graph. Then $L(B)$ and its complement have no odd pair.*

Two vertices x, y are *antitwins* [66] if every vertex distinct from x and y is adjacent to exactly one of them. Clearly, antitwins are a special type of odd pair. Olariu [66] proved that no minimal imperfect graph contains antitwins. More recently, Hoàng [42] proved that no minimal imperfect graph contains a 3-*pair*, i.e., two non-adjacent vertices such that all chordless paths between them contain precisely three edges. Nonetheless the Odd Pair Conjecture remains open in general.

References

[1] S. R. Arikati and U. N. Peled, A polynomial algorithm for the parity path problem on perfectly orientable graphs, *Discrete App. Math.* **65** (1996) 5–20.

[2] S. R. Arikati and C. Pandu Rangan, An efficient algorithm for finding a two-pair, and its applications, *Discrete App. Math.* **31** (1991) 71–74.

[3] C. Berge, Les problèmes de coloration en théorie des graphes, *Publ. Inst. Stat. Univ. Paris* **9** (1960) 123–160.

[4] C. Berge and P. Duchet, Strongly perfect graphs, In *Topics on Perfect Graphs*, C. Berge and V. Chvátal, editors, *Ann. Discrete Math.* **21** (1984) 57–62, North Holland, Amsterdam.

[5] M.E. Bertschi, *La colorabilité unique dans les graphes parfaits*, PhD thesis, Math. Institute, University of Lausanne, Switzerland, 1988.

[6] M. E. Bertschi, Perfectly contractile graphs, *J. Combin. Theory Ser. B* **50** (1990) 222–230.

[7] M. E. Bertschi and B. A. Reed, A note on even pairs, *Discrete Math.* **65** (1987) 317–318.

[8] D. Bienstock, On the complexity of testing for odd holes and odd induced paths, *Discrete Math.* **90** (1991) 85–92.

[9] M. Burlet and J .P. Uhry, Parity graphs, In *Topics on Perfect Graphs*, C. Berge and V. Chvátal, editors, *Ann. Discrete Math.* **21** (1984) 253–278, North Holland, Amsterdam.

[10] V. Chvátal, Perfectly ordered graphs, In *Topics on Perfect Graphs*, C. Berge and V. Chvátal, editors, *Ann. Discrete Math.* **21** (1984) 63–68, North Holland, Amsterdam.

[11] V. Chvátal, Star cutsets, *J.Combin.Theory Ser. B* **39** (1985) 189–199.

[12] V. Chvátal, A class of perfectly orderable graphs, Technical Report 89573-OR, Institut für Ökonometrie und Operations Research, Rheinische Friedrich Wilhelms Universität, Bonn, Germany, May 1989.

[13] V. Chvátal, Which claw-free graphs are perfectly orderable? *Discrete Appl. Math.* **44** (1993) 39–63.

[14] V. Chvátal, J. Fonlupt, L. Sun, A. Zemirline, Recognizing dart-free perfect graphs, Technical Report 92778-OR, Institut für Ökonometrie und Operations Research, Rheinische Friedrich Wilhelms Universität, Bonn, Germany, 1992.

[15] V. Chvátal and N. Sbihi, Bull-free Berge graphs are perfect, *Graphs and Combin.* **3** (1987) 127–139.

[16] V. Chvátal and N. Sbihi, Recognizing claw-free perfect graphs, *J. Combin. Theory Ser. B* **44** (1988) 154–176.

[17] C. De Simone and A. Galluccio, New classes of Berge perfect graphs, *Discrete Math.* **131** (1994) 67–79.

[18] R.D. Dutton and R.C. Brigham, A new graph coloring algorithm, *Computer Journal* **24** (1981) 85-86.

[19] H. Everett, C.M.H. de Figueiredo, C. Linhares-Sales, F. Maffray, O. Porto and B.A. Reed, Path parity and perfection, *Discrete Math.* **165/166** (1997) 223–242.

[20] C. M. H. de Figueiredo and F. Maffray, Optimizing bull-free perfect graphs, Manuscript, Universidade Federal do Rio de Janeiro, Brazil, 1998.

[21] C. M. H. de Figueiredo, F. Maffray and O. Porto. On the structure of bull-free perfect graphs, *Graphs and Combin.* **13** (1997) 31–55.

[22] C. M. H. de Figueiredo, F. Maffray and O. Porto. On the structure of bull-free perfect graphs, 2: the weakly triangulated case, RUTCOR Research Report 45-94, Rutgers University, 1994.

[23] C. M. H. de Figueiredo and K. Vušković, Recognition of quasi-Meyniel graphs, Manuscript, Universidade Federal do Rio de Janeiro, Brazil, 1997.

[24] J. Fonlupt and J.P. Uhry, Transformations which preserve perfectness and h-perfectness of graphs, *Ann. Discrete Math.* **16** (1982) 83–85.

[25] J. Fonlupt and A. Zemirline, A characterization of $K_4 - e$-free graphs, *Maghreb Math. Rev.* **1** (1992) 167–202.

[26] M. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman, San Francisco (1979).

[27] F. Gavril, Algorithms on clique-separable graphs, *Discrete Math.* **19** (1977) 159–165.

[28] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York (1980).

[29] M. Grötschel, L. Lovász and A. Schrijver, Polynomial algorithms for perfect graphs, In *Topics on Perfect Graphs*, C. Berge and V. Chvátal, editors, *Ann. Discrete Math.* **21** (1984) 325–356, North Holland, Amsterdam.

[30] R. Hayward, Weakly triangulated graphs, *J.Combin. Theory Ser. B* **39** (1985) 200–208.

[31] R. Hayward, Bull-free weakly triangulated perfectly orderable graphs, Manuscript, Dept. Comp. Sci., University of Lethbridge, Canada, 1998.

[32] R. Hayward, C .T. Hoàng and F. Maffray, Optimizing weakly triangulated graphs, *Graphs and Combin.*, **5** (1989) 339-349. Erratum in vol. 6 (1990) 33–35.

[33] A. Hertz, Skeletal graphs – a new class of perfect graphs, *Discrete Math.* **78** (1989) 291–296.

[34] A. Hertz, Slim graphs *Graphs and Combin.* **5** (1989) 149–157.

[35] A. Hertz, Slender graphs *J. Combin. Theory ser. B* **47** (1989) 231–236.

[36] A. Hertz, A fast algorithm for coloring Meyniel graphs, *J. Combin. Theory Ser. B* **50** (1990) 231–240.

[37] A. Hertz, COSINE, a new graph coloring algorithm, *Operations Research Letters* **10** (1991) 411–415.

[38] A. Hertz and D. de Werra, Perfectly orderable graphs are quasi-parity graphs: a short proof, *Discrete Math.* **68** (1988) 111–113.

[39] C.T. Hoàng, Alternating orientation and alternating coloration of perfect graphs, *J. Combin. Theory Ser. B* **42** (1987) 264–273.

[40] C.T. Hoàng, On a conjecture of Meyniel. *J. Comb. Theory Ser. B* **42** (1987) 302–312.

[41] C.T. Hoàng, Algorithms for minimum weighted coloring of perfectly ordered, comparability, triangulated and clique-separable graphs, *Discrete Appl. Math.* **55** (1994) 133–143.

[42] C.T. Hoàng, Some properties of minimal imperfect graphs, *Discrete Math.* **160** (1996) 165–175.

[43] C.T. Hoàng and F. Maffray, Weakly triangulated graphs are strict quasi-parity graphs, Rutcor Research Report 6–86, Rutgers University (1986).

[44] C.T. Hoàng and F. Maffray, Opposition graphs are strict quasi-parity graphs, *Graphs and Combin.* **5** (1989) 83–85.

[45] C.T. Hoàng and F. Maffray, On slim graphs, even pairs and star-cutsets, *Discrete Math.* **105** (1992) 93-102.

[46] S. Hougardy, *Perfekte Graphen*, PhD thesis, Institut für Ökonometrie und Operations Research, Rheinische Friedrich Wilhelms Universität, Bonn, Germany, 1991.

[47] S. Hougardy, Counterexamples to three conjectures concerning perfect graphs, *Discrete Math.* **117** (1993) 245–251.

[48] S. Hougardy, Even and odd pairs in line-graphs of bipartite graphs, *European J. Combin.* **16** (1995) 17–21.

[49] S. Hougardy, Even pairs and the strong perfect graph conjecture, *Discrete Math.* **154** (1996) 277–278.

[50] S. Hougardy, V.B. Le, A. Wagler, Wing-triangulated graphs are perfect, *J. Graph Theory* **24** (1997), 25-31.

[51] W. L. Hsu, Decomposition of perfect graphs, *J. Combin. Th. ser. B* **43** (1987) 70–94.

[52] W. L. Hsu, Recognizing planar perfect graphs, *J. Assoc. Comp. Mach.* **34** (1987) 255–288.

[53] A. E. Kézdy and M. Scobee, A proof of Hougardy's conjecture for diamond-free graphs, Manuscript, Dept. Math., University of Lousville, Kentucky, 1998.

[54] C. Linhares Sales and F. Maffray, Even pairs in claw-free perfect graphs, *J. Combin. Theory Ser. B* **74** (1998) 169–191.

[55] C. Linhares Sales, F. Maffray and B. A. Reed, On planar perfectly contractile graphs, *Graphs and Combin.* **13** (1997) 167–187.

[56] C. Linhares Sales, F. Maffray and B.A. Reed, Recognizing planar strict quasi-parity graphs, Manuscript, Laboratoire Leibniz-IMAG, Grenoble, France, 1998.

[57] C. Linhares Sales, F. Maffray and B.A. Reed, On planar quasi-parity graphs, Manuscript, Laboratoire Leibniz-IMAG, Grenoble, France, 1999.

[58] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2** (1972) 253–267.

[59] F. Maffray and M. Preissmann, Perfect Graphs with no P_5 and no K_5, *Graphs and Combin.* **10** (1994) 179-184.

[60] F. Maffray, B.A. Reed, A description of claw-free perfect graphs, *J. Combin. Theory Ser. B* **75** (1999) 134–156.

[61] H. Meyniel, The graphs whose odd cycles have at least two chords, In *Topics on Perfect Graphs*, C. Berge and V. Chvátal, editors, *Ann. Discrete Math.* **21** (1984) 115–120, North-Holland, Amsterdam.

[62] H. Meyniel, A new property of critical imperfect graphs and some consequences, *European J. Combin.* **8** (1987) 313–316.

[63] H. Meyniel, S. Olariu, A new conjecture about minimal imperfect graphs, *J. Combin. Theory Ser. B* **47** (1989) 244–247.

[64] M. Middendorf and F. Pfeiffer, On the complexity of recognizing perfectly orderable graphs, *Discrete Math.* **81** (1990) 327–333.

[65] S. Olariu, *Results on perfect graphs*, PhD thesis, School of Computer Science, McGill University, Montréal, 1986.

[66] S. Olariu, No antitwins in minimal imperfect graphs, *J. Combin. Theory Ser. B* **45** (1988) 255–257.

[67] K. R. Parthasarathy and G. Ravindra, The strong perfect graph conjecture is true for $K_{1,3}$-free graphs, *J. Combin. Theory Ser. B* **21** (1976) 212–223.

[68] B.A. Reed, Problem session on parity problems, (Public communication) *DIMACS Workshop on Perfect Graphs*, Princeton University, New Jersey, 1993.

[69] B. A. Reed. Perfection, parity, planarity, and packing paths, In R. Kannan and W.R. Pulleyblank, editors, *Integer Programming and Combinatorial Optimization*, Math. Program. Soc., University of Waterloo Press, Waterloo, Canada (1990) 407–419.

[70] B. A. Reed and N. Sbihi, Recognizing bull-free perfect graphs, *Graphs and Combin.* **11** (1995) 171–178.

[71] F. Roussel and I. Rusu, An $O(n^2)$ algorithm to color Meyniel graphs, Manuscript, LIFO, University of Orléans, France, 1998.

[72] I. Rusu, Quasi-parity and perfect graphs, *Inf. Proc. Lett.* **54** (1995) 35–39.

[73] I. Rusu, Building counterexamples, *Discrete Math.* **172** (1997) 213–227.

[74] I. Rusu, Even pairs in Artemis graphs, Manuscript, LIFO, Université d'Orléans, France, 1997.

[75] I. Rusu, Perfectly contractile diamond-free graphs, Manuscript, LIFO, Université d'Orléans, France, 1998.

[76] J. Spinrad and R. Sritharan, Algorithms for weakly triangulated graphs, *Discrete Appl. Math.* **59** (1995) 181–191.

[77] A. Tucker, The strong perfect graph conjecture for planar graphs, *Can. J. Math.* **25** (1973) 103–114.

[78] A. Tucker, On Berge's strong perfect graph conjecture, *Ann. N.Y. Acad. Sci.* **319** (1979) 530–535.

[79] A. Tucker, Coloring perfect $(K_4 - e)$-free graphs, *J. Comb. Th. ser. B* **42** (1987) 313–318.

[80] A. Tucker, A reduction procedure for colouring perfect K_4-free graphs, *J. Comb. Th. ser. B* **43** (1987) 151-172.