The 1-join sandwich problem is NP-complete*

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Abstract

A graph is a 1-join composition, if its vertex set can be partitioned into four nonempty sets A_0, A_1, S_0 and S_1 such that, every vertex of A_0 is adjacent to every vertex of A_1; no vertex of S_0 is adjacent to a vertex of $A_1 \cup S_1$; no vertex of S_1 is adjacent to a vertex of $A_0 \cup S_0$. Graph sandwich problems have been introduced recently by Golumbic, Kaplan and Shamir in a paper in the Journal of Algorithms. The graph sandwich problem for property Φ is defined as follows: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property Φ? We prove that the sandwich problem when property Φ is to be a 1-join composition is NP-complete.

Keywords: Computational complexity; Design of algorithms; Sandwich problems; Perfect graphs.

1 Introduction

We say that a graph $G_1 = (V, E_1)$ is a *spanning* subgraph of $G_2 = (V, E_2)$, if $E_1 \subseteq E_2$ and we say that a graph $G = (V, E)$ is a *sandwich* graph for the pair G_1, G_2, if $E_1 \subseteq E \subseteq E_2$. According to [7], the GRAPH SANDWICH PROBLEM FOR PROPERTY Φ is defined as follows:

Problem: GRAPH SANDWICH PROBLEM FOR PROPERTY Φ
Instance: Two graphs, G_1 and G_2, such that G_1 is a spanning subgraph of G_2.
Question: Does there exist a sandwich graph for the pair G_1, G_2 which satisfies property Φ?

Note that graph sandwich problems arise as a natural generalization of recognition problems. The recognition problem for a class of graphs $\mathcal{C}$ is equivalent to the graph sandwich problem in which $G_1 = G_2 = G$, where G is the graph we want to recognize and property Φ is to belong to class $\mathcal{C}$.

In [7], several subclasses of perfect graphs are considered with respect to sandwich problems. In particular, they prove that the GRAPH SANDWICH PROBLEM FOR SPLIT GRAPHS

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remains in P. On the other hand, they prove that the GRAPH SANDWICH PROBLEM FOR PERMUTATION GRAPHS turns out to be NP-complete.

We are interested in graph sandwich problems for properties Φ related to compositions arising in perfect graph theory. Several compositions of graphs are known to preserve perfection: graph substitution [8], join composition [2, 4, 6], clique identification [1].

A graph $G = (V, E)$ is a *1-join composition* (or *join composition*) if its vertex set V can be partitioned into sets V_0 and V_1, so that for $i = 0, 1$, $|V_i| \geq 2$ and V_i contains a nonempty set A_i, with the property that every node of A_0 is adjacent to every node of A_1, no node of $V_0 \setminus A_0$ is adjacent to a node of V_1, and no node of $V_1 \setminus A_1$ is adjacent to a node of V_0. A polynomial-time algorithm, of complexity $\mathrm{O}(n^3)$, for 1-join composition recognition is given in [5]. Let $S_0 = V_0 \setminus A_0$ and $S_1 = V_1 \setminus A_1$. If $S_0 = \emptyset$ then A_0 is a homogeneous set in G, and if $S_1 = \emptyset$ then A_1 is a homogeneous set in G. In [3] a polynomial-time algorithm is given for solving the graph sandwich problem when property Φ is to contain a homogenous set. To distinguish the property of containing a homogenous set from the property of being a 1-join composition graph, we further impose the condition that $S_0 \neq \emptyset$ and $S_1 \neq \emptyset$. We say that $V_0|V_1$ is a *1-join partition* of G, with *special sets* S_0, A_0, S_1, A_1. In this paper we prove that the graph sandwich problem when property Φ is to be a 1-join composition is NP-complete.

2 Proof of NP-completeness

In this section we prove that 1-JOIN SANDWICH is NP-complete, by reducing the NP-complete problem 3-SATISFIABILITY to 1-JOIN SANDWICH. These two decision problems are defined as follows. We denote by $K_{|V|}$ the complete graph on the vertex set V.

3-SATISFIABILITY (3SAT)
INSTANCE: Set $X = \{x_1, \ldots, x_n\}$ of variables, collection $C = \{c_1, \ldots, c_m\}$ of clauses over X such that each clause $c \in C$ has $|c| = 3$ literals.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal?

1-JOIN SANDWICH
INSTANCE: Vertex set V, forced edge set E^1, forbidden edge set E^0.
QUESTION: Is there a graph $G = (V, E)$, such that $E^1 \subseteq E$ and $E \subseteq E(K_{|V|}) \setminus E^0$, which is a 1-join composition?

Theorem 1 *The 1-JOIN SANDWICH problem is NP-complete.*

In order to reduce 3SAT to 1-JOIN SANDWICH we need to construct a particular instance (V, E^1, E^0) of 1-JOIN SANDWICH from a generic instance (X, C) of 3SAT, such that C is satisfiable if and only if (V, E^1, E^0) admits a sandwich graph $G = (V, E)$ which is a 1-join composition.

Construction of particular instance of 1-JOIN SANDWICH.

The vertex set V is the union of sets of vertices: $\{x_1, \ldots, x_n\}$, $\{v_0, v_1, u_0, u_1, s_0, s_1\}$, $\cup_{i=1}^{m}\{z_1^i, z_2^i, z_3^i\}$, $\cup_{i=1}^{m}(V_1^i \cup V_2^i \cup V_3^i)$, where $V_1^i \cup V_2^i \cup V_3^i$ consists of auxiliary vertices corresponding to each clause. See Figures 1– 7 below, where solid edges denote E^1-edges and dashed edges denote E^0-edges.

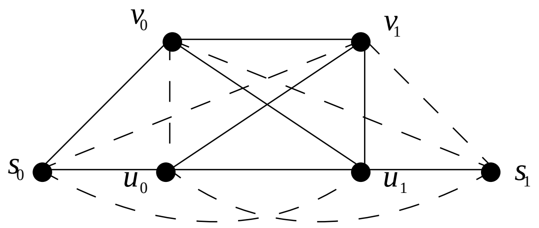

Figure 1: Auxiliary graph for special instance (V, E^1, E^0).

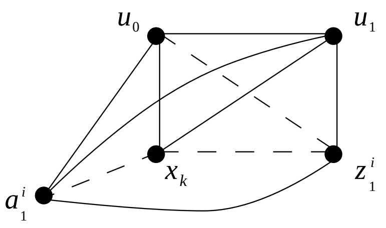

Figure 2: Same side gadget for the case when $l_1^i = x_k$.

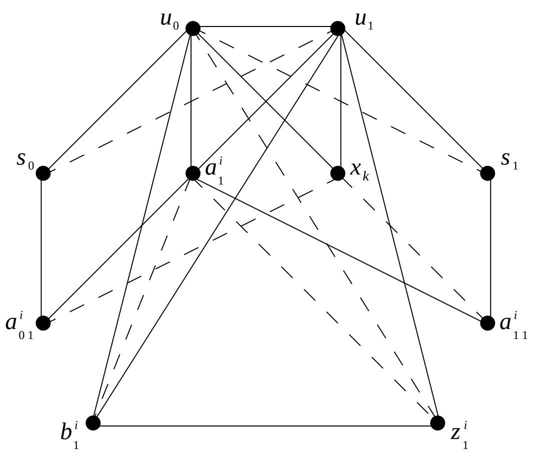

Figure 3: Different side gadget for the case when $l_1^i = \overline{x_k}$.

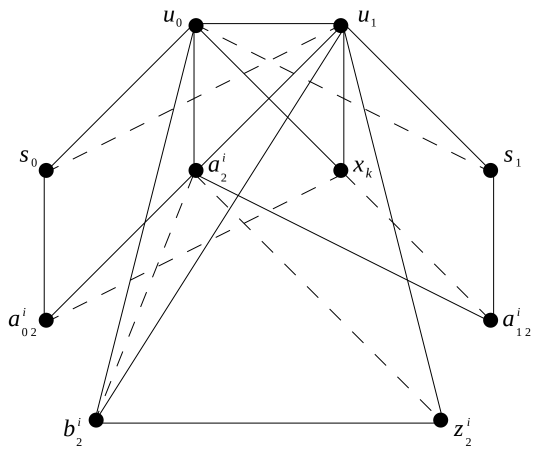

Figure 4: Different side gadget for the case when $l_2^i = x_k$.

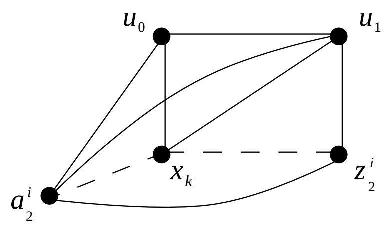

Figure 5: Same side gadget for the case when $l_2^i = \overline{x_k}$.

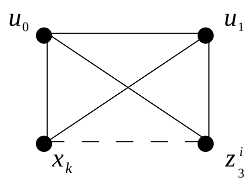

Figure 6: Same side gadget for the case when $l_3^i = x_k$.

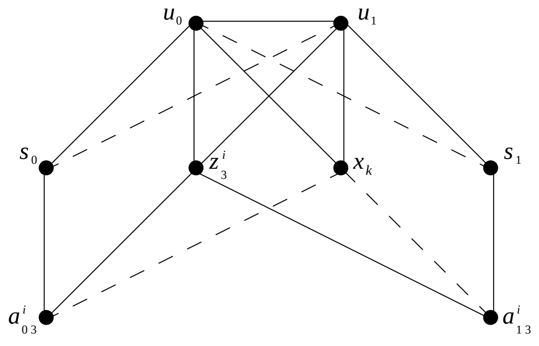

Figure 7: Different side gadget for the case when $l_3^i = \overline{x_k}$.

The forced edge set E^1 is the union of sets of edges:
$\{u_0u_1, s_0u_0, u_1s_1\}$,
$\{v_0w : w \in \cup_{i=1}^m \{z_3^i\} \cup \{x_1, \ldots, x_n, s_0, u_1, v_1\}\}$,
$\{v_1w : w \in \cup_{i=1}^m \{z_1^i, z_2^i, z_3^i\} \cup \{x_1, \ldots, x_n, u_0, u_1, v_0\}\}$,
$\cup_{i=1}^m \{x_iu_0, x_iu_1, z_2^iz_3^i, u_1z_1^i, u_1z_2^i, u_1z_3^i, u_0z_3^i\}$,
$\cup_{i=1}^m (B_1^i \cup B_2^i \cup B_3^i)$, where $B_1^i \cup B_2^i \cup B_3^i$ consists of auxiliary forced edges corresponding to each clause.

The forbidden edge set E^0 is the union of sets of edges:
$\{v_0s_1, v_1s_0, v_1s_1, u_1s_0, u_0s_1, u_0v_0\}$,
$\cup_{i=1}^m \{z_1^iz_2^i, u_0z_1^i\}$,
$\cup_{i=1}^m (D_1^i \cup D_2^i \cup D_3^i)$, where $D_1^i \cup D_2^i \cup D_3^i$ consists of auxiliary forbidden edges corresponding to each clause.

Let $c_i = (l_1^i \vee l_2^i \vee l_3^i)$ be a clause. If $l_1^i = x_k$, then the following nodes and edges are added to construct a *same side* gadget:
$V_1^i = \{a_1^i\}$,
$B_1^i = \{a_1^iu_0, a_1^iu_1, z_1^ia_1^i, v_0a_1^i, v_1a_1^i\}$,
$D_1^i = \{z_1^ix_k, a_1^ix_k\}$.
On the other hand, if $l_1^i = \overline{x_k}$, then the following nodes and edges are added to construct a *different side* gadget:
$V_1^i = \{a_1^i, b_1^i, a_{01}^i, a_{11}^i\}$,
$B_1^i = \{s_0a_{01}^i, s_1a_{11}^i, u_0a_1^i, u_1a_1^i, u_0b_1^i, u_1b_1^i, a_1^ia_{01}^i, a_1^ia_{11}^i, b_1^iz_1^i, v_0a_1^i, v_0b_1^i, v_0a_{01}^i, v_1a_1^i, v_1b_1^i, v_1a_{11}^i\}$,
$D_1^i = \{a_{01}^ix_k, a_{11}^ix_k, z_1^ia_1^i, a_1^ib_1^i\}$.

If $l_2^i = x_k$, then the following nodes and edges are added to construct a *different side* gadget:
$V_2^i = \{a_2^i, b_2^i, a_{02}^i, a_{12}^i\}$,
$B_2^i = \{s_0a_{02}^i, s_1a_{12}^i, u_0a_2^i, u_1a_2^i, u_0b_2^i, u_1b_2^i, a_2^ia_{02}^i, a_2^ia_{12}^i, b_2^iz_2^i, v_0a_2^i, v_0b_2^i, v_0a_{02}^i, v_1a_2^i, v_1b_2^i, v_1a_{12}^i\}$,
$D_2^i = \{a_{02}^ix_k, a_{12}^ix_k, z_2^ia_2^i, a_2^ib_2^i\}$.
On the other hand, if $l_2^i = \overline{x_k}$, then the following nodes and edges are added to construct a *same side* gadget:
$V_2^i = \{a_2^i\}$,
$B_2^i = \{u_0a_2^i, u_1a_2^i, z_2^ia_2^i, v_0a_2^i, v_1a_2^i\}$,
$D_2^i = \{z_2^ix_k, x_ka_2^i\}$.

Finally, if $l_3^i = x_k$, then the following nodes and edges are added to construct a *same side* gadget:
$V_3^i = \emptyset$, $B_3^i = \emptyset$, $D_3^i = \{z_3^ix_k\}$.
On the other hand, if $l_3^i = \overline{x_k}$, then the following nodes and edges are added to construct a *different side* gadget:
$V_3^i = \{a_{03}^i, a_{13}^i\}$,
$B_3^i = \{s_0a_{03}^i, s_1a_{13}^i, z_3^ia_{03}^i, z_3^ia_{13}^i, v_0a_{03}^i, v_1a_{13}^i\}$,
$D_3^i = \{x_ka_{03}^i, x_ka_{13}^i\}$.

Lemmas 1 and 2 prove the required equivalence for establishing Theorem 1.

Lemma 1 *If the particular instance (V, E^1, E^0) of 1-*JOIN SANDWICH *constructed above admits a 1-join composition $G = (V, E)$, such that $E^1 \subseteq E$ and $E \subseteq E(K_{|V|}) \setminus E^0$, then there exists a truth assignment that satisfies (X, C).*

Proof: Suppose there exists a sandwich graph $G = (V, E)$ that has a 1-join partition $V_0|V_1$, where V_0 denotes the side of the partition that contains the node v_0. Let S_0, A_0, S_1 and A_1 be the special sets of this 1-join partition.

Claim 1: *$s_0 \in S_0$, $u_0 \in A_0$, $s_1 \in S_1$ and $u_1 \in A_1$.*

Proof of Claim 1: We first show that $s_0, u_0 \in V_0$ and $s_1, u_1 \in V_1$, by considering the following two cases.

Case 1: $v_1 \in V_0$

Since $\{w : wv_0 \in E^1\} \cup \{w : wv_1 \in E^1\} = V \setminus \{s_1\}$ and $S_1 \neq \emptyset$, we have that $S_1 = \{s_1\}$. Since $s_1 \in S_1$ and $s_1u_1 \in E^1$, $u_1 \in V_1$. Since $\{w : wu_1 \in E^1\} \cup \{w : ws_0 \in E^1\} \cup \{w : ws_1 \in E^1\} = V \setminus \{s_0\}$ and $S_0 \neq \emptyset$, we have that $s_0 \notin V_1$ and hence $s_0 \in V_0$. Since $u_1v_1, u_1v_0 \in E^1$, $v_0, v_1 \in A_0$. Since $u_0v_1 \in E^1$ and $u_0v_0 \in E^0$, we have that $u_0 \notin V_1$ and hence $u_0 \in V_0$.

Case 2: $v_1 \in V_1$

Since $v_0v_1 \in E^1$, we have that $v_0 \in A_0$ and $v_1 \in A_1$. First suppose that $s_0 \in V_1$. Since $\{w : wv_1 \in E^1\} \cup \{w : ws_0 \in E^1\} = V \setminus \{v_1, s_0, s_1\}$ and $S_0 \neq \emptyset$, we have that $S_0 = \{s_1\}$. Since $s_1u_1 \in E^1$, we have that $u_1 \in V_0$. Since $u_1v_1 \in E^1$, we have that $u_1 \in A_0$. But then $s_0v_0 \in E^1$ and $s_0u_1 \in E^0$ contradict the assumption that $V_0|V_1$ is a 1-join partition. Therefore $s_0 \in V_0$. Since $s_0v_1 \in E^0$, we have that $s_0 \in S_0$. Since $u_0s_0 \in E^1$, we have that $u_0 \in V_0$. Suppose that $u_1 \in V_0$. Since $u_1v_1 \in E^1$, we have that $u_1 \in A_0$. Since $s_1u_1 \in E^1$ and $s_1v_0 \in E^0$, we have that $s_1 \notin V_1$ and hence $s_1 \in V_0$. But then, since $\{w : wu_1 \in E^1\} \cup \{w : ws_0 \in E^1\} \cup \{w : ws_1 \in E^1\} = V \setminus \{s_0\}$, it implies that $S_1 = \emptyset$, which contradicts the assumption that $V_0|V_1$ is a 1-join partition. Therefore $u_1 \in V_1$. Since $u_1v_0 \in E^1$, we have that $u_1 \in A_1$. Since $s_1u_1 \in E^1$ and $s_1v_1 \in E^0$, we have that $s_1 \notin V_0$ and hence $s_1 \in V_1$.

Therefore $u_0, s_0 \in V_0$ and $u_1, s_1 \in V_1$. Since $u_0u_1 \in E^1$, we have that $u_0 \in A_0$ and $u_1 \in A_1$. Since $s_0u_1 \in E^0$, we have that $s_0 \in S_0$, and since $s_1u_0 \in E^0$, we have that $s_1 \in S_1$. This completes the proof of Claim 1.

Claim 2: *For $i = 1, \dots, m$ and $j = 1, 2, 3$, if z^i_j and x_k, for some $k \in \{1, \dots, n\}$, are connected by a same side gadget, then z^i_j and x_k are on the same side of 1-join partition $V_0|V_1$ of G.*

Proof of Claim 2: By Claim 1, $u_0 \in A_0$ and $u_1 \in A_1$. Since $x_ku_0, x_ku_1 \in E^1$, $x_k \in A_0 \cup A_1$. First suppose that $j = 3$. Since $z^i_3u_0, z^i_3u_1 \in E^1$, $z^i_3 \in A_0 \cup A_1$. Since $z^i_3x_k \in E^0$, z^i_3 and x_k must be on the same side of 1-join partition $V_0|V_1$. Now assume that $j = 1$ or 2. Since $a^i_ju_0, a^i_ju_1 \in E^1$, $a^i_j \in A_0 \cup A_1$. Since $a^i_jx_k \in E^0$, a^i_j and x_k must be on the same side of 1-join partition $V_0|V_1$. Since $z^i_ja^i_j \in E^1$ and $z^i_jx_k \in E^0$, z^i_j must be on the same side of 1-join partition $V_0|V_1$ as a^i_j and x_k. This completes the proof of Claim 2.

Claim 3: *For $i = 1, \dots, m$ and $j = 1, 2, 3$, if z^i_j and x_k, for some $k \in \{1, \dots, n\}$, are connected by a different side gadget, then z^i_j and x_k are on different sides of 1-join partition $V_0|V_1$ of G.*

Proof of Claim 3: By Claim 1, $s_0 \in S_0$, $u_0 \in A_0$, $s_1 \in S_1$ and $u_1 \in A_1$. Since $s_0 \in S_0$ and $a^i_{0j}s_0 \in E^1$, $a^i_{0j} \in V_0$. Since $s_1 \in S_1$ and $a^i_{1j}s_1 \in E^1$, $a^i_{1j} \in V_1$. Since $x_ku_0, x_ku_1 \in E^1$,

$x_k \in A_0 \cup A_1$. First suppose that $j = 3$. Since $z_3^i u_0, z_3^i u_1 \in E^1$, $z_3^i \in A_0 \cup A_1$. Since $a_{13}^i z_3^i \in E^1$ and $a_{13}^i x_k \in E^0$, z_3^i and x_k cannot both be in V_0. Since $a_{03}^i z_3^i \in E^1$ and $a_{03}^i x_k \in E^0$, z_3^i and x_k cannot both be in V_1. Hence z_3^i and x_k are on different sides of 1-join partition $V_0|V_1$. Now assume that $j = 1$ or 2. By a similar argument as above, x_k and a_j^i must be on different sides of 1-join partition $V_0|V_1$. By a similar argument as in Claim 2 applied to vertices a_j^i, b_j^i and z_j^i, we have that a_j^i and z_j^i must be on the same side of 1-join partition $V_0|V_1$. Hence, x_k and z_j^i must be on different sides of 1-join partition $V_0|V_1$. This completes the proof of Claim 3.

We now define the following truth assignment for (X, C): for $i = 1, \ldots, n$, x_i is false if and only if vertex $x_i \in V_0$. By the construction of (V, E^1, E^0) and by Claims 2 and 3, for $i = 1, \ldots, m$, literal l_1^i is false if and only if $z_1^i \in V_0$, literal l_2^i is false if and only if $z_2^i \in V_1$, and literal l_3^i is false if and only if $z_3^i \in V_0$. Suppose that for some $i \in \{1, \ldots, m\}$, the clause $c_i = (l_1^i \vee l_2^i \vee l_3^i)$ is false. Then $z_1^i, z_3^i \in V_0$ and $z_2^i \in V_1$. By Claim 1, $u_0 \in A_0$ and $u_1 \in A_1$. Since $z_1^i u_1, z_3^i u_1 \in E^1$, $z_1^i, z_3^i \in A_0$. But then since $z_1^i z_2^i \in E^0$ and $z_2^i z_3^i \in E^1$, the assumption that $V_0|V_1$ is a 1-join partition of G is contradicted. Hence, the above defined truth assignment satisfies (X, C). ∎

Lemma 2 *If there exists a truth assignment that satisfies (X, C), then the particular instance (V, E^1, E^0) of 1-JOIN SANDWICH constructed above admits a 1-join composition $G = (V, E)$, such that $E^1 \subseteq E$ and $E \subseteq E(K_{|V|}) \setminus E^0$.*

Proof: Suppose there is a truth assignment that satisfies (X, C). We shall define a partition of V into sets V_0, V_1 that in turn defines a solution G for the particular instance (V, E^1, E^0) of 1-JOIN SANDWICH associated with the 3SAT instance (X, C).

Place vertices v_0, u_0, s_0 in V_0, and vertices v_1, u_1, s_1 in V_1. For $j = 1, \ldots, n$, if x_j is false place it in V_0, and otherwise in V_1. For each clause c_i, $i = 1, \ldots, m$, place the remaining vertices as follows. If $l_1^i = x_k$, place z_1^i and a_1^i on the same side as x_k. If $l_1^i = \overline{x_k}$, place a_{01}^i in V_0, a_{11}^i in V_1, and a_1^i, b_1^i and z_1^i on different side from x_k. If $l_2^i = x_k$, place a_{02}^i in V_0, a_{12}^i in V_1, and a_2^i, b_2^i and z_2^i on different side from x_k. If $l_2^i = \overline{x_k}$, place z_2^i and a_2^i on the same side as x_k. If $l_3^i = x_k$, place z_3^i on the same side as x_k. If $l_3^i = \overline{x_k}$, place a_{03}^i in V_0, a_{13}^i in V_1, and z_3^i on different side from x_k.

Sets S_0 and S_1 are defined as follows. Place s_0 in S_0 and s_1 in S_1. For $i = 1, \ldots m$ and $j = 1, 2, 3$: if z_j^i is connected by a different side gadget, then if z_j^i is in V_0 place a_{0j}^i in S_0, and if z_j^i is in V_1 place a_{1j}^i in S_1. For $i = 1, \ldots, m$: if z_1^i is in V_1 then place z_1^i in S_1, and if z_2^i and z_3^i are both in V_1 then place z_2^i in S_1.

Let $A_0 = V_0 \setminus S_0$ and $A_1 = V_1 \setminus S_1$. Let $E = E^1 \cup \{uv : u \in A_0 \text{ and } v \in A_1\}$. To show that the graph $G = (V, E)$ is a sandwich graph with 1-join partition $V_0|V_1$ we need to show that the following types of edges do not exist: $uv \in E^0$ such that $u \in A_0$ and $v \in A_1$, $uv \in E^1$ such that $u \in S_0$ and $v \in V_1$, $uv \in E^1$ such that $u \in V_0$ and $v \in S_1$.

Let $e \in E^0$ and suppose that one endnode of e is in A_0 and the other is in A_1. Then neither s_0 nor s_1 is an endnode of e. The edge e cannot be of the form $u_0 z_1^i$, for some $i \in \{1, \ldots, m\}$, because $u_0 \in A_0$ and if z_1^i is in V_1 then z_1^i is in S_1. The edge e cannot be of the form $z_1^i z_2^i$, for some $i \in \{1, \ldots, m\}$, because that would imply that z_1^i is in V_0 (since if

z_1^i is in V_1 then it is in S_1) and so z_2^i is in A_1, which means that z_3^i is in V_0 (since z_2^i and z_3^i are both in V_1 then z_2^i is in S_1), which would imply that all three literals in c_i are false. Let $i \in \{1, \ldots, m\}$. If z_1^i and x_k are connected by a same side gadget, then $e \notin D_1^i$ since x_k, z_1^i and a_1^i are all placed on the same side. If z_1^i and x_k are connected by a different side gadget, then a_1^i, b_1^i and z_1^i are all placed on the same side (hence $e \neq a_1^i b_1^i, z_1^i a_1^i$), z_1^i and x_k are placed on different sides, so if $x_k \in V_0$ then $a_{11}^i \in S_1$ (hence $e \neq x_k a_{01}^i, x_k a_{11}^i$), and if $x_k \in V_1$ then $a_{01}^i \in S_0$ (hence $e \neq x_k a_{01}^i, x_k a_{11}^i$). Therefore $e \notin D_1^i$. Similarly $e \notin D_2^i \cup D_3^i$.

Let $e \in E^1$ and suppose that one endnode of e is in S_0 and the other is in V_1. If $s_0 u \in E^1$ then $u = u_0$, v_0 or a_{0j}^i for some $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$. Since all these nodes are in V_0, the edge e cannot have s_0 as an endnode. Thus an endnode of e is a_{0j}^i for some $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$. But then $a_{0j}^i \in S_0$, hence $z_j^i \in V_0$ and so $e \neq a_{03}^i z_3^i$, $a_{02}^i a_2^i$, $a_{01}^i a_1^i$.

Let $e \in E^1$ and suppose that one endnode of e is in S_1 and the other is in V_0. If $s_1 u \in E^1$ then $u = u_1$ or a_{1j}^i for some $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$. Since all these nodes are in V_1, the edge e cannot have s_1 as an endnode. Suppose that for some $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$, a_{1j}^i is an endnode of e. Then $a_{1j}^i \in S_1$, hence $z_j^i \in V_1$ and so $e \neq a_{11}^i a_1^i$, $a_{12}^i a_2^i$, $a_{13}^i z_3^i$. Suppose that for some $i \in \{1, \ldots, m\}$, $z_1^i \in S_1$ and that it is an endnode of e. Then $e \neq z_1^i u_1$, since $u_1 \in V_1$. If z_1^i and x_k are connected by a same side gadget, then $e \neq z_1^i a_1^i$, since z_1^i and a_1^i are placed on the same side. If z_1^i and x_k are connected by a different side gadget, then $e \neq z_1^i b_1^i$, since z_1^i and b_1^i are placed on the same side. Thus we may assume that for some $i \in \{1, \ldots, m\}$, $z_2^i \in S_1$ and that it is an endnode of e. Then $z_3^i \in V_1$ and hence $e \neq z_2^i z_3^i$. Since $u_1 \in V_1$, $e \neq z_2^i u_1$. If z_2^i and x_k are connected by a same side gadget, then $e \neq z_2^i a_2^i$, since z_2^i and a_2^i are placed on the same side. If z_2^i and x_k are connected by a different side gadget, then $e \neq z_2^i b_2^i$, since z_2^i and b_2^i are placed on the same side. ■

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