Optimizing Bull-Free Perfect Graphs

Celina M. H. de Figueiredo[1]
Frédéric Maffray[2]

Abstract. A bull is a graph obtained by adding a pendant vertex at two vertices of a triangle. Here we present polynomial-time combinatorial algorithms for the optimal weighted coloring and weighted clique problems in bull-free perfect graphs. The algorithms are based on a structural analysis and decomposition of bull-free perfect graphs.

Key words. graph algorithms, perfect graphs, analysis of algorithms and problem complexity, combinatorial optimization

AMS subject classifications. 05C85, 05C60, 68Q25, 90C27

1 Introduction

A graph G is called *perfect* if the vertices of every induced subgraph H of G can be colored with $\omega(H)$ colors, where $\omega(H)$ is the maximum clique size in H. Berge [1] introduced perfect graphs and conjectured the following characterization: *A graph is perfect if and only if it contains no odd hole and no odd antihole.* Here a *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complement of a hole. This conjecture is still open and is known as the Strong Perfect Graph Conjecture. Graphs with no odd hole and no odd antihole have become known as *Berge* graphs. Chvátal and Sbihi [3] proved the validity of the Strong Perfect Graph Conjecture for bull-free graphs, i.e., graphs with no induced subgraph isomorphic to a bull, where a *bull* is a graph with five vertices a, b, c, d, e and five edges ab, bc, cd, be, ce. Subsequently, Reed and Sbihi [9] gave a polynomial algorithm for recognizing bull-free Berge graphs.

[1]Universidade Federal do Rio de Janeiro, Instituto de Matemática, Caixa Postal 68530, 21945-970 Rio de Janeiro, RJ, Brazil. E-mail: `celina@cos.ufrj.br`. Partially supported by CNPq, grant 30 1160/91.0.

[2]CNRS, Laboratoire Leibniz-IMAG, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France. E-mail: `frederic.maffray@imag.fr`

In this paper, we present polynomial-time algorithms for solving the following optimization problems for bull-free perfect graphs: find a largest clique, a largest stable set, a minimum coloring, and a minimum clique covering. We actually present algorithms which solve the weighted versions of these problems, defined as follows. We are given a graph G with vertices $v_1, \dots, v_n$ and non-negative weights $w(v_1), \dots, w(v_n)$. We solve the following optimization problems for bull-free perfect graphs:

MAXIMUM WEIGHTED CLIQUE PROBLEM: Find a clique K of G, such that the weight of K, defined as $w(K) = \sum\{w(x) \mid x \in K\}$, is maximum over all cliques of G.

MAXIMUM WEIGHTED STABLE SET PROBLEM: Find a stable set S of G, such that the weight of S, defined as $w(S) = \sum\{w(x) \mid x \in S\}$, is maximum over all stable sets of G.

MINIMUM WEIGHTED COLORING PROBLEM: Find stable sets $S_1, \dots, S_t$ and weights $X(S_1), \dots, X(S_t)$, such that:

$$\sum\{X(S_i) \mid S_i \ni v_j\} \geq w(v_j) \qquad (\forall v_j) \tag{1}$$

and the sum $X(S_1) + \cdots + X(S_t)$ is minimum over all sets of real weights that satisfy (1).

MINIMUM WEIGHTED CLIQUE COVERING PROBLEM: Find cliques $K_1, \dots, K_t$ and weights $X(K_1), \dots, X(K_t)$, such that:

$$\sum\{X(K_i) \mid K_i \ni v_j\} \geq w(v_j) \qquad (\forall v_j) \tag{2}$$

and the sum $X(K_1) + \cdots + X(K_t)$ is minimum over all sets of real weights that satisfy (2).

Let us review some known results about perfect graphs and useful properties of bull-free graphs. A graph is called *weakly triangulated* if it contains no hole and no antihole. Given a subset of vertices S, a vertex is said to be *partial* on S if it has at least one neighbour and at least one non-neighbour in S. A vertex is *impartial on* S if it either sees all vertices of S or misses all vertices of S. A proper subset H of vertices is called *homogeneous* if it has at least two vertices and every vertex not in H sees either all or none of H, in other words every vertex not in H is impartial on H. Notice that if H is a homogeneous set of G then it is also a homogeneous set of the complement graph $\overline{G}$. Notice also that G is bull-free if and only if $\overline{G}$ is bull-free. Hence it is enough to solve MAXIMUM WEIGHTED CLIQUE PROBLEM and MINIMUM WEIGHTED COLORING PROBLEM.

In a previous paper we proved the following theorem about the structure of bull-free Berge graphs:

Theorem 1 ([4]) *Every bull-free perfect graph with no antihole either has an incomplete homogeneous set or is weakly triangulated or is perfectly orderable.*

This theorem yields polynomial algorithms for MAXIMUM WEIGHTED CLIQUE PROBLEM and MINIMUM WEIGHTED COLORING PROBLEM in case we have a bull-free perfect graph with no antiholes. If the graph has an incomplete homogeneous set H, we can recursively apply the algorithm on H and find a minimum weighted coloring for H. This coloring consists of a family of stable sets $T_1, \ldots, T_p$ with weights $W(T_1), \ldots, W(T_p)$. We then substitute H by a clique $Q(H)$ of cardinality p, resulting in a new graph G'. The i-th vertex in $Q(H)$ receives weight $W(T_i)$, while the vertices of $G' - Q$ keep the same weight as in the original graph. Classical polyhedral considerations (see [6]) imply that in a weighted perfect graph there exists a minimum weighted coloring that satisfies (1) with equality for every vertex; moreover if all the vertex weights are integers then the weights of the stable sets in a minimum coloring can be taken to be integers as well. (Actually these two facts will always be evident in the algorithms cited or presented here.) So we can obtain a minimum weighted coloring of the original graph by combining a minimum weighted coloring of the new graph with $T_1, \ldots, T_p$. When the graph is weakly triangulated, we can apply the minimum weighted coloring algorithm from [7], whose complexity is $\mathrm{O}(n^4 m)$. When the graph is perfectly orderable, we can apply the minimum weighted coloring algorithm from [8], whose complexity is $\mathrm{O}(nm)$. The overall complexity is $\mathrm{O}(n^5 m)$. An analogous argument solves MAXIMUM WEIGHTED CLIQUE PROBLEM in this case.

In Section 2, we further exploit Theorem 1 and prove:

Theorem 2 *Every bull-free perfect graph with no antihole either has an incomplete homogeneous set or is weakly triangulated or is a comparability* *graph.*

This stronger result allows us to solve the optimization problems MAXIMUM WEIGHTED CLIQUE PROBLEM and MINIMUM WEIGHTED COLORING PROBLEM in case we have a bull-free perfect graph with an even antihole but with no even hole. The complements of comparability graphs are well-studied perfect graphs [2, 5].

The remaining case is a bull-free perfect graph with an even hole and and even antihole. According to [3] a *homogeneous pair* in a graph is a pair of disjoint sets of vertices Q_1, Q_2 in this graph such that: all Q_1-partial vertices are in Q_2; and all Q_2-partial vertices are in Q_1; at least one of Q_1, Q_2 includes at least two vertices; there are at least two vertices outside $Q_1 \cup Q_2$. In Section 3, we show the existence of a special homogeneous pair with respect to which we can use a decomposition defined in [3] to optimize.

2 Boxes and Transitive Orientations

We present a structural property which essentially consists in a bipartite layout that will be useful for finding a transitive orientation in a bull-free graph. For any subset B of vertices in a graph G, we let $M(B)$ denote the set of vertices of $G - B$ that are partial on B.

Definition 1 The Box Partition *Let G be a graph with vertex set V. We call Box Partition any partition of V into disjoint non-empty subsets called the boxes, inducing connected subgraphs which satisfy the following properties.*

(i) Each box is labelled either "odd" or "even" (each vertex will be labelled odd or even accordingly), and there is no edge between two odd boxes or between two even boxes;

(ii) For each box B such that $M(B) \neq \emptyset$, there exists in $V - B$ two auxiliary adjacent vertices a_B and a'_B, such that a_B sees all of B and misses all of $M(B)$ while a'_B sees all of $M(B)$ and misses all of B.

Remark *When G is bull-free, the fact that a'_B sees every vertex of $M(B)$ is a consequence of the other facts given in Property (ii).* Indeed, if a'_B missed a vertex x of $M(B)$, then there should exist adjacent vertices u, v in B such that x sees u and misses v, and then a_B, u, v, x, a'_B would be a bull.

Let us note that if a bull-free perfect graph G with no homogeneous set and no $\overline{C}_6$ admits a box partition, then two further properties hold:

(iii) *Every box is P_4-free;*

(iv) *Any two adjacent vertices in B have comparable neighbourhoods in $M(B)$.*

To prove (iii), we recall that a *broom* is the graph made up of a P_4, plus a fifth vertex adjacent to all vertices of the P_4, plus a sixth vertex adjacent to the fifth vertex only. We proved the following result:

Lemma 1 (The Broom Lemma [4]) *If a bull-free, C_5-free graph contains a broom then it has a homogeneous set which contains the P_4 of the broom.*

Now observe that if a box B contains a P_4, then adding the vertices a_B and a'_B we obtain a broom, and then by the Broom Lemma G should contain a homogeneous set. To prove (iv), suppose on the contrary that some two adjacent vertices u, v in a box B have incomparable neighbourhoods in $M(B)$. So there exists a vertex x in $M(B) \cap N(u) - N(v)$ and a vertex y in $M(B) \cap N(v) - N(u)$. Recall the auxiliary vertices a_B, a'_B for B, so that a_B sees u, v and a'_B and misses x and y, while a'_B sees x and y and misses u and v. If xy is an edge in G, then a_B, u, v, x, y, a'_B is a $\overline{C}_6$. If xy is not an edge in G, then a_B, u, v, x, y is a bull. So (iv) is proved. $\Box$

Given a box partition, any edge whose endpoints are in different boxes will be called a *vertical edge* (necessarily, for any such edge one endpoint is in an odd box and the other is in an even box). The other edges will be called *horizontal*, i.e., a horizontal edge is any edge whose two endpoints are in the same box.

Lemma 2 (The Transitive Box Lemma) *Let G be a bull-free perfect graph which admits a box partition. If G has no homogeneous set and no $\overline{C}_6$, then G admits a transitive orientation.*

Proof. Recall from (iii) that each box B is P_4-free. Every P_4-free graph admits a transitive orientation [5]. Let $\mathcal{L}(B)$ be a transitive orientation, for each box B. All edges xy of G are oriented according to the following rules:

- *Rule V0.* If an edge is vertical, orient it from its even extremity to its odd extremity.
- *Rule E1.* If x, y are in an even box B and x has strictly more neighbours than y in $M(B)$ then orient xy from x to y.
- *Rule E2.* If x, y are in an even box B and have the same neighbourhood in $M(B)$, and if there exists a P_4 $yxvu$ with $u \in M(B)$ and $v \in B$, then orient the edge yx from y to x.
- *Rule E3.* If x, y are in an even box B and do not satisfy the hypotheses of Rules E1 and E2, then orient xy according to $\mathcal{L}(B)$.
- *Rule O1.* If x, y are in an odd box B and x has strictly more neighbours than y in $M(B)$ then orient xy from y to x.
- *Rule O2.* If x, y are in an odd box B and have the same neighbourhood in $M(B)$, and if there exists a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$, then orient the edge xy from x to y.
- *Rule O3.* If x, y are in an odd box B and do not satisfy the hypotheses of Rules O1 and O2, then orient xy according to $\mathcal{L}(B)$.

After these rules are applied, every edge of G has received an orientation. We claim that this is a transitive orientation of G. To certify this claim, we have to check that these combined rules are consistent, produce no P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$, and no circuit.

Claim 1 *The rules are consistent.*

Proof. It suffices to prove that no edge must be oriented by the rules in two opposite ways. Clearly, the vertical edges are oriented consistently. Without loss of generality, let us assume that a case of inconsistency arises on an edge inside an odd box. Property (iv) of the Box Partition implies that such an edge cannot be oriented in two different ways by Rule O1. Since Rules O1, O2 and O3 apply to different types of edges, they cannot contradict each other. So the only case of inconsistency is the following: an edge xy in an odd box B must be oriented from x to y because there is a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$ (Rule O2), and must also be oriented in the opposite way because there is a P_4 $ztyx$ with $z \in M(B)$ and $t \in B$ (Rule O2), and x and y have the same neighbours in $M(B)$. Clearly $v \neq t$ (but $u = z$ is possible). Let a'_B be the auxiliary vertex of B given by Property (ii) of the Box Partition; so a'_B sees u and z and misses all of v, x, y, t. In addition, v must see t or else $vxyt$ is a P_4 in B. Now, either u sees t or z sees v, or else Property (iv) is contradicted for v, t. By symmetry we may assume that u sees t, but then v, t, u, a'_B, x is a bull. In all cases a contradiction appears, so Claim 1 is proved. □

Claim 2 *The rules produce no P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$.*

Proof. Observe that the definition of the rules is such that no P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$ can be produced with vertices in different boxes. Without loss of generality, let us assume that a P_3 xyz with orientation $\vec{xy}$ and $\vec{yz}$ arises inside an odd box. Note that one of the edges xy, yz must have been oriented by Rule O1 or by Rule O2.

Case 1. The edge xy was oriented from x to y by Rule O1. This hypothesis means that there exists a vertex u in $M(B) \cap N(y) - N(x)$. If u misses z then yz should be oriented by Rule O1 from z to y, a contradiction; so u sees z. Now x, y, z, u, a'_B is a bull.

Since xy is not oriented by Rule O1, then necessarily x and y have the same neighbourhood in $M(B)$.

Case 2. The edge xy was oriented from x to y by Rule O2. This means that x and y have the same neighbourhood in $M(B)$ and that there exists a P_4 $uvxy$ with $u \in M(B)$ and $v \in B$. Because B has no P_4, we have v sees z. In addition, if u sees z, then u, v, z, y, a'_B is a bull. Moreover, y and z have the same neighbourhood in $M(B)$. For, if y has more neighbours than z in $M(B)$, then by Rule O1, we have yz oriented from z to y. If there exists $w \in M(B) \cap N(z) - N(y)$, then w misses x also and Rule O2 orients xy from y to x, a contradiction. Thus we can apply Rule O2 to the P_4 $uvzy$ which forces yz to be oriented from z to y, a contradiction.

Case 3. The edge yz was oriented from y to z by Rule O1. This hypothesis means that there exists a vertex u in $M(B) \cap N(z) - N(y)$. Recall that x and y have the same neighbourhood in $M(B)$. Thus we can apply Rule O2 to the P_4 $uzyx$ which forces xy to be oriented from y to x, a contradiction.

Case 4. The edge yz was oriented from y to z by Rule O2. This means that there exists a P_4 $uvyz$ with $u \in M(B)$, and that y and z have the same neighbourhood in $M(B)$. Recall that x and y also have the same neighbourhood in $M(B)$. Vertex x misses v or else u, v, x, y, z is a bull. Thus we can apply Rule O2 to the P_4 $uvyx$ which forces xy to be oriented from y to x, a contradiction.

In all cases a contradiction arises; so Claim 2 is proved. $\square$

Claim 3 *The rules produce no circuit.*

Proof. By Rule V0, it is clear that a circuit may occur only inside a box. Without loss of generality, let us assume that an odd box B contains a circuit $C = c_1 \cdots c_r$. Observe that if an edge xy in B is oriented from x to y then y has at least as many neighbours as x in $M(B)$, because of Rule O1. Therefore, if somewhere along the circuit two consecutive vertices c_i, c_{i+1} satisfy $N(c_i) \cap M(B) \subset N(c_{i+1}) \cap M(B)$ (where $\subset$ denotes strict inclusion), then necessarily elsewhere on the cycle some two consecutive vertices c_j, c_{j+1} must satisfy $N(c_{j+1}) \cap M(B) \subset N(c_j) \cap M(B)$. But then this inclusion contradicts the fact that the edge $c_j c_{j+1}$ is oriented from c_j to c_{j+1}. So, all vertices along C have the same neighbourhood in $M(B)$. Moreover, since $\mathcal{L}(B)$ has no circuit, at least one edge of C must have been oriented by Rule O2. So let us assume that there is a P_4 uvc_1c_2 with $u \in M(B)$ and $v \in B$. Since all the vertices of C have the same neighbourhood in $M(B)$, in particular they all miss u, and v is not one of the c_i's. Let j be the last subscript such that v misses c_j ($j \geq 2$). Then $uvc_{j+1}c_j$ is a P_4 implying that the edge c_jc_{j+1} is oriented from c_{j+1} to c_j, a contradiction. So Claim 3 is proved. $\square$

Now the proof of Lemma 2 is complete. □

3 Graphs with a hole and an antihole

We now consider the case of a bull-free Berge graph that contains a hole and an antihole. We will proceed to a recursive decomposition of this graph, giving a decomposition tree whose leaves contain no antihole; it will then be possible to obtain a solution for the MAXIMUM WEIGHTED CLIQUE PROBLEM and the MINIMUM WEIGHTED COLORING PROBLEM for the initial graph from solutions of the same problems for the leaves.

Everywhere in this section we reserve the letter G for a bull-free Berge graph that contains an even hole and no homogeneous set. We let k denote the length of a shortest even hole in G. So there are k non-empty sets $V_1, \ldots, V_k$ such that every vertex in V_i sees every vertex in V_{i+1} (modulo k) and there is no other edge between two V_i's. By [4] G admits a box partition built from the V_i's. We will need to use some properties and notation from [4] concerning this box partition. In particular, the boxes of this partition are classified as either "central" or "peripheral" with the following properties that will be used here:

(a) If $k \geq 8$, every central box is a singleton. (To see this, recall from [4] that when $k \geq 8$ the central boxes are the connected components of the k sets $V_1, \ldots, V_k$. If $B \subseteq V_1$, then the proof of [4, Lemma 3] gives $M(B) \subseteq A_2$, where A_2 is the set of vertices that are adjacent to all of $V_1 \cup V_3 \cup \cdots \cup V_{k-1}$; hence every vertex adjacent to B is adjacent to all of B, and B is a homogeneous set. So $|B| = 1$.)

(b) If $k = 6$, there are eight sets $D_1, \ldots, D_6, A_1, A_2$ such that the central boxes are exactly the connected components of these eights subgraphs. The sets $D_1, \ldots, D_6$ play symmetrical roles; the sets A_1, A_2 play symmetrical roles. Moreover, if B is a box in D_1 or in A_1 then $M(B) \subseteq D_4 \cup A_2$. There are vertices v_2, v_6 that see all of $D_1 \cup A_1$ and none of $D_4 \cup A_2$; there are vertices v_3, v_5 that see all of $D_4 \cup A_2$ and none of $D_1 \cup A_1$; v_2v_3 and v_5v_6 are the only edges between v_2, v_3, v_5, v_6.

(c) [4, Lemma 4, Property (v)]: In a peripheral box B any two adjacent vertices have comparable neighbourhoods in $M(B)$.

Lemma 3 *The graph G contains an antihole if and only if it contains a $\overline{C}_6$.*

Proof. The 'if' part is trivial. Conversely, suppose that G contains no $\overline{C}_6$. Then Lemma 2 implies that G is transitively orientable, hence it contains no antihole. □

When a C_4 (a "square") is denoted $uvxy$ it is understood that ux and vy are the two non-adjacent pairs.

Definition 2 Blocking Square *We say that a square $uvxy$ is blocking if u, v belong to one box B and x, y belong to another box B'. The edges uv and xy are called the blocking edges of the square. Likewise any edge uv with both endpoints in one box is called blocking whenever it is one of the two blocking edges of a blocking square.*

Remark In Definition 2, clearly one of B, B' is a left box and the other is a right box. Clearly too, we have $\{x, y\} \subseteq M(B)$ and $\{u, v\} \subseteq M(B')$.

Lemma 4 *The graph G contains a $\overline{C}_6$ if and only if G contains a blocking square.*

Proof. First suppose that G contains a blocking square $uvxy$ with the notation as in Definition 2. Let a_B and a'_B be the auxiliary vertices for B. Then a_B, a'_B, u, v, x, y induce a $\overline{C}_6$.

Conversely, suppose that six vertices $u_1, u_2, \ldots, u_6$ form in G a $\overline{C}_6$, such that the non-adjacent pairs are u_iu_{i+1} (subscripts here are understood modulo 6). In the triangle u_1, u_3, u_5 at least two vertices are on the same side of the box partition, say u_1 and u_3 are in one left box B. If both u_4, u_6 are in a right box then $u_1u_3u_6u_4$ is a blocking square. So let us assume without loss of generality that u_4 is in a left box, hence in B. Then u_2 is in a right box or else $u_3u_1u_4u_2$ would be a P_4 in B. If u_5 is on the right then $u_1u_4u_2u_5$ is a blocking square. So let us assume that u_5 is on the left, hence in B. Then u_6 is on the right or else $u_5u_3u_6u_4$ would be a P_4 in B. But now $u_3u_5u_2u_6$ is a blocking square. □

Lemma 5 *An edge uv in a box B is a blocking edge if and only if the vertices u, v have incomparable neighbourhoods in $M(B)$.*

Proof. The 'only if' part of the lemma is trivial. Conversely, suppose that u, v have incomparable neighbourhoods in $M(B)$, i.e., there exists a vertex $x \in M(B) \cap N(v) - N(u)$ and a vertex $y \in M(B) \cap N(u) - N(v)$. Recall that the auxiliary vertex a_B sees both u, v and misses both x, y. Then x must see y or else a_B, u, v, x, y would be a bull. Now $uvxy$ is a blocking square and uv is a blocking edge. □

From now on we assume that G admits a blocking square $uvxy$, with blocking edges uv in a box B and xy in a box B'.

According to [3] a *homogeneous pair* in a graph is a pair of disjoint sets of vertices Q_1, Q_2 in this graph such that: all Q_1-partial vertices are in Q_2; and all Q_2-partial vertices are in Q_1; at least one of Q_1, Q_2 includes at least two vertices; there are at least two vertices outside $Q_1 \cup Q_2$. Recall the graph called H_0 in [3]. An H_0 is a graph with eight vertices a, b, c, d, e, f, g, h and fourteen edges $ac, ae, ab, af, be, bd, bf, cg, cd, ch, dg, dh, eg, fh$.

Theorem 3 ([3]) *Let G be a bull-free graph. If G contains an H_0, then G contains a homogeneous pair.*

By Lemma 5 the vertices u, v have incomparable neighbourhoods in $M(B)$. By [4, Lemma 4, Property (v)] as recalled above, B cannot be a peripheral box. So B is a central box. If $k \geq 8$ then item (a) above implies $|B| = 1$, which is impossible. So $k = 6$, and we may assume, without loss of generality, that $B \subseteq D_1 \cup A_1$, with the notation of item (b). Now we have

$$u, v \in B \subseteq D_1 \cup A_1,$$

$$x, y \in B' \subseteq D_4 \cup A_2.$$

Using the vertices v_2, v_3, v_5, v_6 whose properties are recalled in (b) above, we note that vertex v_2 sees all of $\{u, v, v_3\}$, vertex v_6 sees all of $\{u, v, v_5\}$, vertex v_3 sees all of $\{x, y, v_2\}$, vertex v_5 sees all of $\{x, y, v_6\}$, and there are no other edges between the vertices $u, v, x, y, v_2, v_3, v_5, v_6$. Hence these eight vertices induce an H_0.

Now the proof of Theorem 3 in [3] implies that there exists a homogeneous pair Q_1, Q_2 such that:

- $u, v \in Q_1$ and $x, y \in Q_2$;
- Q_1 and Q_2 induce connected subgraphs of G;
- $Q_1 \subseteq N(v_2) \cap N(v_6) - (N(v_3) \cap N(v_5))$;
- $Q_2 \subseteq N(v_3) \cap N(v_5) - (N(v_2) \cap N(v_6))$.

These conditions imply $Q_1 \subseteq B$ and $Q_2 \subseteq B'$. Moreover, the proof of Theorem 3 in [3] is actually a polynomial-time algorithm that builds this homogeneous pair Q_1, Q_2 around a blocking square $uvxy$.

As we remarked above, if the graph contains a blocking square, then it contains this auxiliary graph H_0. We show in the sequel how we can adapt the decomposition defined by Chvátal and Sbihi in order to break the graph into two auxiliary bull-free perfect graphs H and F. Finally, we show how to use this decomposition to compute in polynomial time both a maximum weighted clique and a minimum weighted coloring in this case.

Write $H^* = G \setminus (Q_1 \cup Q_2)$. Construct a graph H by adding to H^* vertices u_1, u_2, s_1, s_2, edges u_1s_1, u_2s_2, u_1s_2, u_2s_1, s_1s_2 and edges tu_i, ts_i for each i and each vertex $t \in H^*$ that sees every vertex of Q_i.

Claim 4 *H is perfect.*

Proof. Consider any induced subgraph H' of H. If H' contains at most one of u_1, s_1 and at most one of u_2, s_2 then it is easy to see that H' is isomorphic to an induced subgraph of G (because the homogeneous pair Q_1, Q_2 contains a square of G), so H' is perfect. Suppose now by symmetry that H' contains both u_1 and s_1. Note that s_1 dominates u_1 in H (i.e., $N_H(u_1) \subset N_H(s_1) \cap \{s_1\}$), and thus also in H'. It is well-known (see e.g. [5]) that a minimally imperfect graph cannot contain a pair of vertices such that one dominates the other. So all induced subgraphs of H are perfect, including H itself. □

Claim 5 *H is bull-free.*

Proof. Suppose that H contains a bull B. It is easy to see that an induced subgraph of H that contains none of the two triangles formed by u_1, s_1, s_2 and u_2, s_1, s_2 is isomorphic to an induced subgraph of G (again because the homogeneous pair Q_1, Q_2 contains a square of G). So we may assume by symmetry that B contains u_1, s_1, s_2. Now B must have a pendant vertex adjacent to exactly one of u_1, s_1 and not adjacent to s_2. But H contains no such vertex since u_2 is the only vertex adjacent to exactly one of u_1, s_1. □

Construct a graph F by adding to the subgraph of G induced by $Q_1 \cup Q_2$ adjacent vertices x_1, x_2 and edges $x_i y$ for all vertices y in Q_i ($i = 1, 2$). Note that F is isomorphic to an induced subgraph of G, so F is perfect and bull-free.

We now show how to build solutions of the optimization problems MAXIMUM WEIGHTED CLIQUE PROBLEM and MINIMUM WEIGHTED COLORING PROBLEM for G from solutions of the same problems for the auxiliary graphs H and F. We denote by $w(x)$ the weight of a vertex x in G.

We define weights for vertices in H as follows. Denote by $\omega(X)$ the maximum weight of a clique in X, and set:

$$\begin{aligned}
w_H(u_1) = w_H(u_2) &= \omega(Q_1) + \omega(Q_2) - \omega(Q_1 \cup Q_2), \\
w_H(s_1) &= \omega(Q_1 \cup Q_2) - \omega(Q_2), \\
w_H(s_2) &= \omega(Q_1 \cup Q_2) - \omega(Q_1), \\
w_H(x) &= w(x) \qquad (\forall x \in H^*).
\end{aligned}$$

Let q be the maximum weight of a clique in H with respect to the weighting w_H, and let C_H be a clique of this weight. We may transform C_H into a clique C_G of weight q in G as follows. Say that a set X of vertices in H is of type 0, if $X \cap \{u_1, s_1, u_2, s_2\} = \emptyset$; of type 1, if $X \cap \{u_1, s_1\} \neq \emptyset$ and $X \cap \{u_2, s_2\} = \emptyset$; of type 2, if $X \cap \{u_1, s_1\} = \emptyset$ and $X \cap \{u_2, s_2\} \neq \emptyset$; of type 3, if $X \cap \{u_1, s_1\} \neq \emptyset$ and $X \cap \{u_2, s_2\} \neq \emptyset$.

If C_H is of type 0, then set $C_G = C_H$. If C_H is of type i, with $i = 1, 2$, then let C_G be the union of $H^* \cap C_H$ and a clique of size $\omega(Q_i)$ in Q_i. If

C_H is of type 3, then let C_G be the union of $H^* \cap C_H$ and a clique of size $\omega(Q_1 \cup Q_2)$ in $Q_1 \cup Q_2$.

Now it remains to exhibit a q-weighted coloring of G: that will show both that the clique C_G defined above for G is maximum and that the coloring we are about to define for G is minimum. In order to find such a coloring for G, we start with a q-weighted coloring for H, i.e., a collection of stable sets $S_1^H, \ldots, S_t^H$ with corresponding weights $X(S_1^H), \ldots, X(S_t^H)$ such that

$$\sum\{X(S_i^H) \mid S_i^H \ni x\} = w_H(x) \qquad (\forall x \in H)$$

and $X(S_1^H) + \ldots + X(S_t^H) = q$.

Split the subscripts $1, 2, \ldots, t$ into sets I_0, I_1, I_2, I_3 by writing $j \in I_i$ if and only if S_j^H is of type i. By this definition, we have

$$\begin{aligned} w_H(u_i) &\leq \sum\{X(S_j^H) \mid j \in I_i \cup I_3\}, \\ w_H(s_i) &\leq \sum\{X(S_j^H) \mid j \in I_i\}. \end{aligned}$$

In addition, since u_i and s_i are adjacent,

$$w_H(u_i) + w_H(s_i) \leq \sum\{X(S_j^H) \mid j \in I_i \cup I_3\}.$$

We define weights for vertices in F as follows; set:

$$\begin{aligned} w_F(x_1) &= \sum\{X(S_j^H) \mid j \in I_2\}, \\ w_F(x_2) &= \sum\{X(S_j^H) \mid j \in I_1\}, \\ w_F(y) &= w(y) \qquad (\forall y \in Q_1 \cup Q_2). \end{aligned}$$

We have

$$\begin{aligned} w_F(x_1) + \omega(Q_1) &= w_F(x_1) + w_H(u_1) + w_H(s_1) \\ &\leq \sum\{X(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}, \end{aligned}$$

and similarly

$$\begin{aligned} w_F(x_2) + \omega(Q_2) &= w_F(x_2) + w_H(u_2) + w_H(s_2) \\ &\leq \sum\{X(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}. \end{aligned}$$

In addition,

$$\begin{aligned}\omega(Q_1 \cup Q_2) &= w_H(u_1) + w_H(s_1) + w_H(s_2)\\ &\le \sum\{X(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}.\end{aligned}$$

Hence each clique C_F has weight at most $q_F = \sum\{X(S_j^H) \mid j \in I_1 \cup I_2 \cup I_3\}$. Since F is perfect, there exists a family of stable sets $S_1^F, \ldots, S_r^F$ of F and weights $X(S_1^F), \ldots, X(S_r^F)$ such that

$$X(S_1^F) + \cdots + X(S_r^F) \le q_F$$

and

$$\sum\{X(S_j^F) \mid x \in S_j^F\} = w_F(x) \qquad (\forall x \in F).$$

Since x_1 and x_2 are adjacent, no S_j^F covers both of them. The definition of $w_F(x_1)$ implies

$$\sum\{X(S_j^F) \mid x_1 \in X_j^F\} = \sum\{X(S_j^H) \mid j \in I_2\},$$

and similarly for x_2.

Now we merge the family of stable sets $\{S_i^H \mid i \in I_1 \cup I_2 \cup I_3\}$ with the family of stable sets S_j^F. We do this in three steps. First, we merge the family $\{S_i^H \mid i \in I_1\}$ with the family of stable sets S_j^F covering x_2. Note that the total weight is the same for both families by the definition of $w_F(x_2)$ (though the individual weights may be different). Note also that $(S_j^F \cap (Q_1 \cup Q_2)) \cup (S_i^H \cap H^*)$ is a stable set, for any choice of j, i. In order to be able to merge these two families of stable sets S_i^H and S_j^F such that the individual weights of the merged sets match one-to-one we apply the following procedure: take the heaviest set S of the two families, make two copies S', S'' of S, assign to S' the weight of the heaviest set T of the other family and to S'' the remaining weight from S, and then merge S' and T into $S' \cup T - \{u_1, s_1, x_2\}$; repeat with the remaining families as long as necessary. Secondly and analogously, we merge the family $\{S_i^H \mid i \in I_2\}$ with the family of stable sets S_j^F covering x_1. Thirdly, we merge the family $\{S_j^H \mid j \in I_3\}$ with the remaining family of stable sets S_j^F (i.e., stable sets S_j^F not covering x_1 or x_2). The resulting merged family of stable sets, along with the family $\{S_j^H \mid j \in I_0\}$, cover the vertices of G and give the desired q-weighted coloring for G.

Chvátal and Sbihi describe in [3] a polynomial-time algorithm which, starting with an H_0, finds a homogeneous pair Q_1, Q_2. This algorithm together with the complexity analysis given next justify that our method gives a polynomial decomposition algorithm for solving the optimization problems in this remaining case. We only need to know how many times the decomposition along an H_0 will be performed. Denote by $m(G)$ the number of edges of G.

Claim 6 *There are at most $O(m(G))$ nodes in the decomposition tree of G with respect to an induced H_0.*

Proof. Assume that G is decomposed into two graphs H and F along the homogeneous pair Q_1, Q_2 given by an H_0. Let m' be the number of edges of G with both endpoints in $Q_1 \cup Q_2$ and m'' be the number of edges of G with both endpoints out of $Q_1 \cup Q_2$. Let a_i $(i = 1, 2)$ be the number of vertices outside $Q_1 \cup Q_2$ that see all of Q_i and none of Q_{3-i}, and a be the number of vertices outside $Q_1 \cup Q_2$ that see all of $Q_1 \cup Q_2$. Write $q_i = |Q_i|$. Now,

$$\begin{aligned} m(G) &= m' + m'' + q_1(a_1 + a) + q_2(a_2 + a), \\ m(H) &= 5 + m'' + 2(a_1 + a) + 2(a_2 + a), \\ m(F) &= m' + 1 + q_1 + q_2. \end{aligned}$$

Hence $m(H)+m(F)-m(G) = 10+(2-q_1)(a_1+a-1)+(2-q_2)(a_2+a-1)$. Since $q_1 \geq 2$, $q_2 \geq 2$, $a_1 \geq 1$, and $a_2 \geq 1$ hold, we obtain $m(H) + m(F) \leq m(G) + 10$. In order to admit such a decomposition, G must contain an H_0, thus $m(G) \geq 14$. Moreover, in case $m(G) = 14$ and G is decomposed into graphs H and F, we have $m(H) = 15$ and $m(F) = 9$. Note that H has strictly less blocking squares than G. Let $\beta(G)$ be the number of nodes in the decomposition tree of G with respect to H_0. We show by induction on $m(G)$ and on the number of blocking squares that, for $m(G) \geq 14$, $\beta(G) \leq 2m(G) - 21$. This holds for $m(G) = 14$. When $m(G) \geq 15$ and G is decomposed into graphs H and F, then $\beta(G) \leq 1 + \beta(H) + \beta(F) \leq 1+(2m(H)-21)+(2m(F)-21) = 1+2(m(H)+m(F))-42 \leq 1+2(m(G)+10) - 42 = 2m(G) - 21$. $\square$

This final claim certifies that the whole algorithm is called recursively at most $O(m)$ times. Thus the algorithm is strongly polynomial, with worst-case complexity $O(n^5m^2)$.

References

[1] C. Berge. Les problèmes de coloration en théorie des graphes. *Publ. Inst. Stat. Univ. Paris*, 9:123–160, 1960.

[2] K. Cameron. Antichain sequences. *Order*, 2:249–255, 1985.

[3] V. Chvátal and N. Sbihi. Bull-free Berge graphs are perfect. *Graphs and Combinatorics*, 3:127–139, 1987.

[4] C. M. H. de Figueiredo, F. Maffray, and O. Porto. On the structure of bull-free perfect graphs. *Graphs and Combinatorics*, 13:31–55, 1997.

[5] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York, (1980).

[6] M. Grötschel, L. Lovász, and A. Schrijver. Polynomial algorithms for perfect graphs. In C. Berge and V. Chvátal, editors, *Topics on Perfect Graphs (Ann. Discrete Math. 21)*, pages 325–356. North-Holland, Amsterdam, 1984.

[7] R. Hayward, C. T. Hoàng, and F. Maffray. Optimizing weakly triangulated graphs. *Graphs and Combinatorics*, 5:339–349, 1989. See erratum in vol. 6, 1990, p. 33–35.

[8] C.T. Hoàng. Efficient algorithms for minimum weighted colouring of some classes of perfect graphs. *Disc. Appl. Math.*, 55:133–143, 1994.

[9] B. A. Reed and N. Sbihi. Recognizing bull-free perfect graphs. *Graphs and Combinatorics*, 11:171–178, 1995.