Local conditions for edge-colouring of cographs

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Abstract

In this note, we investigate three versions of the overfull property for colouring graphs and their relation to cographs. Each of these properties implies that the graph cannot be edge-colored with Δ colors, where Δ is the maximum degree. Consequently, the graph is a Class 2 graph. These three versions are not equivalent for general graphs. However, some equivalences hold for the classes of indifference graphs, split graphs, and complete multipartite graphs. Hilton and Chetwynd conjectured that being Class 2 is equivalent to being subgraph-overfull, when the graph has maximum degree greater than $\frac{1}{3}|V(G)|$.

A *cograph* is a graph with no induced subgraph isomorphic to a P_4. We show that connected cographs have maximum degree greater than $\frac{1}{2}|V(G)|$. In view of Hilton and Chetwynd's conjecture, it is interesting to study when a cograph is subgraph-overfull. We prove that for a special class of cographs, obtained by restricting the number of levels in the modular decomposition tree, being subgraph-overfull is equivalent to being either overfull or neighborhood-overfull.

1 Introduction

In this paper, G denotes a simple, undirected, finite, connected graph. The sets $V(G)$ and $E(G)$ are the vertex and edge sets of G. A **clique** is a set of vertices pairwise adjacent in G. A **subgraph** of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote $G[X]$ the **subgraph induced by** X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices that are adjacent to v. In addition, $N(v)$ denotes the **neighbourhood** of v, that

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is, $N(v) = Adj(v) \cup \{v\}$. A subgraph that is induced by the neighbourhood of a vertex is simply called a **neighbourhood**. The **degree** of a vertex v is $deg(v) = |Adj(v)|$. The maximum degree of a graph G is then $\Delta(G) = \max_{v \in V(G)} deg(v)$. A vertex v is **Δ-vertex** if $\deg(v) = \Delta$. For us, K_n denotes the complete graph on $n \geq 1$ vertices and we denote the maximum degree of a graph G by Δ when there is no possible ambiguity.

A **complete multipartite** graph is a graph whose vertex set admits a partition into parts. If two vertices are in the same part, they are not adjacent, whereas if they are in differents parts, they are adjacent.

We denote by P_4 the four-vertex graph on vertex set $\{a,\ b,\ c,\ d\}$ with edge set $\{(a,b),\ (b,c),\ (c,d)\}$. A **cograph** is a graph without induced subgraph isomorphic to P_4.

An **edge-colouring** of a graph is an assignment of colours to its edges such that no adjacent edges have the same colour. The **chromatic index** of a graph is the minimum number of colours required to produce an edge-colouring for that graph. In this paper, we address the computation of the chromatic index for cographs.

An easy lower bound for the chromatic index is the maximum vertex degree. A celebrated theorem by Vizing states that these two quantities differ by at most one [14]. Graphs which chromatic index equals the maximum degree are said to be **Class** 1; graphs whose chromatic index exceeds the maximum degree by one are said to be **Class** 2.

Despite the powerful restriction imposed by Vizing's result, it is very hard to compute the chromatic index in general. Indeed, deciding between these two possibilities is NP-hard [9]. This holds for various classes, such as comparability graphs (a superclass of cographs) [1]. More precisely, we have that Class 1 recognition is in NP while Class 2 recognition is in co-NP. This means that Class 1 graphs always have short certificates. Indeed, to convince someone that a graph is in Class 1, all we have to do is to exhibit a Δ-colouring. In contrast, to show that a graph is in Class 2 we must produce an argument that no Δ-colouring exists.

Nevertheless, for some graphs this argument can be very simple. By definition of edge-colouring, each colour determines a matching and can cover at most $\lfloor n/2 \rfloor$ edges, where n is the number of vertices. Therefore, if the total number of edges is greater than the product of the maximum degree by $\lfloor n/2 \rfloor$, then the graph is necessarily Class 2. Graphs to which this argument can be applied are called **overfull** [12, 13, 7, 6]. More generally, if a graph has an overfull subgraph with the same maximum degree, then the same counting argument shows that the supergraph is Class 2. These graphs are called **subgraph-overfull**.

In addition, we can observe that this argument will not work for a graph

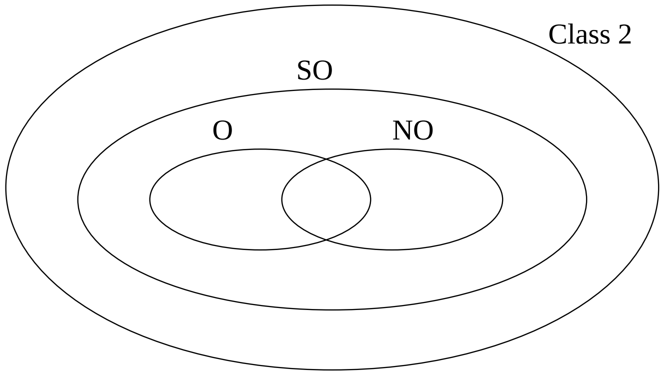

Figure 1: Class containment diagram for general graphs.

with even n. In this case, the following relation is always true:

$$\Delta \frac{n}{2} \geq m,$$

where m is the number of edges.

If the overfull subgraph H happens to be a **neighbourhood**, that is, is induced by a Δ-vertex and all its neighbours, then we say that G is **neighbourhood-overfull** [4]. Again, neighbourhood-overfull graphs are in Class 2.

In the rest of this paper, we shall denote by O, SO, and NO the classes of overfull, subgraph-overfull, and neighbourhood-overfull graphs, respectively. Figure 1 shows how these classes are related.

Hilton and Chetwynd conjectured that being Class 2 is equivalent to being SO, when the graph has a maximum degree greater than $\frac{1}{3}n$ [6]. Every Class 2 graph with maximum degree of at least $n-3$ is SO [2]; every Class 2 complete multipartite graph is O [8]. These classes provide evidence for Hilton and Chetwynd's conjecture.

We show, in section 2, that connected cographs have a maximum degree greater than $\frac{1}{2}n$. Hence, it is interesting to know when a cograph is SO.

The equivalence between SO and NO holds for split and indifference graphs [3]. For any graph class, the equality O = Class 2 implies O = SO; hence, O is equivalent to SO for complete multipartite graph. For cographs, the equality O = SO does not hold, although cograph is a superclass of complete multipartite graphs. In section 4, we prove that for a special class of cographs SO = (O $\cup$ NO) $\setminus$ (O $\cap$ NO). Before that, we will propose a notation for this special subclass of cographs in section 3.

2 Cographs and the Hilton and Chetwynd's conjecture

A cograph has a unique tree representation. This tree is called a **cotree**. In the cotree representation, the leaves represent vertices of the graph. Internal nodes of the cotree are labeled 0 or 1 in such a way that 0-nodes and 1-nodes alternate along every path starting from the root, which is always a 1-node. The root will have only one 0-node child if and only if the represented cograph is disconnected. Two vertices x and y of the graph are adjacent if and only if the unique path from x to the root of tree meets the unique path from y to the root at a 1-node.

Let G, $T(G)$ and $1, 2, \ldots, r$ be, respectively, a connected cograph, its cotree and the $T(G)$ root children. The leaves of the subtrees of $T(G)$ with root at the nodes $1, 2, \ldots, r$ give a partition of $V(G)$. We call such partition an **α-partition of** G. We denote $\alpha_G(i)$ the vertex set of G that belongs to the class of the α-partition defined by child i of $T(G)$ and $a_G(i)$ its cardinality. The subscript G in the notation will be omitted when there is no possible ambiguity.

In the following, we show that cographs satisfy the restriction of Hilton and Chetywind's conjecture.

Theorem 1 *If G is a cograph, then $\Delta(G) \geq n/2$.*

Proof: Let G be a graph and $\{\alpha(1), \alpha(2), \cdots, \alpha(r)\}$ the α-partition of $V(G)$. Without loss of generality, we consider $1 \leq a(1) \leq a(2) \leq \cdots \leq a(r)$.

We denote G' the graph constructed from G in the following way. Take $V(G') = V(G)$ and $E(G') = E(G) \backslash \{(u, w) \mid u, w \in \alpha(i),\ 1 \leq i \leq r\}$. Hence $\Delta(G) \geq \Delta(G')$.

Notice that the vertices in $\alpha(1)$ are Δ-vertices of G' and $\Delta(G') = n - a(1)$. Since $a(1) \leq a(i),\ 2 \leq i \leq r$, we have $a(1) \leq a(2) + \cdots + a(r)$. This implies $\Delta(G') = n - a(1) \geq n/2$. Hence $\Delta(G) \geq n/2$. ■

This result, although simple, shows that cographs may be used to provide another evidence to Hilton and Chetwynd's conjecture.

Next we present known lemmas which will be useful throughout this note [5].

We denote by $\deg^*_G(v)$ the number of $\Delta(G)$-vertices in $N_G(v)$. For a nonempty proper subset S of $V(G)$, we denote $[S, \overline{S}]$ the set of edges with one end in S and the other in $\overline{S}$.

Lemma 1 *Let G be an overfull graph. Then for every vertex v of G,* $\deg^*_G(v) \geq 2$.

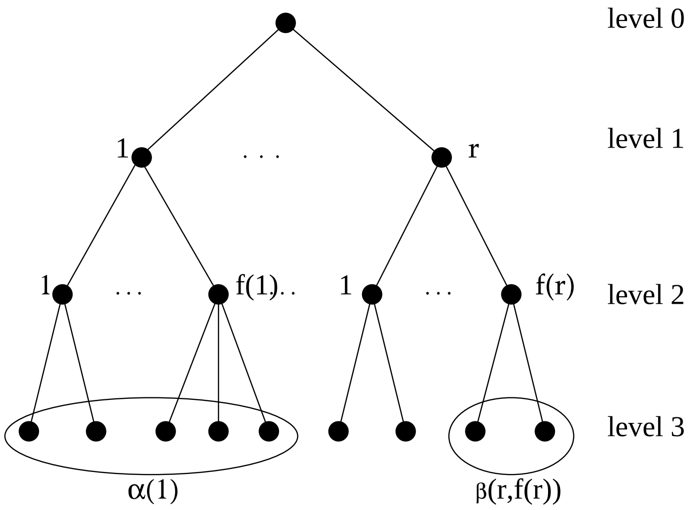

Figure 2: A cotree.

Lemma 2 *Let $S \subseteq V(G)$ so that $\Delta(G[S]) = \Delta(G)$. If $G[S]$ is an overfull graph, then $|[S, \overline{S}]| \leq \Delta(G) - 2$.*

Lemma 3 *[10] Let G be a graph with $2\Delta(G) \geq |V(G)|$. Then G has at most one subgraph which is overfull in G.*

3 Cographs with complete level 3 cotree

A **level k cotree** is a cotree with height k. The root is on level zero. A **complete level k cotree** is a cotree of level k such that each leaf is on level k. A complete graph is a cograph with a level 1 cotree. A nontrivial complete multipartite graph (i.e., with $a(i) \geq 2$ for some i) is a cograph with a level 2 cotree. Hoffman and Roger [8] proved that complete multipartite graphs are Class 2 if and only if they are overfull. We are interested in complete level 3 cotree cographs.

Let G be a cograph with a complete level 3 cotree. We will use the notation introduced in section 2 plus the following terms. The $f_G(i)$ children of i are denoted by $\beta(i,j)$, for each j, $1 \leq j \leq f_G(i)$. We denote $b_G(i,j) = |\beta_G(i,j)|$. The set $\{\beta_G(i,j),\ 1 \leq i \leq r,\ 1 \leq j \leq f_G(i)\}$ is a **β-partition of** G. Note that $r \geq 2$ (G is connected) and $b(i,j) \geq 2$, for $1 \leq j \leq f(i)$ (G has a complete level 3 cotree). Figure 2 shows these notation.

We may see $E(G)$ as the union of the set of edges of a complete multipartite graph $K_{\alpha(1),\cdots,\alpha(r)}$ and the set of the edges of complete graphs

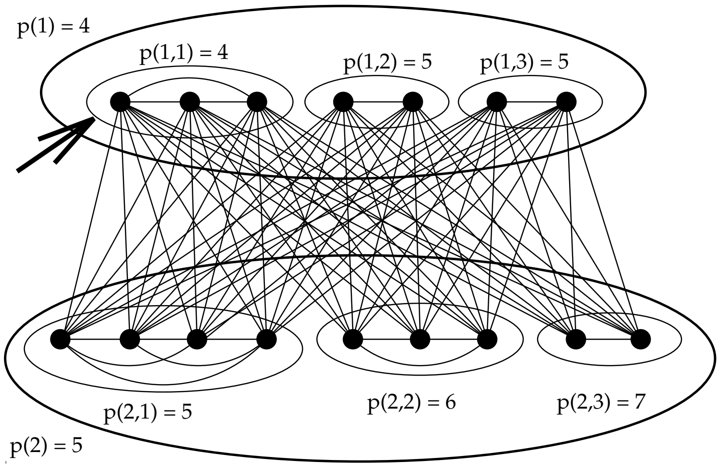

Figure 3: Cograph with ordered α-partition and β-partition. The arrow indicates a class that contains Δ-vertices.

$K_{\beta(i,j)}$ for each i and each j. The degree of a vertex (i,j) of $\beta(i,j)$ is $b(i,j)-1+n-a(i)$, where $b(i,j)-1$ is the degree of (i,j) when we restricted to $K_{\beta(i,j)}$, and $n-a(i)$ is the degree of (i,j) when we restricted to $K_{\alpha(1),\cdots,\alpha(r)}$. Notice that vertices of the same class of the β-partition have the same degree.

If G is a cograph with a complete level 3 cotree, we will write $\Delta(G)$, the cardinality of $V(G)$ and $E(G)$, and the condition for a cograph G to be an overfull graph, using the proposed notation.

For $|V(G)|$ and $\Delta(G)$ we have

$$n \;=\; |V(G)| = \sum_{i=1}^{r} a(i) = \sum_{i=1}^{r}\sum_{j=1}^{f(i)} b(i,j),$$

$$\Delta(G) = max_{i,j}\, deg(i,j) = max_{i,j}(b(i,j)-1+n-a(i)).$$

Let $p(i,j) = a(i) - b(i,j)$, for $1 \le i \le r$, $1 \le j \le f(i)$ and $p(i) = min_{1\le j\le f(i)}\{p(i,j)\}$. We may assume $p(1) \le p(2) \le \cdots \le p(r)$. Then, $p(1) = min_{i,j}\, p(i,j)$.

Therefore,

$$\Delta(G) = n - 1 - p(1),$$

$$m = \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{f(i)} b(i,j)\ \deg(i,j) = \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{f(i)} b(i,j)\ (b(i,j)-1+n-a(1)) =$$

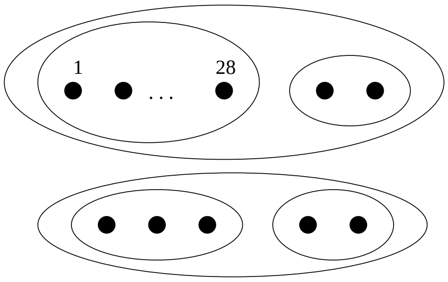

Figure 4: A cograph with complete cotree of level 3 that is SO.

$$= \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{f(i)} b(i,j)(n-1-p(i,j)).$$

and G is overfull if and only if $|V(G)|$ is odd and

$$\sum_{i=1}^{r}\sum_{j=1}^{f(i)} b(i,j)p(i,j) < (n-1)(1+p(1)).$$

Figure 3 shows a cograph with complete cotree of level 3 and its α-partition and β-partition.

4 Cographs SO with complete cotree of level 3

We present in this section a counting argument that shows that the particular structure of cographs with a complete level 3 cotree forces subgraph-overfullness to be equivalent to either neighbourhood-overfullness or overfullness.

We know that for a complete multipartite graph subgraph-overfullness is equivalent to overfullness [3, 8]. This result cannot be extended to cographs, although complete multipartite graphs are subclass of cographs. The cograph in Figure 4 is SO but it is not O.

Lemma 4 *Let G be a cograph with a complete level 3 cotree and H an induced subgraph of G with $\Delta(H) = \Delta(G)$. Then H is a cograph with a level 3 cotree.*

Proof: Let H be an induced subgraph of G with $\Delta(H) = \Delta(G)$. Since H is an induced subgraph of G, H is a cograph. (Obviously, H is a level 3 cotree cograph, if $V(H) = V(G)$.)

Consider $V(H) \neq V(G)$. Since $\Delta(H) = \Delta(G)$, the nonempty set $V(G) \setminus V(H)$ is contained in a unique $\alpha(i)$, $1 \leq i \leq r$, such that $\alpha(i)$ contains Δ-vertices of G. Without loss of generality, let $i = 1$.

By hypothesis, G is a cograph with a complete level 3 cotree, then $r \geq 2$ and, therefore, H contains vertices on the level 3. ■

Lemma 5 *Let G be a cograph with a complete level 3 cotree and H an induced subgraph of G with $V(H) \neq V(G)$ and $\Delta(H) = \Delta(G)$. Then there is an index i such that all Δ-vertices of H belong to $\alpha_H(i)$, and every $v \in V(H) \setminus \alpha_H(i)$ satisfies*

$$deg_H(v) \leq \Delta(G) - (|V(G)| - |V(H)|).$$

Proof: Let u be a Δ-vertex of H; u is also a Δ-vertex of G. The graph H is constructed by removing $x = |V(G)| - |V(H)|$ vertices from G. Let i, j be such that $u \in \beta_G(i,j)$. We have $|V(G)| - 1 - p_G(i,j) = |V(G)| - x - 1 - p_H(i,j)$.

Hence, the vertices that have degree $\Delta(G)$ in H are those such that $p_H(i,j) = p_G(i,j) - x$. This means that the decrease in $p_G(i,j)$ must be equal to the decrease in $|V(G)|$. For this, we must remove each of the x vertices from $\alpha_G(i) \setminus \beta_G(i,j)$. Therefore, each vertex with degree $\Delta(G)$ in H is in the same class of the α-partition of H, which is containned in $\alpha_G(i)$.

It follows that any vertex v outside $\alpha_G(i)$ sees all x vertices removed. Hence $\deg_H(v) \leq \Delta(G) - x$. ■

Lemma 6 *Let G be a cograph with a complete level 3 cotree and H an overfull subgraph of G with $\Delta(H) = \Delta(G)$. If $\deg_G(v) < \Delta(G)$ and v belongs to the class of α-partition of G that contains the Δ-vertices of H, then $v \notin V(H)$.*

Proof: Let $\alpha_G(i)$ be the class of the α-partition of G that contains the Δ-vertices of H and $v \in \alpha_G(i)$, with $\deg_G(v) < \Delta$. Then, by Lemma 5, $\deg^*_H(v) = 0$. Since H is an overfull graph, by Lemma 1, $v \notin V(H)$. ■

Theorem 2 *Let G be a cograph with a complete level 3 cotree. If G is subgraph-overfull, then G is either overfull or neighbourhood-overfull (but not both).*

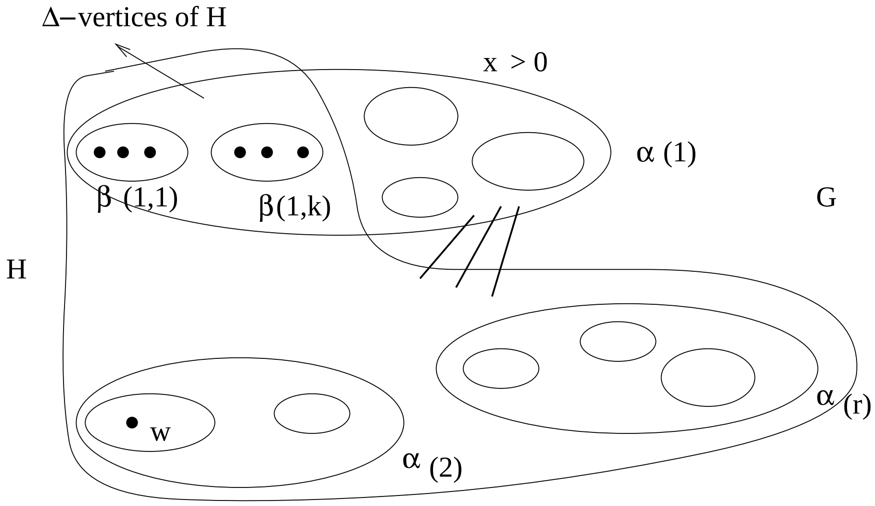

Figure 5: The proof of the Theorem 2.

Proof: Let G be a cograph with a complete level 3 cotree that is not an overfull graph itself. If G is subgraph-overfull, then G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$ and $V(H) \neq V(G)$. By Lemma 4, H is a cograph with a level 3 cotree.

Let $D(H)$ be the set of Δ-vertices of H. It follows, from Lemma 5 and Lemma 6, that $D(H) = \beta_G(1,1) \cup \cdots \cup \beta_G(1,k)$ for some $k \geq 1$, with $b_G(1,1) = b_G(1,2) = \cdots = b_G(1,k)$. Therefore, $|D(H)| = kb_G(1,1)$. Also each vertex $v \in V(H) \setminus \alpha_H(1)$ has $\deg_H(v) \leq \Delta(G) - x$, for $x = |V(G)| - |V(H)|$. Because $V(H) \neq V(G)$, x is a positive integer.

Since H is an overfull graph, then $|[V(H), \overline{V(H)}]| \leq \Delta(G) - 2$ (Lemma 2). On the other hand, $|[V(H), \overline{V(H)}]| \geq x(\Delta(G) - b_G(1,1) + 1)$. Then we obtain $\Delta(G) - 2 \geq x(\Delta(G) - b_G(1,1) + 1)$, or equivalently,

$$\Delta(G) \leq \frac{x(b_G(1,1) - 1) - 2}{x - 1}.$$

By hypothesis, the cotree of G is complete, then $x > 1$. This implies $1 < \frac{x}{x-1} \leq 2$. Hence $\Delta(G) \leq 2b_G(1,1) - 1$.

We now consider $w \in V(H) \setminus D(H)$. The degree of w in H is $kb_G(1,1) + q$, where q denotes the number of vertices that are in $N(w) \setminus D(H)$. From Lemma 5, we have that $\deg_H(w) \leq \Delta(G) - x$. Then $\deg_H(w) < \Delta(G)$ and

$$kb_G(1,1) + q < 2b_G(1,1) - 1$$

$$kb_G(1,1) - 2b_G(1,1) < -(q + 1)$$

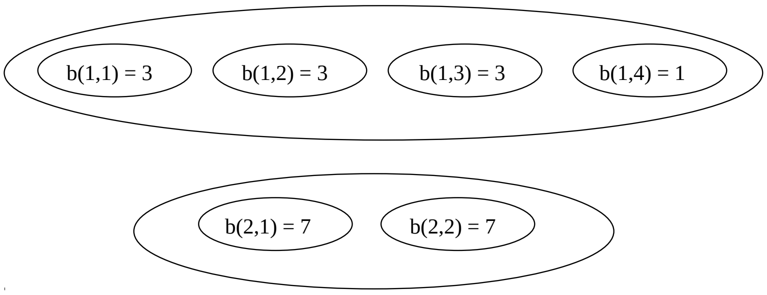

Figure 6: The subgraph H of G induced by $V(G) \setminus \beta(1,4)$ is overfull and has $\Delta(H) = \Delta(G)$.

$$b_G(1,1)(k-2) < -(q+1).$$

This inequality does imply $k < 2$. Therefore, $k = 1$ and H is induced by the neighbourhood of a Δ-vertice.

Now, let G be a cograph with a complete level 3 cotree and with $G \in O \cap NO$. Since G is NO, there is an overfull subgraph H of G with $\Delta(H) = \Delta(G)$ that is a neighbourhood. Hence the cograph H does not have a complete level 3 cotree. Since G is O, by Theorem 1 and Lemma 3, $G \cong H$. It follows that G does not have a complete level 3 cotree, a contradiction. ■

5 Conclusion

We know that for a level 1 or 2 cotree cograph, we have SO=O=Class 2. We proved that for a cograph with a complete level 3 cotree, SO = (O ∪ NO) \ (O ∩ NO). This provides a very simple algorithm for checking whether such a graph is SO, without having to resort to general techniques [11]. This equivalence does not hold for a general cograph G, even when G is a cograph with a level 3 cotree. The cograph G in Figure 6 is not overfull, and has a level 3 cotree and a subgraph H induced by $V(G) \setminus \beta(1,4)$ that is an overfull graph with $\Delta(H) = \Delta(G)$. But H is not induced by a neighbourhood of a vertex of H. It follows that G is not an NO graph.

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