The clique operator on cographs and serial graphs

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Abstract

The **clique graph** of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. The iterated clique graphs $K^n(G)$ are defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$ and K is the clique operator. A **cograph** is a graph with no induced subgraph isomorphic to P_4. In this article we describe the K-behaviour of cographs and give some partial results for the larger class of serial (i.e. complement-disconnected) graphs.

Key words: Iterated clique graphs; Clique divergence; Cographs.

1 Introduction

The **clique graph** of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. The **iterated clique graphs** $K^n(G)$ are defined by $K^0(G) = G$ and $K^i(G) = K(K^{i-1}(G))$, $i > 0$. We refer to [21] and [24] for the literature on iterated clique graphs. Graphs behave in a variety of ways under the iterates of the clique operator K, the main distinction being between K-convergence and K-divergence. A graph G is said to be K-**null** if $K^n(G)$ is the trivial graph K_1 for some n. We say that G is K-**periodic** if $K^n(G) \cong G$ for some $n \geq 1$; the smallest such n is the **period** of G. More generally, a

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graph G is said to be K**-convergent** if $K^n(G) \cong K^m(G)$ for some pair of non-negative integers $n < m$. If n and m are the smallest such integers, we say that n is the **transition index** and $m-n$ is the **period** of G. A graph G is K-convergent iff G is, in the obvious sense, **eventually** K**-periodic**. This last is equivalent to the boundedness of the sequence of the orders $|V(K^n(G))|$. A graph G is said to be K**-divergent** if $\lim_{n\to\infty} |V(K^n(G))| = \infty$. The graph G is K-divergent if and only if it is not K-convergent.

A graph is **clique-Helly** if its (maximal) cliques satisfy the Helly property: each family of mutually intersecting cliques has non-trivial intersection. For instance, all triangleless graphs are clique-Helly. Clique-Helly graphs have been introduced in [6,7] and studied in [19,20], among others. Clique-Helly graphs are always K-convergent: indeed, they are all eventually K-periodic of period 1 or 2 [6]. Clique-Helly graphs can be recognized in polynomial time [23]. So far, most of the results on convergence of iterated clique graphs are on the domain of clique-Helly graphs. In particular, several classes of clique-Helly graphs have been shown to be K-null, as is the case of interval graphs [9]. More generally, Bandelt and Prisner characterized the clique-Helly graphs which are K-null [1]. In general, much less is known about clique convergence, when non clique-Helly graphs are considered. Some results on convergence of graphs which are not clique-Helly can be found in [1,3,4]. All the graphs which will be shown to be K-convergent in this paper are clique-Helly.

The first examples of K-divergent graphs were given by Neumann-Lara (see [6,17]). For $n \geq 2$, define the n-dimensional octahedron $\mathcal{O}_n$ as the complement of a perfect matching on $2n$ vertices. Then $\mathcal{O}_n$ is a complete multipartite graph $K_{2,2,\ldots,2}$. Neumann-Lara showed that $K(\mathcal{O}_n) \cong \mathcal{O}_{2^{n-1}}$ and hence, for $n \geq 3$, $\mathcal{O}_n$ is K-divergent with superexponential growth. Moreover, he showed that all complete multipartite graphs $K_{p_1,\ldots,p_n}$, with $n \geq 3$ and $p_i \geq 2,\ 1 \leq i \leq n$, are K-divergent with superexponential growth. This last result completed the determination of the behaviour under K (or K-**behaviour**) of the class $\mathcal{CM}$ of complete multipartite graphs. In fact, the rest of the graphs in $\mathcal{CM}$ were previously known to be K-convergent: the complete graphs K_n are in $\mathcal{CM}$ and $K(K_n)$ is trivial. The other graphs in $\mathcal{CM}$ that have a universal vertex are those of the form $G = K_{1,p_2,\ldots,p_n} \neq K_n$ and here $K(G)$ is non-trivial but $K^2(G)$ is trivial. The remaining graphs in $\mathcal{CM}$ are those which have exactly two parts and at least two vertices in each part; they are triangleless and without terminal vertices, so by [8] they are K-periodic of period one (only for the square $K_{2,2} = C_4$) or two (all the rest).

In [18] it has been asked whether there are divergent graphs with polynomial growth. Recently, affirmative answers have been given in [11–13]. The graphs that will be shown to be K-divergent in this work have all superexponential growth.

Modular decompositions (which will be reviewed in §3) play an important rôle in this paper. The class of cographs properly contains the class of complete multipartite graphs, and it is known to be characterized by the absence of neighbourhood nodes in the modular decomposition tree (MDT). The wider class of serial graphs is defined by the fact that they are connected, non-trivial, and the root of the MDT is not neighbourhood: this are just the graphs whose complement is disconnected. In §4 we will characterize the serial graphs which are clique-Helly (a sufficient condition for K-convergence). An important class of serial graphs is that of parallel-decomposable serial graphs: for these, the MDT does not contain neither leaves nor neighbourhood modules in the root and the first level. In §5 we will give some sufficient conditions for K-divergence for such graphs. In §6 we shall determine completely the K-behaviour for the class of cographs. An interesting result is that a cograph is K-convergent if and only if it is clique-Helly. Finally, in §7 we characterize the cographs whose clique graph is also a cograph.

2 Preliminaries and definitions

We consider simple, undirected, finite graphs. The sets $V(G)$ and $E(G)$ are the vertex and edge sets of a graph G. A **trivial** graph is a graph with a single vertex. A **complete** is a set of pairwise adjacent vertices in G. A **clique** of G is a complete not properly contained in any other complete. A **subgraph** of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the **subgraph induced by** X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X. We often identify induced subgraphs with their vertex sets. If v is a vertex of a subgraph H of G adjacent to every other vertex of H, then we say that v is **universal** in H, or that H is a **cone** with **apex** v.

Let X be a subset of $V(G)$ and x any vertex of X. The **quotient graph** G/X is defined as $V(G/X) = (V(G) - X) \cup \{x\}$ and $E(G/X) = E(G[V(G) - X]) \cup \{\{x, v\} \mid \{u, v\} \in E(G),\ u \in X,\ v \in V(G) - X\}$.

Let H and H' be vertex-disjoint graphs. The **union** or **parallel composition** of H and H' is the graph $G = H \cup H'$ defined as $V(G) = V(H) \cup V(H')$ and $E(G) = E(H) \cup E(H')$. The **join**, **sum**, or **serial composition** of H and H' is the graph $G = H + H'$ defined as $V(G) = V(H) \cup V(H')$ and $E(G) = E(H) \cup E(H') \cup \{\{x, y\} \mid x \in V(H),\ y \in V(H')\}$. Finally, the symbol $\bar{G}$ represents the **complement** of G. Notice for instance that $\overline{G + H} = \bar{G} \cup \bar{H}$.

In order to study K-convergence and K-divergence the following results are useful tools. We recall them for the reader's convenience.

Let T be a triangle of a graph G. The **extended triangle of G, relative to T,** is the subgraph $\widehat{T}$ of G induced by the vertices which form a triangle with at least one edge of T. The following theorem characterizes clique-Helly graphs [23].

Theorem 1 *A graph G is clique-Helly if and only if each of its extended triangles is a cone.* $\square$

Let G, H be graphs. A *morphism* $\alpha : G \to H$ is a vertex-function $\alpha : V(G) \to V(H)$ such that the images under α of adjacent vertices of G either coincide or are adjacent in H. A morphism $\alpha : G \to H$ is a **retraction from G to H** if there exists a morphism $\beta : H \to G$ such that $\alpha\beta$ is the identity function on $V(H)$. In this case, H is a **retract** of G. Notice that H must be isomorphic to an induced subgraph of G. If H is a subgraph of G, then a retraction from G to H is just a morphism $G \to H$ whose restriction to H is the identity. Notice also that, if v is a vertex of G, there is always a **total retraction** from G to v. The following theorem describes the relationship between retracts and K-divergence [17].

Theorem 2 *Let G and H be graphs such that H is a retract of G. If H is K-divergent, then so is G.* $\square$

For instance, if $n \geq 3$ and $p_i \geq 2$ $(1 \leq i \leq n)$, the complete multipartite graph $K_{p_1,\ldots,p_n}$ can be retracted to the octahedron $\mathcal{O}_n = K_{2,2,\ldots,2}$ and is therefore K-divergent.

3 Modular decompositions

One promising paradigm for studying properties of a class of graphs involves partitioning the set of vertices of a graph into subsets called modules, and the decomposition process is called modular decomposition. This decomposition has been studied by many researchers.

A **module** of G is a set of vertices M of $V(G)$ such that all the vertices of M have the same neighbours outside of M, that is, each vertex in $V(G) - M$ is either adjacent to all vertices of M, or to none. For instance, every singleton as well as the whole G are modules. Two vertices $x, y \in G$ are **twins** if they are neighbours and $\{x, y\}$ is a module of G, and any module M such that $G[M]$ is complete is just a set of twins in G. We say that M is a **strong** module if for any other module A the intersection $M \cap A$ is empty or equals either M or A. For non-trivial G, the family $\{G_1, G_2, \ldots, G_p\}$ of all maximal (proper) strong modules is a partition of $V(G)$ and $p \geq 2$. This partition is the **modular decomposition of** G. We will often identify the modules G_i

with the induced subgraphs $G[G_i]$.

If a module M contains vertices from two different connected components of G, then M must contain those components. Therefore, for disconnected G, the maximal strong modules are the connected components. In this case $G = G_1 \cup G_2 \cup \cdots \cup G_p$ and G is called **parallel**.

The modules, strong modules and maximal strong modules of G are the same as those of $\bar{G}$. Therefore, if $\bar{G}$ is disconnected, the maximal strong modules of G are the vertex sets of the connected components of $\bar{G}$. In this case $G = G_1 + G_2 + \cdots + G_p$ and G is called **serial**.

The modular decomposition of a non-trivial graph G is used recursively in order to define its unique **modular decomposition tree** $T(G)$. The module M is **parallel** if $G[M]$ is disconnected, M is **serial** if $\bar{G}[M]$ is disconnected and M is **neighbourhood** if both $G[M]$ and $\bar{G}[M]$ are connected. The root of $T(G)$ is G, the first-level vertices of $T(G)$ are the maximal strong modules of G, and so on. The leaves of $T(G)$ are the vertices of G and the internal nodes of $T(G)$ are modules labeled with P, S or N (for parallel, serial, or neighbourhood module, respectively). A linear time algorithm that produces the modular decomposition tree is given in [16].

Modular decomposition has been extensively used for the class of cographs. It is in fact the basis for finding fast algorithms for problems on cographs which are NP-hard in general [5]. We denote by P_4 the four-vertex path. A **cograph** is a graph having no induced subgraph isomorphic to P_4. Cographs were introduced by Lerchs [14] and rediscovered under different names: D^*-graphs, Hereditary Dacey graphs, 2-parity graphs [2,10,22]. In [14], Lerchs showed that cographs have a unique tree representation, called **cotree**. The leaves represent vertices of the graph and the internal nodes correspond to the union and join operation. This cotree is just the modular decomposition tree, but for cographs it has no neighbourhood nodes. In fact, the class of cographs can also be defined as the class of those graphs for which the modular decomposition tree does not have neighbourhood nodes (see §6), or as the smallest class of graphs containing the single vertex graph and closed under serial and parallel composition.

Lemma 3 *Let G be a graph and M a module of G. Let R be a subgraph of M which is a retract of M. Then any retraction $\rho : M \to R$ can be extended to a retraction $\rho' : G \to G[(G - M) \cup R]$.*

Proof Define $\rho'(v) = \rho(v)$ for $v \in M$, and $\rho'(v) = v$ for $v \in G - M$. Since M is a module of G, ρ' is a morphism. Since the restriction of ρ to R is the identity, the restriction of ρ' to $G[(G - M) \cup R]$ is the identity. $\square$

Lemma 4 *Let G be a graph and M a module of G. Then the quotient graph G/M is a retract of G.*

Proof Let x be any vertex of M and take $V(G/M) = (V(G) - M) \cup \{x\}$. By Lemma 3, the total retraction $M \to x$ can be extended to a retraction $G \to G/M$. Hence, G/M is a retract of G. $\square$

Remark 5 *In the particular case in which the module M of G is complete we have that $K(G) \cong K(G/M)$; indeed, if $u, v \in G$ are twins, any clique which contains u also contains v, so $K(G) \cong K(G-v)$. In the general case we know by Theorem 2 that G is K-divergent if G/M is so.*

Lemma 6 *Let G be a graph. If $P = S_1 \cup S_2 \cup \cdots \cup S_q$ is a parallel module of G and some S_i is a single vertex v, then $G - v$ is a retract of G.*

Proof Fix some $x \in P - v$. Since P is parallel, v has no neighbours in P. Then there is a retraction $\alpha : P \to P - v$ defined by $\alpha(v) = x$ and $\alpha(y) = y$ for each $y \in P - v$. By Lemma 3, α can be extended to a retraction $\alpha : G \to G - v$. Thus $G - v$ is a retract of G. $\square$

Notice that any graph obtained by iterated application of Lemma 4 or Lemma 6 is a retraction of the original graph G.

4 Serial graphs

Since for a disconnected graph $G = H \cup H'$ we clearly have that $K(G) = K(H) \cup K(H')$, in order to investigate the K-behaviour we can restrict ourselves to connected non-trivial graphs G. If we make the additional assumption that G is not neighbourhood, then G must be serial. By definition, a graph is serial iff its complement is disconnected (iff it is a sum of two graphs). Serial graphs are just those connected non-trivial graphs such that the root of the modular decomposition tree is not neighbourhood. We will characterize in this section those serial graphs which are clique-Helly, and describe their K-behaviour.

Recall that a graph G is a cone if G has a universal vertex (apex). Note that each clique of the cone G contains the apex, so G is certainly clique-Helly. Furthermore, $K(G)$ is a complete graph (G is **clique-complete**) and thus $K^2(G)$ is trivial: all cones are K-null. Of course, any clique-complete and clique-Helly graph must be a cone, but there exist clique-complete graphs which are not cones (see [15]). Notice that if G is a cone, then any sum $G+H$ is also a cone (and serial), but if $G = G_1 + G_2 + \cdots + G_p$ is the modular decomposition of a serial graph G, then G is a cone if and only if some G_i is trivial: indeed, if G is a cone the apex is isolated in $\bar{G}$.

Lemma 7 *Let $G = H + H'$ be a clique-Helly graph, and assume that H' is not a cone. Then each connected component of H is a cone.*

Proof Let C be a connected component of H, and let the vertex x of C have maximum degree as a vertex of C. Assuming that x is not universal in C, we will get a contradiction. Let y and v be vertices of C such that x and v are neighbours of y but $\{x, v\} \notin E(H)$. Choose any $z \in H'$ and consider the triangle $T = \{x, y, z\}$ of G. By Theorem 1, the extended triangle $\widehat{T}$ relative to T has a universal vertex u. Since $H' \subset \widehat{T}$ and H' is not a cone, $u \in H$, but then $u \in C$. Since all the neighbours of x in C and also v are in $\widehat{T}$, $\deg_C(u) > \deg_C(x)$. This contradiction shows that C is a cone with apex x. □

Theorem 8 *Let $G = G_1 + G_2 + \cdots + G_p$ be the modular decomposition of a serial graph. Then G is clique-Helly if and only if it satisfies one of the following conditions:*

(1) G is a cone, or
(2) $p = 2$ and all the connected components of G_1 and G_2 are cones.

Proof If G is a cone we know that it is clique-Helly. If the second condition holds, choose an apex in each of the connected components of both G_1 and G_2, and let H be the subgraph of G induced by all these apices. Then H is a complete bipartite graph. Since H has no triangles its cliques are its edges and it is clique-Helly. Each clique of G contains a unique edge of H, and two cliques of G meet if and only if their edges in H meet. Therefore G is clique-Helly.

Assume now that G is clique-Helly.

Suppose first that $p \geq 3$. Choose $x_i \in G_i$ for $i = 1, 2, 3$. Consider the triangle $T = \{x_1, x_2, x_3\}$ and observe that its extended triangle is $\widehat{T} = G$. By Theorem 1, G is a cone. Thus, $p \geq 3$ implies condition 1.

If $p = 2$, we can assume that condition 1 does not hold. In particular, G_1 and G_2 are not cones. By Lemma 7, each connected component of both G_1 and G_2 is a cone. □

If G is a clique-Helly graph, we know that G is eventually K-periodic of period

1 or 2, but if G is also serial the previous theorem and its proof enable us to be more specific. In case 1: If G is complete (i.e. all the G_i are trivial) then $K(G)$ is trivial; if G is not complete, then $K(G)$ is non-trivial and $K^2(G)$ is trivial. In case 2: We can assume that case 1 does not hold, and then G_1 and G_2 are disconnected. Thus, the complete bipartite subgraph H of G has at least two vertices in each part. If $H = G$, we know that either $K(G) \cong G$ (if $G \cong K_{2,2}$) or else $K(G) \not\cong G$ but $K^2(G) \cong G$. If $H \neq G$, we use the fact that each clique Q of G contains a unique edge (i.e. clique) E_Q of H, and that $Q \cap Q' \neq \emptyset$ iff $E_Q \cap E_{Q'} \neq \emptyset$. For a given edge E of H, all the cliques Q of G for which $E_Q = E$ are twins in $K(G)$. Keeping just one Q for each E, we get an induced subgraph S of $K(G)$ such that $S \cong K(H)$. By Remark 5 we have $K(K(G)) \cong K(S)$, and then $K^2(G) \cong K^2(H) \cong H$. Thus G is not K-periodic, but it is eventually K-periodic of period one or two according to whether both G_1 and G_2 have exactly two connected components or not.

The serial graphs such that all maximal strong modules are disconnected (such as those satisfying condition 2 in the previous theorem) will be the subject of our next section.

5 Parallel-decomposable serial graphs

Consider the modular decomposition

$$G = G_1 + G_2 + \cdots + G_p$$

of a serial graph G. If some G_i is trivial, then G is a cone and we already know that G is K-null. We can therefore assume that all the G_i are non-trivial. If we make the additional assumption that no G_i is neighbourhood, then all the G_i must be parallel. In this case each of the G_i has a modular decomposition of the form

$$G_i = \cup_{j=1}^{p_i} G_j^i, \ p_i \geq 2,$$

and we say that G is a **parallel-decomposable serial graph.** Since a module of a module of G is again a module of G, all the graphs G_j^i are (connected) modules of G. All the G_j^i are trivial precisely for the complete multipartite graphs which are not cones (i.e. with non-singular parts). Parallel-decomposable serial graphs are just those non-trivial connected graphs such that the modular decomposition tree does not contain neither leaves nor neighbourhood modules in the root and the first level. Parallel-decomposable serial graphs with at least 3 maximal strong modules are always K-divergent:

Theorem 9 *Let G be a parallel-decomposable serial graph. If $p \geq 3$, then G is K-divergent.*

Proof Applying Lemma 4 to each non-trivial G^i_j we obtain a complete multipartite graph $G' \cong K_{p_1,\ldots,p_p}$, with $p_i \geq 2$ for all $1 \leq i \leq p$. Since $p \geq 3$, we already know that G' is K-divergent by [17]. By Theorem 2, G is also K-divergent. $\square$

For parallel-decomposable serial graphs G, it only remains to study the K-behaviour in the case $p = 2$. If G is a cone or all the G^i_j are cones, we know by Theorem 8 that G is K-convergent. But there are also parallel-decomposable serial graphs with $p = 2$ which are K-divergent:

Theorem 10 *Let $G = G_1 + G_2$ be a parallel-decomposable serial graph. If at least one G^i_j is also a parallel-decomposable serial graph, then G is K-divergent.*

Proof Without loss of generality, we assume that G^1_1 is a parallel-decomposable serial graph, so we have a decomposition $G^1_1 = \sum_{r=1}^{q} \cup_{s=1}^{q_r} T^r_s$.

Let us first consider G_2. By Lemma 4 we can retract each module G^2_j to a single vertex $u \in V(G^2_j)$, for $1 \leq j \leq p_2$. Then G_2 retracts to the union of p_2 vertices.

Let us now consider G_1. By Lemma 4 we can retract each T^r_s to a single vertex. Then the graph $K_{q_1,\ldots,q_q}$ is a retract of G^1_1, where $q \geq 2$ and each $q_i \geq 2$. Again by Lemma 4, we can retract each G^1_j to a single vertex u_j for $2 \leq j \leq p_1$. Therefore G_1 retracts to the union of $K_{q_1,\ldots,q_q}$ and $p_1 - 1$ single vertices. By Lemma 6, $K_{q_1,\ldots,q_q}$ is a retract of G_1.

Hence the graph $K_{q_1,\ldots,q_q,p_2}$ is a retract of G. By Theorem 2, since $K_{q_1,\ldots,q_q,p_2}$ is K-divergent, G is also K-divergent. $\square$

In conclusion, the K-behaviour of a non-clique-Helly serial graph G remains unknown if G is not parallel-decomposable or it is, but $p = 2$ and no G^i_j is a parallel-decomposable graph. Notice that in the latter case G cannot be a cone and some G^i_j must be non-trivial, for otherwise G is clique-Helly.

6 Cographs

Recall that a cograph is a graph without induced paths of length 3. The class of cographs is clearly closed under complements, induced subgraphs, and serial and parallel compositions. Any connected cograph which is not 2-connected is a cone: any cut vertex is universal.

We mentioned in Section 3 that cographs are those graphs such that there are no neighbourhood nodes in the modular decomposition tree T. We give a proof of this: If T has no neighbourhood nodes, then G is a cograph because it can be reconstructed from its vertices using serial and parallel compositions. The converse follows from the fact that the complement of a connected non-trivial cograph is always disconnected. This is proved by induction on the order n of G, the case $n = 2$ being obvious. If G has a cut vertex then G is a cone and $\bar{G}$ is disconnected. If G is 2-connected, take any $v \in G$. Since $G - v$ is a connected cograph, $\overline{G - v}$ is disconnected. Supposing that $\bar{G}$ is connected we get a contradiction, because v is then a cut vertex of $\bar{G}$ and then $G = \overline{\overline{G}}$ is disconnected.

Complete multipartite graphs are precisely those cographs G such that the cotree has at most two levels. If all the vertices lie at the first level, then G is complete. If there is some vertex at the first level, then G is a cone. If G is not a cone all the vertices are at the second level.

The following results determine the K-behaviour of cographs. Notice that any cograph which is not a cone is a parallel-decomposable serial graph.

Theorem 11 *A cograph G is K-divergent if and only if both the following conditions hold.*

(1) G is not a cone, and
(2) either $p \geq 3$, or $p = 2$ and at least one of G^i_j is not a cone.

Proof Sufficiency: Since G is not a cone, G is a parallel-decomposable serial graph. If $p \geq 3$, G is K-divergent by Theorem 9. If $p = 2$, G is K-divergent by Theorem 10 because some G^i_j is parallel-decomposable.

Necessity: Assume that G is K-divergent. Then G is not a cone, for we know that cones are K-convergent. If $p = 2$ and each G^i_j is a cone, then G is K-convergent by Theorem 8. □

Theorem 11 implies that the K-divergence or K-convergence of a cograph G can be decided in linear time. In fact, the cotree of G can be obtained in linear time [5] and the conditions of Theorem 11 can be also checked in linear time.

Corollary 12 *A cograph G is K-convergent if and only if G is clique-Helly.*

Proof It follows immediately from theorems 8 and 11. $\square$

The example of Figure 1 shows that in the class of serial graphs there are K-convergent graphs that are not clique-Helly.

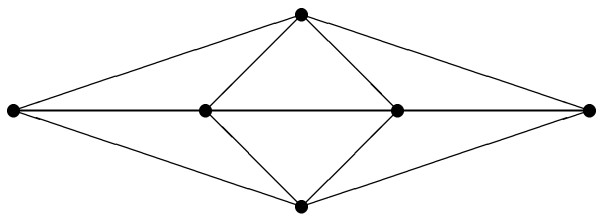

Fig. 1. A K-convergent serial graph that is not clique-Helly.

7 Cographs whose clique graph is a cograph

The product $G \times G'$ of two graphs G and G' is given by $V(G \times G') = V(G) \times V(G')$ and $E(G \times G') = \{\{(u,u'),(v,v')\} : \{u,v\} \in E(G),\ \{u',v'\} \in E(G')\}$. Notice that $G \times (G' \cup G'') = G \times G' \cup G \times G''$ and that a product $G = G_1 \times G_2 \times \cdots \times G_s$ is discrete (has no edges) if and only if some G_i is discrete. We will also use the fact that $\bar{K}(G_1 + G_2) = \bar{K}(G_1) \times \bar{K}(G_2)$, where $\bar{K}(G) = \overline{K(G)}$ (see [17,24]).

Lemma 13 *Let $s \geq 2$ and $G = G_1 \times G_2 \times \cdots \times G_s$ with G_i connected and non-trivial for each i. Then G is a cograph if and only if each G_i is complete bipartite.*

Proof If G is a cograph, each G_i is a cograph (resp.: bipartite): Indeed, otherwise some G_i would contain an induced P_4 (resp.: an induced odd cycle γ). Then G would contain an induced subgraph isomorphic to $P_4 \times K_2 \times \cdots \times K_2$ (resp.: to $\gamma \times K_2 \times \cdots \times K_2$). But the connected components of this subgraph are isomorphic to P_4 (resp.: to a cycle whose length is the double of that of γ) and so G would not be a cograph. A connected bipartite cograph must be complete bipartite. For the converse just note that $K_{m,n} \times K_{p,q} \cong K_{mp,nq} \cup K_{mq,np}$. $\square$

Corollary 14 *Let $s \geq 2$ and $G = G_1 \times G_2 \times \cdots \times G_s$. Then G is a cograph if and only if either some G_i is discrete or every non-trivial connected component of each G_i is complete bipartite.* $\square$

Theorem 15 *Let G be a cograph. Then G is clique-complete if and only if G is a cone.*

Proof Suppose that the non-trivial part is false, and let G be a non-conical cograph with minimal order such that $K(G)$ is complete. Clearly G is connected, so $G = G_1 + G_2$ for some non-conical cographs G_1 and G_2. Since $\bar{K}(G) = \bar{K}(G_1) \times \bar{K}(G_2)$ is discrete, either $\bar{K}(G_1)$ or $\bar{K}(G_2)$ is discrete, contradicting the minimality of the order of G. □

Theorem 16 *Let G be a connected, non-conical cograph such that $K(G)$ is a cograph. Let $G = G_1 + G_2 + \cdots + G_p$ be the modular decomposition of G. Then each G_i is a cograph with exactly two connected components and these are cones.*

Proof We have that $\bar{K}(G) = \bar{K}(G_1) \times \bar{K}(G_2) \times \cdots \times \bar{K}(G_s)$. Since each G_i is non-trivial, it is disconnected and the same happens with $K(G_i)$, so $\bar{K}(G_i)$ is non-trivial and connected for each i. By Lemma 13, each $\bar{K}(G_i)$ is complete bipartite, and thus each $K(G_i)$ is the union of two complete connected components. Therefore, by Theorem 15, each G_i is the union of two connected components which are conical cographs. □

We shall also use the following construction. Let the family of graphs $(G_i)_{i \in V(H)}$ be indexed by the vertices of a graph H. The **sum over** H of the graphs G_i is the graph $G = \sum_H G_i$ which is obtained from $\bigcup_{i \in V(H)} G_i$ by adding all possible edges of the form $\{u, v\}$ where $u \in G_i$, $v \in G_j$ and $\{i, j\} \in E(H)$. For instance, any sum $\sum_{i \in I} G_i$ is the sum of the G_i over the complete graph on the index set I, and any union $\bigcup_{i \in I} G_i$ is the sum of the G_i over the discrete graph on I. Observe that each G_i is a module of $\sum_H G_i$. In case that all G_i are the same graph G', then $\sum_H G_i$ is just the composition $H[G']$.

If G and H are arbitrary graphs, a vertex-surjective morphism $\pi : G \to H$ is said to be an **additive projection** if $\pi^{-1}(H[\{i, j\}]) = \pi^{-1}(i) + \pi^{-1}(j)$ whenever $\{i, j\} \in E(H)$. It is easy to see that G is (isomorphic to) a sum over H if and only if there exists an additive projection from G to H.

If $\pi : G \to H$ is an additive projection and Q is an induced subgraph of G, then Q is a clique of G if and only if $\pi(Q)$ is a clique of H and $\pi^{-1}(i) \cap Q$ is a clique of the **fiber** $G_i = \pi^{-1}(i)$ for each $i \in V(\pi(Q))$; therefore, π induces a vertex-surjective morphism $\pi_K : K(G) \to K(H)$ given by $\pi_K(Q) = \pi(Q)$. The following result is easily verified:

Theorem 17 *Let $\pi : G \to H$ be an additive projection.*

(1) G is a cograph if and only if H and each fiber of π are cographs.
(2) If each fiber of π is complete, then π_K is an isomorphism.
(3) If each fiber of π is clique-complete, then π_K is an additive projection with complete fibers and therefore $K^2(G) \cong K^2(H)$. □

Theorem 18 *The graph G is a connected cograph such that $K(G)$ is a cograph if and only if G is either a conic cograph or a sum of conic cographs over an octahedron $\mathcal{O}_n$ for some $n \geq 2$.*

Proof Let G be a non-conical connected cograph whose clique graph is also a cograph. Let $G = G_1 + G_2 + \cdots + G_p$ be the modular decomposition of G. By Theorem 16, each G_i is of the form $G_i = G_1^i \cup G_2^i$ where G_1^i and G_2^i are conical cographs. For each $i = 1, \ldots, p$ and $j = 1, 2$, let $x_j^i \in G_j^i$ and let $G' = G[\{x_j^i : i = 1, \ldots, p,\ j = 1, 2\}]$. Clearly, the mapping which sends each G_j^i into x_j^i is an additive projection from G onto G'. Since $G' = G[x_1^1, x_2^1] + G[x_1^2, x_2^2] + \cdots + G[x_1^p, x_2^p] \cong \mathcal{O}_p$, it follows that G is the sum over $\mathcal{O}_p$ of the conical cographs G_j^i. On the other hand, if G is a sum of conic cographs over an octahedron $\mathcal{O}_n$ with $n \geq 2$, then G is a cograph by Theorem 17(1) because all octahedra are cographs. By Theorem 17(3), $\pi_K : K(G) \to K(\mathcal{O}_n)$ is an additive projection with complete fibers. Since $K(\mathcal{O}_n) \cong \mathcal{O}_{2^{n-1}}$ and the complete graphs are cographs, Theorem 17(1) implies that $K(G)$ is a cograph. □

Corollary 19 *If G is a non-conical connected cograph such that $K(G)$ is a cograph and the modular decomposition of G has p summands, then $K^2(G) \cong \mathcal{O}_n$ with $n = 2^{2^{p-1}-1}$.*

Proof It follows directly from Theorem 17(3), Theorem 18 and the fact that $K(\mathcal{O}_p) \cong \mathcal{O}_{2^{p-1}}$. □

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