Fast Forward Property
and
Decompositions of Graph of Functions

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Definition
Let \( V = \{0, 1, \ldots, N - 1\} \). A function \( f : V \to V \) is fast forward if for each \( m \in \mathbb{N} \) and each \( x \in V \) the computational complexity of evaluating the \( m \)th iterate of \( f \) at \( x \),

\[
f^m(x), \quad \text{\((\text{e.g.} \ f^2(x) = f(f(x))\))},
\]

is small (polynomial in \( \log N \)).
**Fast forward property**

**Definition** Let $V = \{0, 1, \ldots, N - 1\}$. A function $f : V \to V$ is fast forward if for each $m \in \mathbb{N}$ and each $x \in V$ the computational complexity of evaluating the $m$th iterate of $f$ at $x$,

$$f^m(x), \quad (\text{e.g. } f^2(x) = f(f(x)))$$

is small (polynomial in $\log N$).

**Example** The cyclic permutation $(0, 1, 2, \ldots, N - 1)$ is fast forward since $f^m(x) = x + m \mod N$. 
Evaluation of iteration of functions is useful in cryptographic applications.
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**Example**


There compute \( f^m(x) \) repeatedly for the function

\[
 f(x) = x^2 - 1 \mod N.
\]
The context

The scale of $N$ is usually very large in cryptographic applications.

Here we consider the case that $N$ is not very large.
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Here we consider the case that $N$ is not very large.

Why?

In the computation of the iteration $f^m(x)$, we need to use $O(N)$ information which is generated in preprocess and stored in memory.
Preliminary

(Recall definitions and results in Tsaban 2003, 2007)
Fast forward permutation

**Definition** The fast forward permutation coded by $(m_0, m_1, ..., m_{\ell-1})$ is

$$
\pi = \begin{pmatrix}
(0, ..., s_0 - 1) \\
m_0
\end{pmatrix}
\begin{pmatrix}
(s_0, ..., s_1 - 1) \\
m_1
\end{pmatrix}
\cdots
\begin{pmatrix}
(s_{\ell-2}, ..., N - 1) \\
m_{\ell-1}
\end{pmatrix},
$$

where $s_i = m_0 + \cdots + m_i$ for $0 \leq i \leq \ell - 2$. 
Fast forward permutation

**Definition** The fast forward permutation coded by 
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\[
\pi = \underbrace{(0, \ldots, s_0 - 1)}_{m_0} \underbrace{(s_0, \ldots, s_1 - 1)}_{m_1} \ldots \underbrace{(s_{\ell-2}, \ldots, N - 1)}_{m_{\ell-1}},
\]

where \(s_i = m_0 + \cdots + m_i\) for \(0 \leq i \leq \ell - 2\).

**Computation of the iteration**

\[
\pi^m(x) = s_i(x) + \left(x - s_i(x) + m \mod (s_i(x) + 1 - s_i(x))\right),
\]

where \(s_i(x) \leq x < s_i(x) + 1\) and the assignments \(x \rightarrow i(x)\) and \(i \rightarrow s_i\) are preprocessed as lookup tables.
For any permutation

\[ f = \underbrace{(b_0, b_1, \ldots, b_{s_0-1})}_{m_0} \underbrace{(b_{s_0}, \ldots, b_{s_1-1})}_{m_1} \ldots \underbrace{(b_{s_{\ell-2}}, \ldots, b_{N-1})}_{m_{\ell-1}}, \]

define \( \sigma(x) = b_x \) for \( x \in V \). Then

\[ f = \sigma \circ \pi \circ \sigma^{-1} \quad \text{and} \quad f^m = \sigma \circ \pi^m \circ \sigma^{-1}. \]
For any permutation

\[ f = (b_0, b_1, \ldots, b_{s_0-1}) \cdot (b_{s_0}, \ldots, b_{s_1-1}) \cdot \ldots \cdot (b_{s_{\ell-2}}, \ldots, b_{N-1}) \cdot \cdot \cdot (b_{s_{\ell-1}}, \ldots, b_{s_0-1}) \cdot \cdot m_{0} \cdot m_{1} \cdot \ldots \cdot m_{\ell-1} \]  

define \( \sigma(x) = b_x \) for \( x \in V \). Then

\[ f = \sigma \circ \pi \circ \sigma^{-1} \quad \text{and} \quad f^m = \sigma \circ \pi^m \circ \sigma^{-1}. \]

Thus, \( f \) is fast forward if lookup tables for \( \sigma, \sigma^{-1}, i(x) \) and \( s_i \) are stored in memory.
Fast forward function

Definition The fast forward function coded by
\((m_0, m_1, ..., m_{\ell-1})\) and an auxiliary sequence
\((p_0, ..., p_{\ell-1}), 0 \leq p_i < s_i,\) is

\[
\theta(x) = \begin{cases} 
  p_i & \text{if } x = s_i - 1, \\
  \pi(x) & \text{otherwise}, 
\end{cases}
\]

where \(\pi\) is the fast forward permutation codes by
\((m_0, m_1, ..., m_{\ell-1})\).
**Fast forward function**

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where \(\pi\) is the fast forward permutation codes by 
\((m_0, m_1, ..., m_{\ell-1})\).

Note that a cycle of \(\pi\), \((s_{i-1}, ..., s_i - 1)\), is connected
to a previous cycle by the mapping \(\theta(s_i - 1) = p_i\) if 
\(p_i < s_{i-1}\). We call this a descent.
Evaluating $\theta^m(x)$

Case 1. if $m$ is small $(s_i(x) \leq x + m < s_i(x)+1)$, then

$$\theta^m(x) = x + m.$$
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$$\theta^m(x) = x + m.$$ 

Case 2. if $x + m \geq s_{i(x)+1}$ and $p_{i(x)+1} \geq s_i(x)$. Then

$$\theta^m(x) = p_{i(x)+1} + \left((x + m - s_{i(x)+1}) \mod (s_{i(x)+1} - p_{i(x)+1})\right).$$
Evaluating $\theta^m(x)$

**Case 1.** if $m$ is small ($s_i(x) \leq x + m < s_i(x)+1$), then

$$\theta^m(x) = x + m.$$  

**Case 2.** if $x + m \geq s_i(x)+1$ and $p_i(x)+1 \geq s_i(x)$. Then

$$\theta^m(x) = p_i(x)+1 + ((x + m - s_i(x)+1) \mod (s_i(x)+1 - p_i(x)+1)).$$

**Case 3.** if $x + m \geq s_i(x)+1$ and $p_i(x)+1 < s_i(x)$. Then

$\theta^m(x)$ is computed recursively by

$$\theta^m(x) = \theta^{x+m-s_i(x)+1}(p_i(x)+1)$$

(here $\theta^0 = I$). (case of descent)
The complexity

**Theorem** (Tsaban, 2007)

The complexity of evaluating $\theta^m(x)$ is measured by the number of descents (recursions) on the tour $x, \theta(x), \theta^2(x), \theta^3(x), \ldots$.

Notation of the number of descents: $D_\theta(x)$. 
Example for number of descents

$$D_\theta(x) = 0 \text{ for } x = 0, \ldots, 8;$$
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Example for number of descents

\[ D_{\theta}(x) = 0 \text{ for } x = 0, \ldots, 8; \quad D_{\theta}(x) = 1 \text{ for } x = 9, \ldots, 15; \]
\[ D_{\theta}(x) = 2 \text{ for } x = 16, 17. \]
If there is a fast forward function $\theta$ such that

$$f = \sigma \circ \theta \circ \sigma^{-1},$$

then

$$f^m = \sigma \circ \theta^m \circ \sigma^{-1}$$

and the complexity of evaluating $f^m(x)$ is $O(D_\theta(x) + 1)$. 
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How to find such a $\theta$?
An approach proposed by Tsaban

**Orbit decomposition**

**Definition** The orbit of an element $x$ in $U \subset V$ is the simple tour $\{x, f(x), f^2(x), \ldots, f^k(x)\}$ such that $f^{k+1}(x) = f^i(x)$ for some $0 \leq i \leq k$ or $f^{k+1}(x) \notin U$. 
The orbit of an element \( x \) in \( U \subset V \) is the simple tour \( \{ x, f(x), f^2(x), \ldots, f^k(x) \} \) such that \( f^{k+1}(x) = f^i(x) \) for some \( 0 \leq i \leq k \) or \( f^{k+1}(x) \notin U \).

An orbit decomposition of \( f \) is \( C_0, C_1, \ldots, C_{\ell-1} \) defined as follows: \( C_0 \) is an orbit in \( V \) and \( C_i \) is an orbit in \( V - C_0 \cup \cdots \cup C_{i-1} \) for \( i > 0 \).
Given an arbitrary function $f$, and

\[
(b_0, b_1, \ldots, b_{s_0-1}) \quad (b_{s_0}, \ldots, b_{s_1-1}) \quad \ldots \quad (b_{s_{\ell-2}}, \ldots, b_{N-1})
\]

\[
C_0 \quad C_1 \quad \ldots \quad C_{\ell-1}
\]

is an orbit decomposition of $f$.  

Given an arbitrary function \( f \), and
\[
\begin{align*}
(b_0, b_1, \ldots, b_{s_0-1}) & \quad (b_{s_0}, \ldots, b_{s_1-1}) & \ldots & \quad (b_{s_{\ell-2}}, \ldots, b_{N-1}) \\
C_0 & \quad C_1 & \ldots & \quad C_{\ell-1}
\end{align*}
\]
is an orbit decomposition of \( f \). Then
\[
f = \sigma \circ \theta \circ \sigma^{-1},
\]
where \( \sigma(x) = b_x \) and \( \theta \) is the fast forward function coded by \((|C_0|, |C_1|, \ldots, |C_{\ell-1}|)\) and the auxiliary sequence \((p_0, \ldots, p_{\ell-1})\) such that \( f(b_{s_i-1}) = b_{p_i} \).
Example Let $f$ be the function:

$9 \rightarrow 8 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0 \circ$
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or \[(4, 3, 2, 1, 0)(9, 8, 7, 6, 5), \quad D_θ(9) = 1,\]
Example Let $f$ be the function:

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or $(4, 3, 2, 1, 0)(9, 8, 7, 6, 5)$, $D_\theta(9) = 1$,

or $(1, 0)(4, 3, 2)(6, 5)(9, 8, 7)$, $D_\theta(9) = 3$,
**Example** Let $f$ be the function:

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or \[(1, 0)(4, 3, 2)(6, 5)(9, 8, 7), \quad D_\theta(9) = 3,\]

... or even the worst

\[(0)(1)(2)(3)(4)(5)(6)(7)(8)(9), \quad D_\theta(9) = 9.\]
Example Two different orbit decompositions:

(here the numbers indicate the numbers of descents)

good

poor
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good

poor
Example Two different orbit decompositions:

(here the numbers indicate the numbers of descents)

good

0 1 1 1 1 1 1 1

poor

4 3 2 1 0
Example Two different orbit decompositions:
(here the numbers indicate the numbers of descents)

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$C_0$ is the maximal length orbit in $V$.
Definition (Tsaban, 2007)

The greedy orbit decomposition of $f$ is $C_0, C_1, \ldots, C_{\ell-1}$ defined as follows:

$C_0$ is the maximal length orbit in $V$ and

$C_i$ is the maximal length orbit in $V - C_0 \cup \cdots \cup C_{i-1}$ for $i > 0$. 

The worst case

The maximal number of descents for greedy orbit decomposition is about $\sqrt{2N}$. 
The average case

The experiment in Tsaban 2007 indicates that the expected number of descents for the greedy orbit decomposition is about

$$\frac{\log_2 N}{5}$$

for random functions and random points.
1. “Does there exist an efficient algorithm to find an orbit decomposition for which the maximal number of descents is as small as it can be?”
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2. “Finding an approach which reduces the average number of descents seems to be of great practical interest.” That is to find a $\theta$ which minimizes

$$\sum_{x \in V} D_\theta(x).$$
New Results
Orbit: graphical point of view

The graph of a function is a disjoint union of components of graph with one cycle.
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The orbit is a line ending at the root (we can consider a cycle as a root).
Orbit: graphical point of view

The graph of a function is a disjoint union of components of graph with one cycle.

The orbit is a line ending at the root (we can consider a cycle as a root).

The first orbit in an original component ends on the circle. When we erase an orbit from a component, the remainder is a disjoint union of components of rooted tree.
**Typical graph of function**

a giant component

rendered by Quisquater and Delescaille 1988.
Descent: graphical point of view

Erase an orbit from each component each time.

The number of descents of $x$ is the time right before $x$ being erased.
Erase an orbit from each component each time.

The number of descents of $x$ is the time right before $x$ being erased.

The number of descents only depends on the decomposition of a graph (independent of the order of components).
Example for numbers of descents

red: 0
Example for numbers of descents

red: 0  green: 1
Example for numbers of descents

red: 0    green: 1    brown: 2
Example for numbers of descents

red: 0  green: 1  brown: 2  purple: 3
Now, the problem is to find a decomposition of a tree that minimizes the number of descents.
Bottom-up construction

Lemma 1 Let $T$ be a rooted tree:

\[ \text{r} \]

\[ T_1 \quad T_2 \quad \ldots \quad T_k \]

Give each $T_i$ a decomposition. Let $D_{\text{old}}(x)$ be the number of descents of $x$. 
**Lemma 1** Let $T$ be a rooted tree:

$$
\begin{array}{c}
\text{r} \\
\text{T}_1 & \text{T}_2 & \ldots & \text{T}_k
\end{array}
$$

Give each $T_i$ a decomposition. Let $D_{old}(x)$ be the number of descents of $x$.

Now, extend the decomposition to $T$: $r$ and the root of $T_a$ in a same orbit. Then $D_{new}(r) = 0$ and

$$
D_{new}(x) = \begin{cases} 
D_{old}(x), & \text{if } x \in T_a, \\
D_{old}(x) + 1, & \text{otherwise.}
\end{cases}
$$
Which one is the best connection?

Choose the subtree which has the largest number of descents.
Which one is the best connection?

Choose the subtree which has the largest number of descents. If there are more than one subtree with the same largest number of descents, then choose anyone of them. By Lemma 1, the maximal number of descents increases 1 in this case.
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The function characterized by above rule has been studied in other areas.
The Horton-Strahler number

It’s originally used to classify river system. Later also appeared in computer science as register function.
The Horton-Strahler number

It’s originally used to classify river system. Later also appeared in computer science as register function.

It was defined on binary trees. Now, we extend the definition to rooted trees with branching factor $\geq 1$:

$$S(r) = \begin{cases} 
0 & \text{if } r \text{ has no child}, \\
M(r) & \text{if only one child with } S(u) = M(r), \\
M(r) + 1 & \text{otherwise}, 
\end{cases}$$

where $M(r) = \max\{S(u) : u \text{ is a child of } r\}$. 
Example of the Horton-Strahler number
The least maximal number of descents

Find a decomposition with minimal $\max_{x \in V} D_\theta(x)$. 
The least maximal number of descents

Find a decomposition with minimal \( \max_{x \in V} D_\theta(x) \).

**Theorem 1**

The optimal orbit decomposition is constructed componentwise by the orbits, a path top-down from the root pass nodes with the largest Horton-Strahler number locally.
The least maximal number of descents

Find a decomposition with minimal $\max_{x \in V} D_\theta(x)$.

Theorem 1

The optimal orbit decomposition is constructed componentwise by the orbits, a path top-down from the root pass nodes with the largest Horton-Strahler number locally.

Moreover, the least maximal number of descents is equal to the largest Horton-Strahler number in the tree.
Horton-Strahler number for binary tree

Let $T$ be a binary tree with $N$ nodes. Let $S_N$ be the Horton-Strahler number of the root of $T$. 
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$$\mathbb{E}S_N = \log_4 N + O(1).$$
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Recently, Auber et al. (2004) studied a different extension.
The worst case

– the tree with the largest least maximal number of descents
The worst case

– the tree with the largest least maximal number of descents

The complete binary tree is the worst case.
The worst case
– the tree with the largest least maximal number of descents

The complete binary tree is the worst case.

Let $d$ be the number of levels. Then the maximal number of descents is $d - 1$ and the total number of nodes is $2^d - 1 = N$.

Thus, generally, the least maximal number of descents is bounded by $\log_2(N + 1) - 1$. 
To minimize average number of descents

- find a decomposition with minimal $\sum_{x \in V} D_\theta(x)$. 
To minimize average number of descents

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Which one is the best child to be connected with?
To minimize average number of descents

– find a decomposition with minimal $\sum_{x \in V} D_\theta(x)$.

Which one is the best child to be connected with?
The one with the most offsprings!
To minimize average number of descents

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Which one is the best child to be connected with?
The one with the most offsprings!

**Orbit decomposition:**

Compute for each node the number of subtree size.

For each component, starting from the root we choose the path of the orbit through the nodes with the largest number of subtree size.
Example: The number is the size of the subtree rooted at the node.
Compare to greedy orbit decomposition

The main difference is the rule to connect the nodes and their children:

```
     r
    /|
   / |  
  /  |   
 T1  T2  ...  Tk
```
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The main difference is the rule to connect the nodes and their children:

```
    r
   /|
  /  
 T_1 T_2 ...
   |
   |
 T_k
```

**greedy** the tallest subtree (with most levels)
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```
   r
  / \
T_1  T_2  ...  T_k
```

greedy the tallest subtree (with most levels)
least max the subtree with largest H-S number
Compare to greedy orbit decomposition

The main difference is the rule to connect the nodes and their children:

```
            r
           /|
          / |`
         /  |  
        T1  T2  ...  Tk
```

greedy the tallest subtree (with most levels)
least max the subtree with largest H-S number
least avg the heaviest subtree (with most nodes)
Theorem 2

Let $T$ be a rooted tree with $N$ nodes:

```
  r
 /   \
/     \
T_1   T_2  ...  T_k
```

Let $m_i$ be the total number of descents in $T_i$ and $n_i$ be the total number of nodes in $T_i$. 
Theorem 2

Let $T$ be a rooted tree with $N$ nodes:

$$
\begin{array}{c}
\text{r} \\
/ & / & / & / \\
T_1 & T_2 & \ldots & T_k
\end{array}
$$

Let $m_i$ be the total number of descents in $T_i$ and $n_i$ be the total number of nodes in $T_i$.

Then the best choice is the $T_i$ with the maximal $n_i$ and the total number of descents of $T$ is

$$
\sum_{i=1}^{k} m_i + (N - 1 - \max\{n_1, \ldots, n_k\}).
$$
The worst case

The worst case is also complete binary tree.
The worst case

The worst case is also complete binary tree.

Here the number indicates the number of descents.
Let $c_k$ be the total number of descents at level $k$.

Then $c_1 = 0$ and $c_k = 2c_{k-1} + 2^{k-2}$ for $k \geq 2$. (Why?)
Let $c_k$ be the total number of descents at level $k$. Then $c_1 = 0$ and $c_k = 2c_{k-1} + 2^{k-2}$ for $k \geq 2$. (Why?)

Thus $c_k = (k - 1)2^{k-2}$ and the average number of descents in this case is

$$\frac{\sum_{k=1}^{d} (k - 1)2^{k-2}}{\sum_{k=1}^{d} 2^{k-1}} = \frac{d}{2} - 1 + \frac{d}{2^d - 1},$$

where $d$ is the number of levels.
Conclusion

1. For any function $f$ on $\{0, 1, \ldots, N - 1\}$, the iteration $f^m(x)$ for each $m$ and $x$ can be computed in time $O(\log N)$ by using preprocessed information. The preprocess need linear space and linear time.
1. For any function $f$ on $\{0, 1, ..., N - 1\}$, the iteration $f^m(x)$ for each $m$ and $x$ can be computed in time $O(\log N)$ by using preprocessed information. The preprocess need linear space and linear time.

2. Open problems: analysis of the Horton-Strahler number and the average number of descents for random functions.
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2. Open problems: analysis of the Horton-Strahler number and the average number of descents for random functions.


4. Another interpretation of our work. (next page)
Another interpretation of our work

**Transportation system**

Consider orbit as bus line and descent as transfer.

Our work is to design a bus system that minimizes the number of transfers.
Thank You!