Graphical models, from graphs to trees (and back)

Andrea Montanari

Stanford University

April 14, 2008
Outline

1. What is this talk about, and why should one care

2. Uniform decorrelation

3. Non-Uniform decorrelation

4. Trees vs graphs: from reconstruction to pure states
Outline

1. What is this talk about, and why should one care
2. Uniform decorrelation
3. Non-Uniform decorrelation
4. Trees vs graphs: from reconstruction to pure states
Outline

1. What is this talk about, and why should one care
2. Uniform decorrelation
3. Non-Uniform decorrelation
4. Trees vs graphs: from reconstruction to pure states
Outline

1. What is this talk about, and why should one care

2. Uniform decorrelation

3. Non-Uniform decorrelation

4. Trees vs graphs: from reconstruction to pure states
What is this talk about, and why should one care
Graphical models

$G = (V, E), \ V = [n], \ \underline{x} = (x_1, \ldots, x_n), \ x_i \in \mathcal{X}$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(ij) \in G} \psi_{ij}(x_i, x_j).$$
1. $G$ has bounded degree (on average).

2. $G$ has girth $\ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. $G$ is random.
1. $G$ has bounded degree (on average).

2. $G$ has girth $\ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. $G$ is random.
1. $G$ has bounded degree (on average).

2. $G$ has girth $\ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. $G$ is random.
Example 1: \( q \)-coloring

\[ G = (V, E) \] graph.

\[ \mathbf{x} = (x_1, x_2, \ldots, x_n), \ x_i \in \{1, \ldots, q\} \] variables
Example 1: $q$-coloring

$G = (V, E)$ graph.

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{1, \ldots, q\}$ variables
Uniform measure over proper colorings

\[
\mu(x) = \frac{1}{Z} \prod_{(i,j) \in E} \psi(x_i, x_j), \quad \psi(x, y) = \mathbb{I}(x \neq y).
\]
Example 2: $k$-satisfiability

$n$ variables: $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0, 1\}$

$m$ $k$-clauses

$$(x_1 \lor \overline{x_5} \lor x_7) \land (x_5 \lor x_8 \lor \overline{x_9}) \land \cdots \land (\overline{x_{66}} \lor \overline{x_{21}} \lor \overline{x_{32}})$$
Uniform measure over solutions

\[ F = \cdots \land (x_{i_1(a)} \lor \overline{x_{i_2(a)}} \lor \cdots \lor x_{i_k(a)}) \land \cdots \]
\[ \mu(\mathbf{x}) = \frac{1}{Z} \prod_{a=1}^{M} \psi_a(x_{i_1(a)}, \cdots, x_{i_k(a)}) \]
Many other examples

- Communications/signal processing (technologically relevant)
- ...

Probability, physics, computer science, information theory, ...
Many other examples

- Communications/signal processing (technologically relevant)
- ...

Probability, physics, computer science, information theory, …
Emerging conceptual unity:

- Approximation of sparse graph models by trees.
- ...

[cf. Aldous’ local weak convergence]
Emerging conceptual unity:

- Approximation of sparse graph models by trees.
- ... 

[cf. Aldous’ local weak convergence]
Is this motivating enough? (a personal view)

Emerging conceptual unity:

- Approximation of sparse graph models by trees.
- ...

[cf. Aldous’ local weak convergence]
Emerging conceptual unity:

- Approximation of sparse graph models by trees.
- ... 

[cf. Aldous’ local weak convergence]
Uniform decorrelation
Each clause is uniformly random among the $2^k \binom{n}{k}$ possible ones.

$n \to \infty$, $m = \alpha n$
Each clause is uniformly random among the $2^k \binom{n}{k}$ possible ones.

$n \to \infty$, $m = \alpha n$
Number of ‘good’ truth assignments

\[ Z_n(\beta) = \sum_x \exp \{-2\beta \# \text{[clauses violated by } x]\} \]

Theorem (Montanari, Shah, 2007)

If \( \alpha < \alpha_{\text{u}}(k) = (2 \log k) k^{-1} \left[ 1 + o_k(1) \right] \) then

\[ \frac{1}{n} \log Z_n(\beta) \xrightarrow{a.s.} \phi(\alpha, \beta), \]

where \ldots
\[ \phi(\alpha, \beta) = -k\alpha \mathbb{E} \log[1 + \tanh h \tanh u] + \alpha \mathbb{E} \log \left\{ 1 - \frac{1}{2^k}(1 - e^{-\beta}) \prod_{i=1}^{k}(1 - \tanh h_i) \right\} + \]

\[ + \mathbb{E} \log \left\{ \prod_{i=1}^{l_+}(1 + \tanh u_i^+) \prod_{i=1}^{l_-}(1 - \tanh u_i^-) + \prod_{i=1}^{l_+}(1 - \tanh u_i^+) \prod_{i=1}^{l_-}(1 + \tanh u_i^-) \right\} , \]

and \( h, u \) are the unique solution of

\[ h \overset{\text{d}}{=} \sum_{a=1}^{l_+} u_a - \sum_{b=1}^{l_-} u'_b , \quad u \overset{\text{d}}{=} f_\beta(h_1, \ldots, h_{k-1}). \]
Proof: 1st preliminary remark

Sufficient to prove

\[ \frac{1}{n} \mathbb{E} \log Z_n(\beta) \longrightarrow \phi(\alpha, \beta), \]
The tree ensemble $T(\ell)$, $T(\infty)$

\[ \text{Poisson}(k\alpha/2) \quad \text{Poisson}(k\alpha/2) \quad \circ \]
The tree ensemble $T(\ell), T(\infty)$

$$\text{Poisson}(k\alpha/2) \quad \text{Poisson}(k\alpha/2)$$
Proof: 2nd preliminary remark

The tree ensemble $T(\ell), T(\infty)$

$$
\text{Poisson}(k\alpha/2) \quad \text{Poisson}(k\alpha/2)
$$

Andrea Montanari

Graphical models, from graphs to trees (and back)
The tree ensemble $T(\ell), T(\infty)$

$$\text{Poisson}(k\alpha/2) \quad \text{Poisson}(k\alpha/2)$$
The tree ensemble $T(\ell), T(\infty)$

\[ \text{Poisson}(k\alpha/2) \]

\[ \text{Poisson}(k\alpha/2) \]
1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1}\mathbb{E}\log Z_n(0)$.

2. Write $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations. ????
1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1} \mathbb{E} \log Z_n(0)$.

2. Write $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations.
Proof strategy

1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1} \mathbb{E} \log Z_n(0)$.

2. Write $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations. ????
Local expectations

Variables $x_i \in \{0, 1\}$

Clauses, e.g. $(x_5 \lor x_7 \lor \overline{x_9} \lor \overline{x_{10}})$

$$\mu(x) = \frac{1}{Z_n(\beta)} \prod_{a=1}^{m} \psi_{\beta,a}(x_{i_1(a)}, \ldots, x_{i_k(a)})$$

$$\psi_{\beta,a}(\cdots) = \begin{cases} 1 & \text{if clause } a \text{ is satisfied} \\ e^{-2\beta} & \text{otherwise} \end{cases}$$
\[
\frac{d}{d\beta} \log Z_n(\beta) = -2 \sum_{a=1}^{M} \mu(\text{clause } a \text{ is not satisfied})
\]
Proof strategy

1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1} \mathbb{E} \log Z_n(0)$.

2. Express $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations.

3. Prove that local expectations on $G$ converge to expectations on $T(\infty)$.

4. Show $\frac{d\phi(\alpha, \beta)}{d\beta}$ is equal to the same expectations on $T(\infty)$. 
1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1} \mathbb{E} \log Z_n(0)$.

2. Express $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations.

3. Prove that local expectations on $G$ converge to expectations on $T(\infty)$.

4. Show $\frac{d\phi(\alpha, \beta)}{d\beta}$ is equal to the same expectations on $T(\infty)$. 
Proof strategy

1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1}E \log Z_n(0)$.

2. Express $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations.

3. Prove that local expectations on $G$ converge to expectations on $T(\infty)$.

4. Show $\frac{d\phi(\alpha, \beta)}{d\beta}$ is equal to the same expectations on $T(\infty)$. 
Proof strategy

1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1}E \log Z_n(0)$.

2. Express $\frac{d}{d\beta} \log Z_n(\beta)$ in terms of local expectations.

3. Prove that local expectations on $G$ converge to expectations on $T(\infty)$.

4. Show $\frac{d\phi(\alpha, \beta)}{d\beta}$ is equal to the same expectations on $T(\infty)$. 
Convergence to tree values

\[ T(\infty) \] infinite k-SAT tree

\[ T(\ell) \] first \( \ell \) generations

\[ \mu^{\ell,z}(\cdot) \] Boltzmann measure on \( T(\ell) \) boundary condition \( z \)

\[ \mu_{r}^{\ell,z}(\cdot) \] root variable marginal
Convergence to tree values

$T(\infty)$ infinite $k$-SAT tree

$T(\ell)$ first $\ell$ generations

$\mu^{\ell,z}(\cdot)$ Boltzmann measure on $T(\ell)$ boundary condition $z$

$\mu_{r}^{\ell,z}(\cdot)$ root variable marginal
Convergence to tree values

- $T(\infty)$: infinite $k$-SAT tree
- $T(\ell)$: first $\ell$ generations
- $\mu^{\ell,z}(\cdot)$: Boltzmann measure on $T(\ell)$ boundary condition $z$
- $\mu_r^{\ell,z}(\cdot)$: root variable marginal

Andrea Montanari
Graphical models, from graphs to trees (and back)
Uniform decorrelation (Gibbs measure uniqueness)

\[ \mathbb{E} \left\{ \max_{z(1), z(2)} \| \mu_r^{t,z(1)}(\cdot) - \mu_r^{t,z(2)}(\cdot) \|_{TV} \right\} \to 0. \]

'Easy' sufficient condition

True only at very small \( \alpha \)

(\( \alpha_u(k) \approx (2 \log k)/k \), conjecture up to \( \sim 2^k \log 2 \))
Uniform decorrelation (Gibbs measure uniqueness)

\[ \mathbb{E} \left\{ \max_{z(1), z(2)} \| \mu_{t, z(1)}^r (\cdot) - \mu_{t, z(2)}^r (\cdot) \|_{TV} \right\} \to 0. \]

‘Easy’ sufficient condition

True only at very small \( \alpha \)

\( (\alpha_u(k) \simeq (2 \log k)/k, \text{conjecture up to } \simeq 2^k \log 2) \)
Uniform decorrelation (Gibbs measure uniqueness)

\[ \mathbb{E} \left\{ \max_{z(1), z(2)} \left\| \mu^{t,z(1)}_{r}(\cdot) - \mu^{t,z(2)}_{r}(\cdot) \right\|_{TV} \right\} \to 0. \]

'Easy' sufficient condition

True only at very small \( \alpha \)

\( (\alpha_u(k) \simeq (2 \log k)/k, \text{ conjecture up to } \simeq 2^k \log 2) \)
Non-Uniform decorrelation
Ferromagnetic Ising model
Ferromagnetic Ising model

\[ G_n = (V_n \equiv [n], E_n) \]
\[ x_i \in \{+1, -1\} \]

\[ \mu(x) = \frac{1}{Z_n(\beta, B)} \exp \left\{ \beta \sum_{(ij) \in E_n} x_i x_j + B \sum_i x_i \right\} \]

Ferromagnetic Ising model

\[ G_n = (V_n \equiv [n], E_n) \]

\[ x_i \in \{+1, -1\} \]

\[ \mu(x) = \frac{1}{Z_n(\beta, B)} \exp \left\{ \beta \sum_{(ij) \in E_n} x_i x_j + B \sum_i x_i \right\} \]

Theorem (Dembo, Montanari 2008)

If $G_n$ converges locally to $T(P)$, then

$$\frac{1}{n} \log Z_n(\beta, B) \xrightarrow{a.s.} \phi(P, \beta, B).$$

where...
Theorem (Dembo, Montanari 2008)

If $G_n$ converges locally to $T(P)$, then

$$
\frac{1}{n} \log Z_n(\beta, B) \xrightarrow{a.s.} \phi(P, \beta, B).
$$

where...
For $B \geq 0$, let $\theta \equiv \tanh \beta$, $h^{(0)} > 0$, and

$$h^{(\ell+1)} \overset{d}{=} \tanh \left\{ B + \sum_{i=1}^{K-1} \text{atanh}(\theta h_i^{(\ell)}) \right\},$$

Then $h^{(\ell)} \overset{d}{\to} h^*$ and

$$\phi(P, \beta, B) \equiv \log \cosh B + \frac{P}{2} \log \cosh \beta - \frac{P}{2} \mathbb{E} \log(1 + \theta h_1^* h_2^*) +$$

$$+ \mathbb{E} \log \left\{ (1 + \tanh B) \prod_{i=1}^{L} (1 + \theta h_i^*) + (1 - \tanh B) \prod_{i=1}^{L} (1 - \theta h_i^*) \right\}.$$
For $B \geq 0$, let $\theta \equiv \tanh \beta$, $h^{(0)} > 0$, and

$$h^{(\ell+1)} \overset{d}{=} \tanh \left\{ B + \sum_{i=1}^{K-1} \text{atanh}(\theta h^{(\ell)}_{i}) \right\},$$

Then $h^{(\ell)} \overset{d}{\to} h^*$ and

$$\phi(P, \beta, B) \equiv \log \cosh B + \frac{P}{2} \log \cosh \beta - \frac{P}{2} \mathbb{E} \log (1 + \theta h_1^* h_2^*) +$$

$$+ \mathbb{E} \log \left\{ (1 + \tanh B) \prod_{i=1}^{L} (1 + \theta h_i^*) + (1 - \tanh B) \prod_{i=1}^{L} (1 - \theta h_i^*) \right\}.$$
\( T(P, \ell) \)
$T(P, \ell)$
$T(P, \ell)$
$T(P, \ell)$

Andrea Montanari

Graphical models, from graphs to trees (and back)
$T(P, \ell)$

Diagram:

- $P_k$
- $\rho_k$
- Further nodes and branches representing a tree structure.
\[ \begin{align*}
P & \equiv \{ P_k \}_{k \geq 0} \quad \text{Degree distribution} \\
T(P, \ell) & \quad \ell\text{-generations Galton-Watson tree} \\
B_i(\ell) & \quad \text{Ball of radius } \ell \text{ around uniformly random node}
\end{align*} \]

**Definition**

\( G_n \) converges locally to \( T(P) \) if uniform bound on the edge number distribution and, for any \( \ell \),

\[ B_i(\ell) \text{ converges in distribution to } T(P, \ell). \]
For $\beta > \beta_c \equiv \text{atanh}(1/\bar{\rho})$

$$\lim_{B \to 0^+} \lim_{n \to \infty} E_i \langle x_i \rangle = - \lim_{B \to 0^-} \lim_{n \to \infty} E_i \langle x_i \rangle > 0$$
\[ z = (+1, +1, \ldots, +1) \quad \Rightarrow \quad \lim_{\ell \to \infty} \langle x_r \rangle_\ell > 0 \]
\[ z = (-1, -1, \ldots, -1) \quad \Rightarrow \quad \lim_{\ell \to \infty} \langle x_r \rangle_\ell < 0 \]
A different case: Ising spin glass

\[ \mu(x) = \frac{1}{Z} \exp \left\{ \beta \sum_{(ij) \in E_n} J_{ij} x_i x_j + B \sum_i x_i \right\} \]

\( J_{ij} \in \{+1, -1\} \) uniformly random

[Viana, Bray 1985]
[Other approaches: Talagrand 2001, Guerra, Toninelli 2003]
[Approximation by trees: Montanari, Gerschenfeld, in preparation]
Trees vs graphs: from reconstruction to pure states
Alice, Bob and $G$

Andrea Montanari

Graphical models, from graphs to trees (and back)
Alice samples a proper coloring (uniformly)…
... and hides a ball $B(\text{root}, t)$
Bob...
Andrea Montanari
Graphical models, from graphs to trees (and back)

...guesses right!
The problem

Does Bob have a chance?
Formally

\[ X = \{X_i : i \in V\} \text{ uniformly random proper coloring.} \]

\[ \mu_U(\cdot | G) \] distribution of \( X_U \equiv \{X_i : i \in U \subseteq V\} \)

\[ \overline{B}(r, t) = \{i \in V : d(i, r) \geq t\} \]

**Definition**

The reconstruction problem is solvable for the sequence of random rooted graphs \( G_n = (V_n = [n], E_n) \) if for some \( \varepsilon > 0 \),

\[ \|\mu_{r, \overline{B}(r, t)}(\cdot, \cdot | G_n) - \mu_r(\cdot | G_n)\mu_{\overline{B}(r, t)}(\cdot | G_n)\|_{TV} \geq \varepsilon , \]

with positive probability (bounded away from 0 as \( n \to \infty \)).
When $G =$Tree


→ Evans, Kenyon, Peres, Schulman (2000): Ising on general trees

→ Mossel, Peres (2003): Non binary variables


→ Chayes et al. (2006): Asymmetric Ising.
Pure states decomposition in $q$-COL

\[
\gamma_d(q) \quad \gamma_c(q) \quad \gamma_s(q)
\]

[Biroli, Monasson, Weigt 2001]
[Mézard, Parisi, Zecchina 2003]
[Achlioptas, Ricci 2007]
[Krzakala, Montanari, Ricci, Semerjian, Zdeborova 2007]
Conjecture (Mézard, Montanari, 05)

\[ \gamma_d(q) = \]

= Multiple pure states

= Graph reconstruction threshold

= Tree reconstruction threshold
A general sufficient condition

Theorem (Gerschenfeld, Montanari, 2007)

If $\mu(\cdot|G)$ is roughly spherical then

$$\text{Graph solvable} \iff \text{Tree solvable.}$$

If $\mu(\cdot|G)$ is not roughly spherical then

$$\text{Graph reconstruction is solvable}$$
Theorem (Gerschenfeld, Montanari, 2007)

If $\mu(\cdot | G)$ is \textit{roughly spherical} then

\[ \text{Graph solvable} \iff \text{Tree solvable}. \]

If $\mu(\cdot | G)$ is \textit{not} roughly spherical then

\[ \text{Graph reconstruction is solvable} \]
A general sufficient condition

Theorem (Gerschenfeld, Montanari, 2007)

If \( \mu(\cdot | G) \) is \textit{roughly spherical} then

Graph solvable \iff Tree solvable.

If \( \mu(\cdot | G) \) is \textit{not} roughly spherical then

Graph reconstruction is solvable
Roughly spherical???

\[ X_i \in \{0, 1\}. \]
\[ X^{(1)} = \{X^{(1)}_i\}, \ X^{(2)} = \{X^{(2)}_i\} \] independent with distribution \( \mu(\cdot | G_n) \)

\[ \mu(\cdot | G_n) \] is roughly spherical if \( d(X^{(1)}, X^{(2)}) \approx n/2 \) with high probability.
$X_i \in \{0, 1\}$.

$X^{(1)} = \{X_i^{(1)}\}$, $X^{(2)} = \{X_i^{(2)}\}$ independent with distribution

$\mu(\cdot | G_n)$

$\mu(\cdot | G_n)$ is *roughly spherical* if $d(X^{(1)}, X^{(2)}) \approx n/2$ with high probability.
Theorem

1. *q*-coloring, $\gamma < (q - 1) \log(q - 1)$: roughly spherical.
2. Ising spin glass $2\gamma(\tanh \beta)^2 < 1$: roughly spherical.
3. Ising ferromagnet: not roughly spherical.

[Tree reconstruction threshold]

2. Evans, Kenyon, Peres, Schulman 2000
3. Tree ≠ Graph]
Can you check this condition?

**Theorem**

1. *q-coloring*, $\gamma < (q - 1) \log(q - 1)$: roughly spherical.
2. *Ising spin glass* $2\gamma(tanh \beta)^2 < 1$: roughly spherical.
3. *Ising ferromagnet*: not roughly spherical

[Tree reconstruction threshold]

2. Evans, Kenyon, Peres, Schulman 2000
3. Tree $\neq$ Graph
Conclusion

Combinatorics/Probability problems on random sparse graphs.

Unifying approach: approximation by trees.

Naturally leads to analytic questions.
Combinatorics/Probability problems on random sparse graphs.

Unifying approach: approximation by trees.

Naturally leads to analytic questions.
Combinatorics/Probability problems on random sparse graphs.

Unifying approach: approximation by trees.

Naturally leads to analytic questions.