Graphical models, from graphs to trees (and back)

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April 14, 2008

Outline

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1. What is this talk about, and why should one care
2. Uniform decorrelation
3. Non-Uniform decorrelation
4. Trees vs graphs: from reconstruction to pure states

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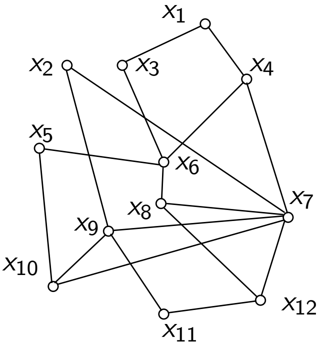
What is this talk about, and why should one care

Graphical models



$$G = (V, E),\ V = [n], \qquad \underline{x} = (x_1, \dots, x_n),\ x_i \in \mathcal{X}$$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(ij) \in G} \psi_{ij}(x_i, x_j)\,.$$

This talk

1. G has bounded degree (on average).

2. G has girth $\ell(n) \to \infty$ (apart from $o(n)$ vertices).

3. G is random.

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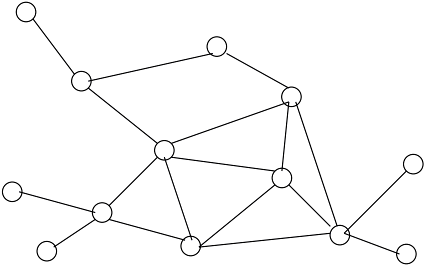
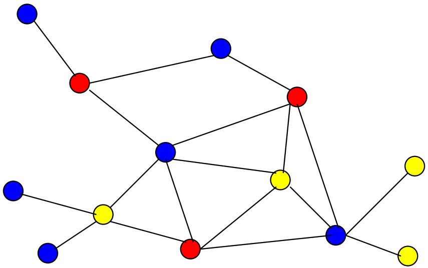
Example 1: q-coloring



$G = (V, E)$ graph.

$\underline{x} = (x_1, x_2, \dots, x_n)$, $x_i \in \{1, \dots, q\}$ variables

Example 1: q-coloring



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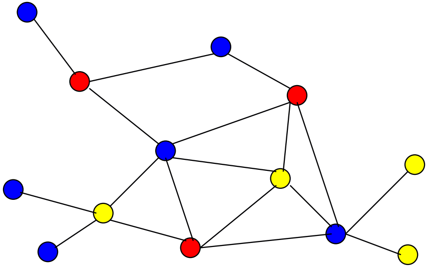
Uniform measure over proper colorings



$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(i,j)\in E} \psi(x_i, x_j)\,, \qquad\qquad \psi(x,y) = \mathbb{I}(x \neq y)\,.$$

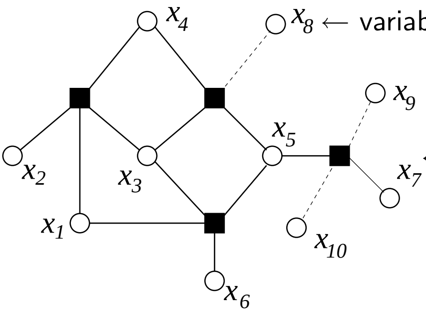
Example 2: k-satisfiability

n variables: $\underline{x} = (x_1, x_2, \dots, x_n)$, $x_i \in \{0, 1\}$

m k-clauses

$$(x_1 \vee \overline{x_5} \vee x_7) \wedge (x_5 \vee x_8 \vee \overline{x_9}) \wedge \cdots \wedge (\overline{x_{66}} \vee \overline{x_{21}} \vee \overline{x_{32}})$$

Uniform measure over solutions



$\leftarrow$ variables $x_i \in \{0, 1\}$

$\leftarrow$ clauses, e.g. $(x_5 \vee x_7 \vee \overline{x_9} \vee \overline{x_{10}})$

$$F = \cdots \wedge \underbrace{(x_{i_1(a)} \vee \overline{x}_{i_2(a)} \vee \cdots \vee x_{i_k(a)})}_{a\text{-th clause}} \wedge \cdots$$

$$\mu(\underline{x}) = \frac{1}{Z} \prod_{a=1}^{M} \psi_a(x_{i_1(a)}, \ldots, x_{i_k(a)})$$

Many other examples

- Communications/signal processing (technologically relevant)
- . . .

Probability, physics, computer science, information theory,. . .

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Probability, physics, computer science, information theory,...

Is this motivating enough? (a personal view)

Emerging conceptual unity:

- Approximation of sparse graph models by trees.
- ...

[cf. Aldous' local weak convergence]

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Uniform decorrelation

Random k-satisfiability

Each clause is uniformly random among the $2^k\binom{n}{k}$ possible ones.

$$n \longrightarrow \infty,\ m = \alpha n$$

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$$n \longrightarrow \infty, \; m = \alpha n$$

Number of 'good' truth assignments

$$Z_n(\beta) = \sum_{\underline{x}} \exp\Big\{ -2\beta \, \#[\text{clauses violated by } \underline{x}] \Big\}$$

Theorem (Montanari, Shah, 2007)

If $\alpha < \alpha_{\rm u}(k) = (2 \log k) k^{-1} \, [1 + o_k(1)]$ *then*

$$\frac{1}{n} \log Z_n(\beta) \xrightarrow{a.s.} \phi(\alpha, \beta) \, ,$$

where ...

. . . where. . .

$$\phi(\alpha,\beta) = -k\alpha\mathbb{E}\log[1+\tanh h\tanh u] + \alpha\mathbb{E}\log\left\{1-\frac{1}{2^k}(1-e^{-\beta})\prod_{i=1}^{k}(1-\tanh h_i)\right\} + \\ +\mathbb{E}\log\left\{\prod_{i=1}^{\ell_+}(1+\tanh u_i^+)\prod_{i=1}^{\ell_-}(1-\tanh u_i^-)+\prod_{i=1}^{\ell_+}(1-\tanh u_i^+)\prod_{i=1}^{\ell_-}(1+\tanh u_i^-)\right\},$$

and h, u are the *unique* solution of

$$h \stackrel{\mathrm{d}}{=} \sum_{a=1}^{l_+} u_a - \sum_{b=1}^{l_-} u'_b\,, \qquad u \stackrel{\mathrm{d}}{=} f_\beta(h_1,\dots,h_{k-1})\,.$$

[Conjectured by Monasson, Zecchina 1999]

Proof: 1st preliminary remark

Sufficient to prove

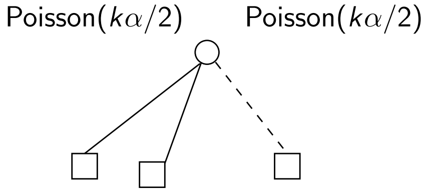
$$\frac{1}{n}\mathbb{E}\log Z_n(\beta) \longrightarrow \phi(\alpha, \beta)\,,$$

Proof: 2nd preliminary remark

The tree ensemble $\mathsf{T}(\ell)$, $\mathsf{T}(\infty)$

$\text{Poisson}(k\alpha/2)$ $\quad$ $\text{Poisson}(k\alpha/2)$

The tree ensemble $\mathsf{T}(\ell)$, $\mathsf{T}(\infty)$



Proof: 2nd preliminary remark

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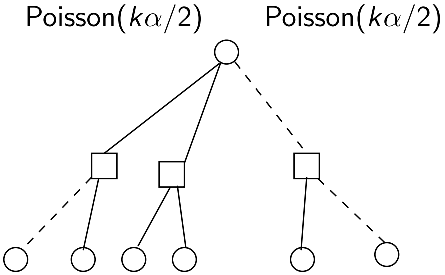

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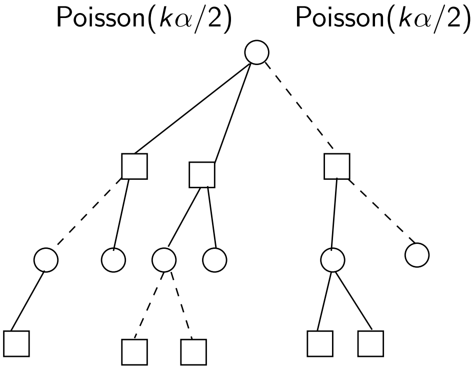

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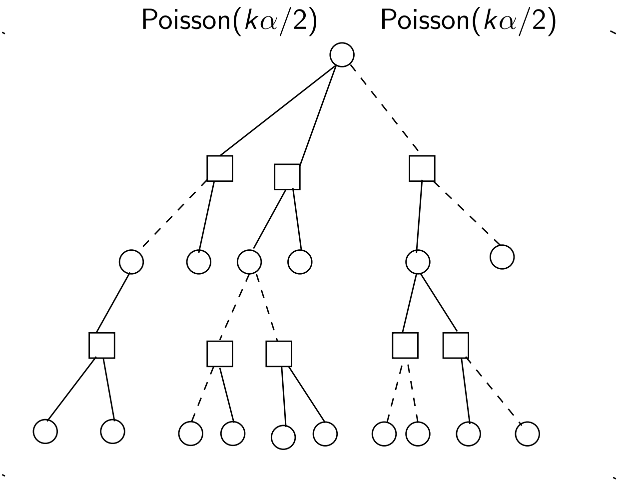

Proof strategy

1. Check it for $\beta = 0$: $\phi(\alpha, 0) = \log 2 = n^{-1}\mathbb{E} \log Z_n(0)$.
2. Write $\frac{\mathrm{d}}{\mathrm{d}\beta} \log Z_n(\beta)$ in terms of local expectations. ????

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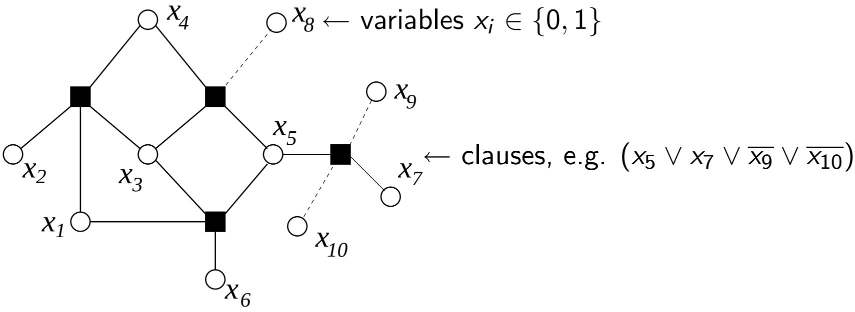
Local expectations ????



$$\mu(\underline{x}) = \frac{1}{Z_n(\beta)} \prod_{a=1}^{m} \psi_{\beta,a}(x_{i_1(a)}, \ldots, x_{i_k(a)})$$

$$\psi_{\beta,a}(\cdots) = \begin{cases} 1 & \text{if clause } a \text{ is satisfied} \\ e^{-2\beta} & \text{otherwise} \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \log Z_n(\beta) = -2 \sum_{a=1}^{M} \mu(\text{clause } a \text{ is not satisfied})$$

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3. Prove that local expectations on G converge to expectations on $\mathsf{T}(\infty)$.
4. Show $\frac{\mathrm{d}\phi(\alpha,\beta)}{\mathrm{d}\beta}$ is equal to the same expectations on $\mathsf{T}(\infty)$.

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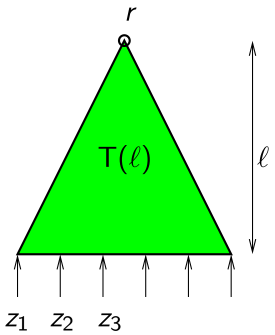
Convergence to tree values



$T(\infty)$	infinite k-SAT tree
$T(\ell)$	first ℓ generations
$\mu^{\ell,z}(\,\cdot\,)$	Boltzmann measure on $T(\ell)$ boundary condition z
$\mu_r^{\ell,z}(\,\cdot\,)$	root variable marginal

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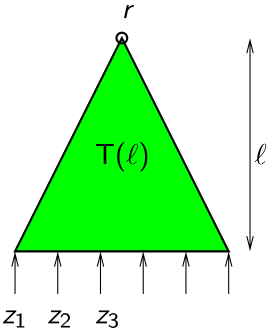

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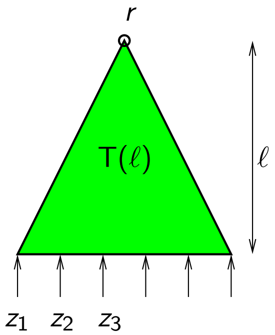

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$$\mathbb{E}\left\{\max_{z(1),z(2)} ||\mu_r^{t,z(1)}(\,\cdot\,) - \mu_r^{t,z(2)}(\,\cdot\,)||_{\mathrm{TV}}\right\} \to 0\,.$$

'Easy' sufficient condition

True only at very small α
$(\alpha_{\mathrm{u}}(k) \simeq (2\log k)/k$, conjecture up to $\simeq 2^k \log 2)$

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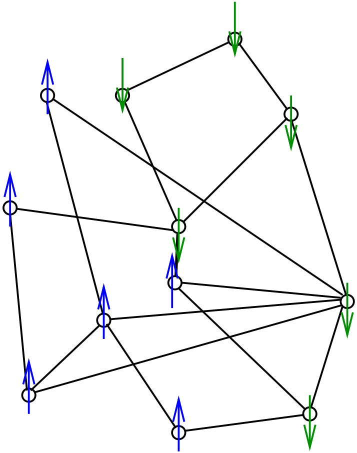
Non-Uniform decorrelation

Ferromagnetic Ising model



Ferromagnetic Ising model

$G_n = (V_n \equiv [n], E_n)$

$x_i \in \{+1, -1\}$

$$\mu(\underline{x}) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(ij)\in E_n} x_i x_j + B \sum_i x_i\right\}$$

[Johnston, Plecháč 1998, Leone et al 2004]

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Free energy density

Theorem (Dembo, Montanari 2008)

If G_n converges locally to $\mathsf{T}(P)$, then

$$\frac{1}{n}\log Z_n(\beta, B) \xrightarrow{a.s} \phi(P, \beta, B)\,.$$

where...

Free energy density

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$\phi(P, \beta, B)$

For $B \geq 0$, let $\theta \equiv \tanh\beta$, $h^{(0)} > 0$, and

$$h^{(\ell+1)} \stackrel{\rm d}{=} \tanh\left\{B + \sum_{i=1}^{K-1} \mathrm{atanh}(\theta\, h_i^{(\ell)})\right\},$$

Then $h^{(\ell)} \xrightarrow{\rm d} h^*$ and

$$\phi(P,\beta,B) \equiv \log\cosh B + \frac{\overline{P}}{2}\log\cosh\beta - \frac{\overline{P}}{2}\,\mathbb{E}\log(1+\theta h_1^* h_2^*) +$$
$$+\mathbb{E}\,\log\left\{(1+\tanh B)\prod_{i=1}^{L}(1+\theta h_i^*) + (1-\tanh B)\prod_{i=1}^{L}(1-\theta h_i^*)\right\}.$$

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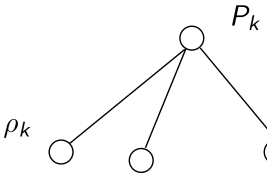
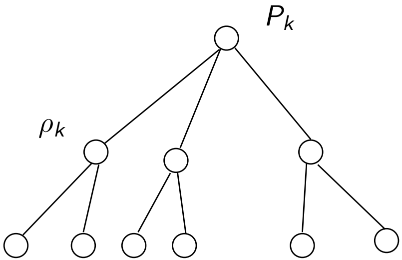
P_k

ρ_k


P_k
ρ_k

$\mathsf{T}(P, \ell)$



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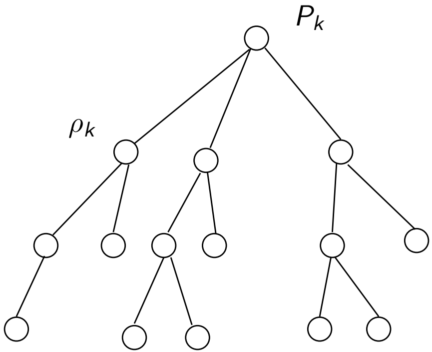

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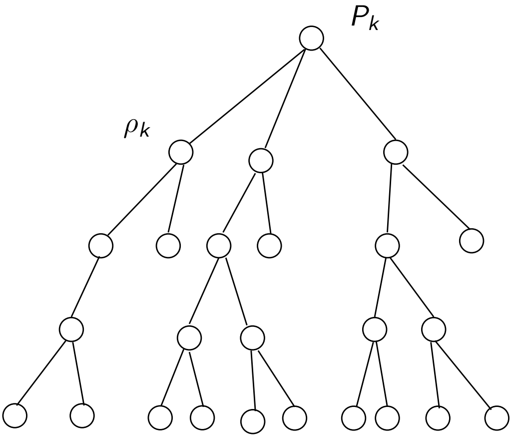


‘Converges locally’

$P \equiv \{P_k\}_{k \geq 0}$	Degree distribution
$\mathsf{T}(P, \ell)$	ℓ-generations Galton-Watson tree
$\mathsf{B}_i(\ell)$	Ball of radius ℓ around uniformly random node

Definition

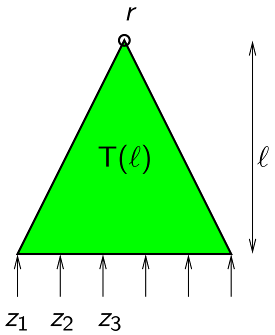
G_n *converges locally* to $\mathsf{T}(P)$ if uniform bound on the edge number distribution and, for any ℓ,

$$\mathsf{B}_i(\ell) \text{ converges in distribution to } \mathsf{T}(P, \ell).$$

Why non-uniform control? Phase transition...

For $\beta > \beta_{\rm c} \equiv {\rm atanh}(1/\overline{\rho})$

$$\lim_{B\to 0+}\lim_{n\to\infty}\mathbb{E}_i\langle x_i\rangle = -\lim_{B\to 0-}\lim_{n\to\infty}\mathbb{E}_i\langle x_i\rangle > 0$$



$$z = (+1, +1, \dots, +1) \qquad \Rightarrow \qquad \lim_{\ell \to \infty} \langle x_r \rangle_\ell > 0$$

$$z = (-1, -1, \dots, -1) \qquad \Rightarrow \qquad \lim_{\ell \to \infty} \langle x_r \rangle_\ell < 0$$

A different case: Ising spin glass

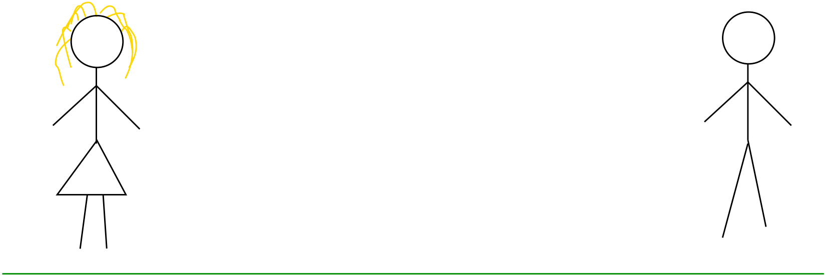
$$\mu(\underline{x}) = \frac{1}{Z} \exp\left\{ \beta \sum_{(ij)\in E_n} J_{ij} x_i x_j + B \sum_i x_i \right\}$$

$J_{ij} \in \{+1, -1\}$ uniformly random

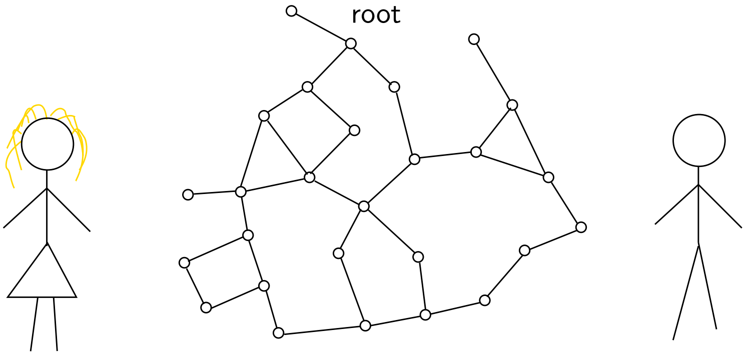
[Viana, Bray 1985]
[Other approaches: Talagrand 2001, Guerra, Toninelli 2003]
[Approximation by trees: Montanari, Gerschenfeld, in preparation]

Trees vs graphs: from reconstruction to pure states

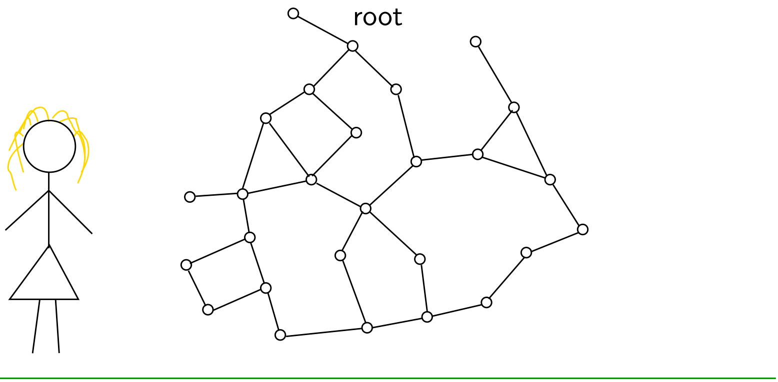
Alice and Bob



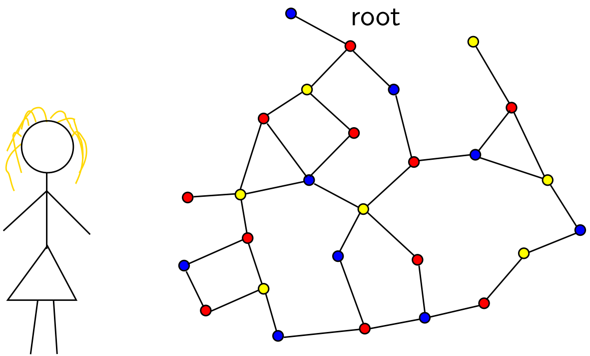
Alice, Bob and G



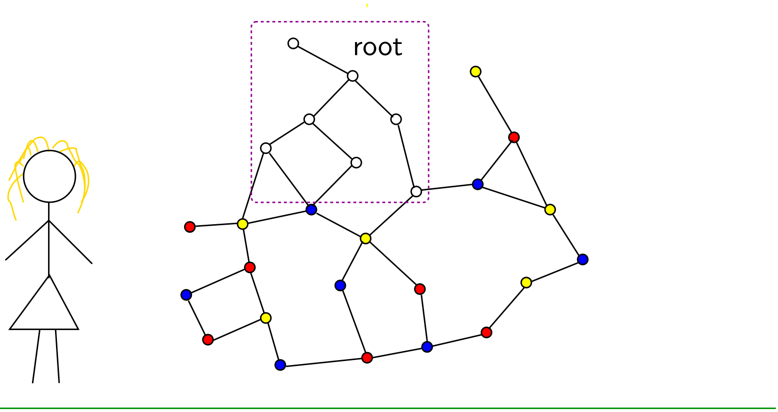
Exit Bob



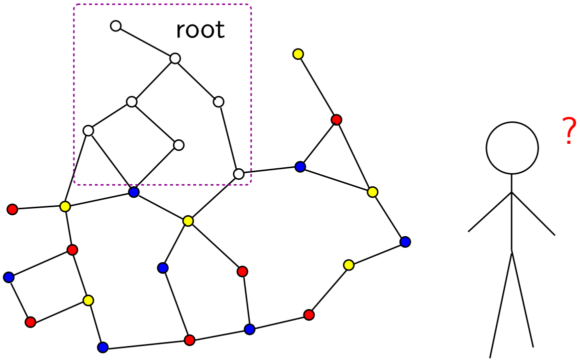
Alice samples a proper coloring (uniformly). . .



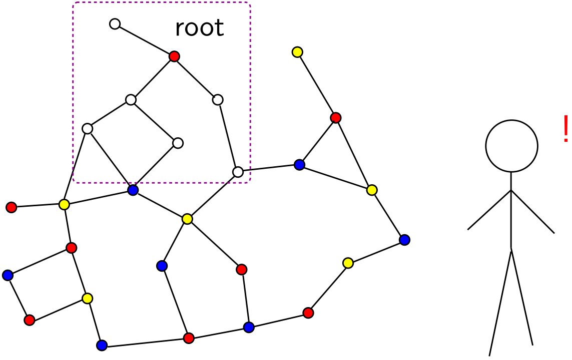
. . . and hides a ball B(root, t)



Bob. . .



. . . guesses right!



The problem

Does Bob have a chance?

Formally

$X = \{X_i : i \in V\}$ uniformly random proper coloring.

$\mu_U(\,\cdot\,|G)$ distribution of $X_U \equiv \{X_i : i \in U \subseteq V\}$

$\overline{\mathrm{B}}(r,t) = \{i \in V : d(i,r) \geq t\}$

Definition

The reconstruction problem is solvable for the sequence of random rooted graphs $G_n = (V_n = [n], E_n)$ if for some $\varepsilon > 0$,

$$||\mu_{r,\overline{\mathrm{B}}(r,t)}(\,\cdot\,,\,\cdot\,|G_n) - \mu_r(\,\cdot\,|G_n)\mu_{\overline{\mathrm{B}}(r,t)}(\,\cdot\,|G_n)||_{\mathrm{TV}} \geq \varepsilon\,,$$

with positive probability (bounded away from 0 as $n \to \infty$).

When G =Tree

$\longrightarrow$ Bleher, Ruiz, Zagrebenov (1995): Ising model on b-ary trees

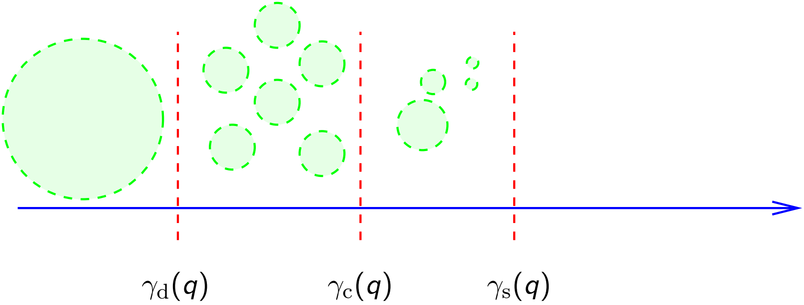
$\longrightarrow$ Evans, Kenyon, Peres, Schulman (2000): Ising on general trees

$\longrightarrow$ Mossel, Peres (2003): Non binary variables

$\longrightarrow$ Brightwell, Winkler (2004), Martin (2004): Independent sets.

$\longrightarrow$ Chayes et al. (2006): Asymmetric Ising.

Pure states decomposition in q-COL



[Biroli, Monasson, Weigt 2001]
[Mézard, Parisi, Zecchina 2003]
[Achlioptas, Ricci 2007]
[Krzakala, Montanari, Ricci, Semerjian, Zdeborova 2007]

Pure states decomposition in q-COL

Conjecture (Mézard, Montanari, 05)

$$\begin{aligned}
\gamma_{\mathrm{d}}(q) &= \\
&= \textit{Multiple pure states} \\
&= \textit{Graph reconstruction threshold} \\
&= \textit{Tree reconstruction threshold}
\end{aligned}$$

A general sufficient condition

Theorem (Gerschenfeld, Montanari, 2007)

If $\mu(\,\cdot\,|G)$ is roughly spherical then

$$\text{Graph solvable} \Leftrightarrow \text{Tree solvable.}$$

If $\mu(\,\cdot\,|G)$ is not roughly spherical then

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Roughly spherical???

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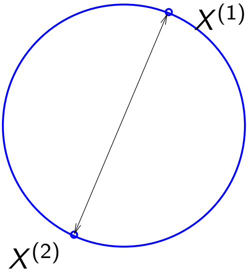
$X^{(1)} = \{X_i^{(1)}\}$, $X^{(2)} = \{X_i^{(2)}\}$ independent with distribution $\mu(\,\cdot\,|G_n)$



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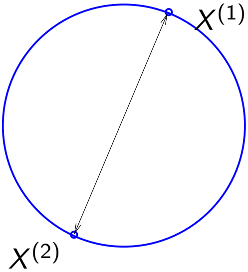

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Can you check this condition?

Theorem

1. *q-coloring,* $\gamma < (q-1)\log(q-1)$*: roughly spherical.*
2. *Ising spin glass* $2\gamma(\tanh\beta)^2 < 1$*: roughly spherical.*
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[Tree reconstruction threshold

1. Bhatayangar, Vera, Vigoda 2008, Sly 2008
2. Evans, Kenyon, Peres, Schulman 2000
3. Tree$\neq$Graph]

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Unifying approach: approximation by trees.

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