ON THE NUMBER OF SUBTREES ON THE FRINGE OF RANDOM TREES
(partly joined with Huilan Chang)

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The number of subtrees

\[ X_{n,k} = \text{number of subtrees of size } k \text{ on the fringe of random binary search trees of size } n. \]
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**Example:** Input: 4, 7, 6, 1, 8, 5, 3, 2

\[ X_{8,4} = 4 \]
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**Example:** Input: 4, 7, 6, 1, 8, 5, 3, 2

![Diagram of a tree with nodes 4 and 7 connected](attachment:image.png)
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![Diagram of a tree with nodes 4, 1, 7, 6]
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\( X_{n,k} = \) number of subtrees of size \( k \) on the fringe of random binary search trees of size \( n \).

**Example:** Input: 4, 7, 6, 1, 8, 5, 3, 2

![Diagram of a tree with nodes labeled 4, 1, 7, 6, 8]
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![Binary tree diagram](image-url)
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```
4
/   \
1 7
/ \ /\
3 6 8
/  \
2 5
```
The number of subtrees

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**Example:** Input: 4, 7, 6, 1, 8, 5, 3, 2

\[ X_{8,1} = 2 \]
The number of subtrees

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**Example**: Input: 4, 7, 6, 1, 8, 5, 3, 2

\[
X_{8, 1} = 2 \\
X_{8, 2} = 2
\]
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Example: Input: 4, 7, 6, 1, 8, 5, 3, 2

\[ X_{8,1} = 2 \]
\[ X_{8,2} = 2 \]
\[ X_{8,3} = 1 \]
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\[ X_{8,1} = 2 \]
\[ X_{8,2} = 2 \]
\[ X_{8,3} = 1 \]
\[ X_{8,4} = 0 \]
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**Example:** Input: 4, 7, 6, 1, 8, 5, 3, 2

\[
\begin{align*}
X_{8,1} &= 2 \\
X_{8,2} &= 2 \\
X_{8,3} &= 1 \\
X_{8,4} &= 0 \\
X_{8,5} &= 1
\end{align*}
\]
The number of subtrees

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**Example**: Input: 4, 7, 6, 1, 8, 5, 3, 2

![Tree Diagram]

- $X_{8,1} = 2$
- $X_{8,2} = 2$
- $X_{8,3} = 1$
- $X_{8,4} = 0$
- $X_{8,5} = 1$
- $X_{8,6} = 0$
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\[
\begin{align*}
X_{8,1} &= 2 \\
X_{8,2} &= 2 \\
X_{8,3} &= 1 \\
X_{8,4} &= 0 \\
X_{8,5} &= 1 \\
X_{8,6} &= 0 \\
X_{8,7} &= 0
\end{align*}
\]
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\[ X_{8,3} = 1 \]
\[ X_{8,4} = 0 \]
\[ X_{8,5} = 1 \]
\[ X_{8,6} = 0 \]
\[ X_{8,7} = 0 \]
\[ X_{8,8} = 1 \]
Mean value and variance

\(X_{n,k}\) satisfies

\[X_{n,k} \overset{d}{=} X_{I_{n},k} + X_{n-1-I_{n},k},\]

where \(X_{k,k} = 1\), \(X_{I_{n},k}\) and \(X_{n-1-I_{n},k}\) are conditionally independent given \(I_{n}\), and \(I_{n} = \text{Unif}\{0, \ldots, n-1\}\).
Mean value and variance

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where \( X_{k,k} = 1, X_{I_n,k} \) and \( X_{n-1-I_n,k} \) are conditionally independent given \( I_n \), and \( I_n = \text{Unif}\{0, \ldots, n-1\} \).

This yields

\[
\mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{2(n + 1)}{(k + 1)(k + 2)}, \quad (n > k),
\]

and

\[
\sigma_{n,k}^2 := \text{Var}(X_{n,k}) = \frac{2k(4k^2 + 5k - 3)(n + 1)}{(k + 1)(k + 2)^2(2k + 1)(2k + 3)}
\]

for \( n > 2k + 1 \).
Some previous results

- Aldous (1991): Weak law of large numbers

\[ \frac{X_{n,k}}{\mu_{n,k}} \rightarrow 1 \quad \text{in probability.} \]
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  \]

- Devroye (1991): Central limit theorem
  \[
  \frac{X_{n,k} - \mu_{n,k}}{\sigma_{n,k}} \overset{d}{\rightarrow} \mathcal{N}(0, 1).
  \]
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  Central limit theorem with optimal Berry-Esseen bound and LLT
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- Flajolet, Gourdon, Martinez (1997):
  Central limit theorem with optimal Berry-Esseen bound and LLT
  \[ \longrightarrow \quad \text{All the above results are for fixed } k. \]
Results for $k = k_n$

Theorem (Feng, Mahmoud, Panholzer (2008))

(i) (Normal range) Let $k = o(\sqrt{n})$ and $k \to \infty$ as $n \to \infty$. Then,

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{2n/k^2}} \xrightarrow{d} N(0, 1).$$

(ii) (Poisson range) Let $k \sim c\sqrt{n}$ as $n \to \infty$. Then,

$$X_{n,k} \xrightarrow{d} \text{Poisson}(2c^{-2}).$$

(iii) (Degenerate range) Let $k < n$ and $\sqrt{n} = o(k)$ as $n \to \infty$. Then,

$$X_{n,k} \xrightarrow{L_1} 0.$$
Why are we interested in $X_{n,k}$?

- $X_{n,k}$ is a new kind of profile of a tree.
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- The phase change from normal to Poisson is a universal phenomenon expected to hold for many classes of random trees.
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- The methods for proving phase change results might be applicable to other parameters which are expected to exhibit the same phase change behavior as well.
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- The methods for proving phase change results might be applicable to other parameters which are expected to exhibit the same phase change behavior as well.

- $X_{n,k}$ is related to parameters arising in genetics.
Example:

Yule generated random genealogical trees
Example:

Random model:

At every time point, two yellow nodes uniformly coalescent.
Yule generated random genealogical trees

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1 2 3 4 5 6
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Example:

```
      4
     / \  
   3   5
  / \ / \
1  2 6  1
```

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Random model:

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Example:

```
  5
 / \
/   \
4   2
 |   |
3   3
|   |
2   4
|   |
1   5
|   |
1   6
```

Random model:
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Example:

```
5
4
3
2
1
```

Random model:

At every time point, two yellow nodes uniformly coalescent.

Same model as random binary search tree model!
Shape parameters of genealogical trees

- $k$-pronged nodes (Rosenberg 2006):
  
  Nodes with an induced subtree with $k - 1$ internal nodes.
Shape parameters of genealogical trees

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  Correspond to nodes whose induced subtree has $k - 1$ internal nodes all of them with out-degree either 0 or 1.
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  Correspond to nodes whose induced subtree has $k - 1$ internal nodes all of them with out-degree either 0 or 1.

- Nodes with minimal clade size $k$ (Blum and François (2005)):
  
  If $k \geq 3$, then they are internal nodes with induced subtree of size $k - 1$ and either an empty right subtree or empty left subtree.
Consider $X_{n,k}$ with

$$X_{n,k} \overset{d}{=} X_{I_n,k} + X_{n-1-I_n,k}^*,$$

where $X_{k,k} = \text{Bernoulli}(p_k)$, $X_{I_n,k}$ and $X_{n-1-I_n,k}^*$ are conditionally independent given $I_n$, and $I_n = \text{Unif}\{0, \ldots, n-1\}$. 


Counting pattern in random binary search trees

Consider $X_{n,k}$ with

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where $X_{k,k} = \text{Bernoulli}(p_k)$, $X_{I_n,k}$ and $X_{n-1-I_n,k}^*$ are conditionally independent given $I_n$, and $I_n = \text{Unif}\{0, \ldots, n-1\}$.

Then,

<table>
<thead>
<tr>
<th>$p_k$</th>
<th>shape parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td># of $k+1$-pronged nodes</td>
</tr>
<tr>
<td>$2/k$</td>
<td># of nodes with minimal clade size $k+1$</td>
</tr>
<tr>
<td>$2^{k-1}/k!$</td>
<td># of $k+1$ caterpillars</td>
</tr>
</tbody>
</table>
Underlying recurrence and solution

All (centered or non-centered) moments satisfy

\[ a_{n,k} = \frac{2}{n} \sum_{j=0}^{n-1} a_{j,k} + b_{n,k}, \]

where \( a_{k,k} \) is given and \( a_{n,k} = 0 \) for \( n < k \).
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where \( a_{k,k} \) is given and \( a_{n,k} = 0 \) for \( n < k \).

We have

\[ a_{n,k} = \frac{2(n+1)}{(k+1)(k+2)} a_{k,k} + 2(n+1) \sum_{k<j<n} \frac{b_{j,k}}{(j+1)(j+2)} + b_{n,k}, \]

where \( n > k \).
Mean value and variance

We have

\[ E(X_{n,k}) = \frac{2(n + 1)}{(k + 1)(k + 2)} p_k, \quad (n > k), \]

and

\[ \text{Var}(X_{n,k}) = \frac{2p_k(4k^3 + 16k^2 + 19k + 6 - (11k^2 + 22k + 6)p_k)(n + 1)}{(k + 1)(k + 2)^2(2k + 1)(2k + 3)} \]

for \( n > 2k + 1 \).
Mean value and variance

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\[
E(X_{n,k}) = \frac{2(n+1)}{(k+1)(k+2)} p_k, \quad (n > k),
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\]

for \( n > 2k + 1 \).

Note that

\[
E(X_{n,k}) \sim Var(X_{n,k}) \sim \frac{2p_k}{k^2} n
\]

for \( n > 2k + 1 \) and \( k \to \infty \) as \( n \to \infty \).
Higher moments

Denote by

\[ A^{(m)}_{n,k} := E(X_{n,k} - E(X_{n,k}))^m. \]
Higher moments

Denote by

\[ A_{n,k}^{(m)} := \mathbb{E}(X_{n,k} - \mathbb{E}(X_{n,k}))^m. \]

Then,

\[ A_{n,k}^{(m)} = \frac{2}{n} \sum_{j=0}^{n-1} A_{j,k}^{(m)} + B_{n,k}^{(m)}, \]

where

\[ B_{n,k}^{(m)} := \sum_{i_1+i_2+i_3=m \atop 0 \leq i_1, i_2 < m} \binom{m}{i_1, i_2, i_3} \frac{1}{n} \sum_{j=0}^{n-1} A_{j,k}^{(i_1)} A_{n-1-j,k}^{(i_2)} \Delta_{i_3}^{n,j,k} \]

and

\[ \Delta_{n,j,k} = \mathbb{E}(X_{j,k}) + \mathbb{E}(X_{n-1-j,k}) - \mathbb{E}(X_{n,k}). \]
Theorem

Let $Z_n, Z$ be random variables. Assume that, as $n \to \infty$,

$$
\mathbb{E}(Z_n^m) \longrightarrow \mathbb{E}(Z^m)
$$

for all $m \geq 1$ and $Z$ is uniquely determined by its moment sequence. Then,

$$
Z_n \xrightarrow{d} Z.
$$
Methods of moments

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Let $Z_n, Z$ be random variables. Assume that, as $n \to \infty$,

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Inductive approach based on asymptotic transfers was extensively used in the AofA to prove numerous result (“moment pumping”).
Methods of moments

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Inductive approach based on asymptotic transfers was extensively used in the AofA to prove numerous result ("moment pumping").

Fuchs, Hwang, Neininger (2007): variation of the above scheme to study the profile of random binary search trees and random recursive trees.
Proposition

Uniformly for \( n, k, m \geq 1 \) and \( n > k \)

\[
A^{(m)}_{n,k} = \max \left\{ \frac{2p_k n}{k^2}, \left( \frac{2p_k n}{k^2} \right)^{m/2} \right\}.
\]
Normal range

Proposition

Uniformly for \( n, k, m \geq 1 \) and \( n > k \)

\[
A_{n,k}^{(m)} = \max \left\{ \frac{2p_k n}{k^2}, \left( \frac{2p_k n}{k^2} \right)^{m/2} \right\}.
\]

Proposition

For \( \mathbf{E}(X_{n,k}) \to \infty \) as \( n \to \infty \),

\[
A_{n,k}^{(2m-1)} = o \left( \left( \frac{2p_k n}{k^2} \right)^{m-1/2} \right), \quad A_{n,k}^{(2m)} \sim g_m \left( \frac{2p_k n}{k^2} \right)^m,
\]

where

\[
g_m = (2m)!/(2^m m!).
\]
Consider

\[ \bar{A}_{n,k}^{(m)} = E(X_{n,k}(X_{n,k} - 1) \cdots (X_{n,k} - m + 1)). \]
Poisson range

Consider

\[
\bar{A}_{n,k}^{(m)} = \mathbf{E}(X_{n,k}(X_{n,k} - 1) \cdots (X_{n,k} - m + 1)).
\]

Then, similarly as before:

**Proposition**

(i) *Uniformly for* \( n, k, m \geq 1 \) *and* \( n > k \)

\[
\bar{A}_{n,k}^{(m)} = \max \left\{ \frac{2p_k n}{k^2}, \left( \frac{2p_k n}{k^2} \right)^m \right\}.
\]

(ii) *For* \( \mathbf{E}(X_{n,k}) \to c \) *and* \( k < n \) *as* \( n \to \infty \),

\[
\bar{A}_{n,k}^{(m)} \to c^m.
\]
The phase change

Theorem

(i) *(Normal range)* Let $\mathbb{E}(X_{n,k}) \to \infty$ and $k \to \infty$ as $n \to \infty$. Then,

$$
\frac{X_{n,k} - \mathbb{E}(X_{n,k})}{\sqrt{2p_k n / k^2}} \overset{d}{\to} \mathcal{N}(0, 1).
$$

(ii) *(Poisson range)* Let $\mathbb{E}(X_{n,k}) \to c > 0$ and $k < n$ as $n \to \infty$. Then,

$$
X_{n,k} \overset{d}{\to} \text{Poisson}(c).
$$

(iii) *(Degenerate range)* Let $\mathbb{E}(X_{n,k}) \to 0$ as $n \to \infty$. Then,

$$
X_{n,k} \overset{L_1}{\to} 0.
$$
A comparison of the phase change

For $k$-caterpillars, we have

$$E(X_{n,k}) = \frac{2^{k-1}n}{(k + 2)!}.$$ 

Note that either

$$E(X_{n,k}) \to \infty \quad \text{or} \quad E(X_{n,k}) \to 0.$$ 

So, there is no Poisson range.
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<th>location</th>
<th>phase change</th>
</tr>
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<tbody>
<tr>
<td>$k$-pronged nodes</td>
<td>$\sqrt{n}$</td>
<td>normal - poisson - degenerate</td>
</tr>
<tr>
<td>minimal clade size $k$</td>
<td>$\sqrt[3]{n}$</td>
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</tr>
<tr>
<td>$k$-caterpillars</td>
<td>$\ln n/(\ln \ln n)$</td>
<td>normal - degenerate</td>
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</tbody>
</table>
Define

\[ \phi_{n,k}(y) = e^{-\sigma_{n,k}^2 y^2 / 2} \mathbb{E} \left( e^{(X_{n,k} - \mu_{n,k})y} \right). \]

and

\[ \phi_{n,k}^{(m)} \left|_{y=0} \right. = \frac{d^m \phi_{n,k}(y)}{dy^m} \left|_{y=0} \right. . \]
Refined results (for \# of subtrees)

Define
\[ \phi_{n,k}(y) = e^{-\sigma^2_{n,k}y^2/2} \mathbb{E} \left( e^{(X_{n,k} - \mu_{n,k})y} \right). \]

and
\[ \phi_{n,k}^{(m)} = \frac{d^m \phi_{n,k}(y)}{dy^m} \bigg|_{y=0}. \]

Proposition

Uniformly for \( n, k \geq 1 \) and \( m \geq 0 \)

\[ |\phi_{n,k}^{(m)}| \leq m! A^m \max \left\{ \frac{n}{k^2}, \left( \frac{n}{k^2} \right)^{m/3} \right\} \]

for a suitable constant \( A \).
Characteristic function

Let $\varphi_{n,k}(y) = \mathbb{E}(\exp\{(X_{n,k} - \mu_{n,k})iy/\sigma_{n,k}\})$.

Proposition

Let $1 \leq k = o(\sqrt{n})$.

(i) For $n$ large

$$
\varphi_{n,k}(y) = e^{-y^2/2} \left( 1 + O\left(|y|^3 \frac{k}{\sqrt{n}}\right) \right),
$$

uniformly for $y$ with $|y| \leq \epsilon n^{1/6}/k^{1/3}$.

(ii) For $n$ large

$$
|\varphi_{n,k}(y)| \leq e^{-\epsilon y^2/2},
$$

where $|y| \leq \pi \sigma_{n,k}$. 
Theorem (Rate of convergency)

For $1 \leq k = o(\sqrt{n})$ as $n \to \infty$,

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{X_{n,k} - \mu_{n,k}}{\sigma_{n,k}} < x \right) - \Phi(x) \right| = O \left( \frac{k}{\sqrt{n}} \right).$$
Berry-Esseen bound and LLT for the normal range

**Theorem (Rate of convergency)**

For \(1 \leq k = o(\sqrt{n})\) as \(n \to \infty\),

\[
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\]

**Theorem (LLT)**

For \(1 \leq k = o(\sqrt{n})\) as \(n \to \infty\),

\[
P(X_{n,k} = \lfloor \mu_{n,k} + x\sigma_{n,k} \rfloor) = \frac{e^{-x^2/2}}{\sqrt{2\pi\sigma_{n,k}}} \left( 1 + O\left( (1 + |x|^3) \frac{k}{\sqrt{n}} \right) \right),
\]

uniformly in \(x = o(n^{1/6}/k^{1/3})\).
LLT for the Poisson range

Define

\[ \bar{\phi}_{n,k}(y) = e^{-\mu_{n,k}(y-1)} E(y^{X_{n,k}}). \]

and

\[ \phi_{n,k}(m) = \frac{d^m \bar{\phi}_{n,k}(y)}{dy^m} \bigg|_{y=1}. \]
Define
\[ \bar{\phi}_{n,k}(y) = e^{-\mu_{n,k}(y-1)} E \left( y^{X_{n,k}} \right). \]
and
\[ \phi_{n,k}^{(m)} = \frac{d^m \bar{\phi}_{n,k}(y)}{dy^m} \bigg|_{y=1}. \]

**Proposition**

Uniformly for \( n > k \) and \( m \geq 0 \)
\[ |\bar{\phi}_{n,k}^{(m)}| \leq m! A^m \left( \frac{n}{k^3} \right)^{m/2} \]
for a suitable constant \( A \).
Poisson approximation

Theorem (LLT)

For \( k < n \) and \( n \to \infty \),

\[
P(X_{n,k} = l) = e^{-\mu_{n,k}} \frac{(\mu_{n,k})^l}{l!} + \mathcal{O} \left( \frac{n}{k^3} \right)
\]

uniformly in \( l \).
Poisson approximation

Theorem (LLT)

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\[
P(X_{n,k} = l) = e^{-\mu_{n,k}} \frac{(\mu_{n,k})^l}{l!} + O\left(\frac{n}{k^3}\right)
\]

uniformly in \( l \).

Theorem (Poisson approximation)

Let \( k < n \) and \( k \to \infty \) as \( n \to \infty \). Then,

\[
d_{TV}(X_{n,k}, \text{Poisson}(\mu_{n,k})) \longrightarrow 0.
\]

Remark: A rate can be given as well.
Other types of random trees

- Random recursive trees
  Non-plane, labelled trees with every label sequence from the root to a leaf increasing; random model is the uniform model.
  Methods works as well (with minor modifications) and similar results can be proved.
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  Non-plane, labelled trees with every label sequence from the root to a
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  Methods works as well (with minor modifications) and similar results
  can be proved.

- Plane-oriented recursive trees (PORTs)
  Plane, labelled trees with every label sequence from the root to a
  leaf increasing; random model is the uniform model.
  Method works as well, but details more involved.
The number of subtrees for PORTs is given by:

\[ X_{n,k} \text{ satisfies} \]

\[ X_{n,k} \overset{d}{=} \sum_{i=1}^{N} X_{I_i,k}^{(i)}, \]

where \( X_{k,k} = 1 \) and \( X_{I_i,k}^{(i)} \) are conditionally independent given \((N, I_1, I_2, \ldots)\).
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This can be simplified to

\[ X_{n,k} \overset{d}{=} X_{I_n,k} + X_{n-I_n,k}^* - 1\{n-I_n=k\}, \]

where \( X_{k,k} = 1 \), \( X_{I_n,k} \) and \( X_{n-I_n,k}^* \) are conditionally independent given \( I_n \) and

\[ P(I_n = j) = \frac{2(n - j)C_j C_{n-j}}{nC_n}. \]
All (centered or non-centered) moments satisfy

\[ a_{n,k} = 2 \sum_{j=1}^{n-1} \frac{C_j C_{n-j}}{C_n} a_{j,k} + b_{n,k}, \]

where \( a_{k,k} \) is given and \( a_{n,k} = 0 \) for \( n < k \).
Underlying recurrence and solution

All (centered or non-centered) moments satisfy

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where \( a_{k,k} \) is given and \( a_{n,k} = 0 \) for \( n < k \).

We have

\[ a_{n,k} = \frac{C_k (n + 1 - k) C_{n+1-k}}{C_n} a_{k,k} + \sum_{k<j \leq n} \frac{C_j (n + 1 - j) C_{n+1-j}}{C_n} b_{n,k}, \]

where \( n > k \).
Mean value and variance of PORTs

We have,

\[ \mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{2n - 1}{4k^2 - 1}, \quad (n > k). \]
Mean value and variance of PORTs

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\[ \mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{2n - 1}{4k^2 - 1}, \quad (n > k). \]

Moreover, for fixed \( k \) as \( n \to \infty \),
\[ \text{Var}(X_{n,k}) \sim c_k n, \]
where
\[ c_k = \frac{8k^2 - 4k - 8}{(4k^2 - 1)^2} - \frac{((2k - 3)!!)^2}{((k - 1)!)^2 4^{k-1} k(2k + 1)}, \]
and, for \( k < n \) and \( k \to \infty \) as \( n \to \infty \),
\[ \mathbb{E}(X_{n,k}) \sim \text{Var}(X_{n,k}) \sim \frac{n}{2k^2}. \]
The phase change

**Theorem**

(i) *Normal range* Let $k = o\left(\sqrt{n}\right)$ and $k \to \infty$ as $n \to \infty$. Then,

$$
\frac{X_{n,k} - \mu_{n,k}}{\sqrt{n/(2k^2)}} \xrightarrow{d} N(0, 1).
$$

(ii) *Poisson range* Let $k \sim c\sqrt{n}$ as $n \to \infty$. Then,

$$
X_{n,k} \xrightarrow{d} \text{Poisson}\left((2c^2)^{-1}\right).
$$

(iii) *Degenerate range* Let $k < n$ and $\sqrt{n} = o(k)$ as $n \to \infty$. Then,

$$
X_{n,k} \xrightarrow{L_1} 0.
$$
Parameters of genealogical trees under different random models
More results and future research

- Parameters of genealogical trees under different random models
- Universality of the phase change for the number of subtrees
  
  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)
Polynomial varieties

Bergeron, Flajolet, Salvy (1992): classes of increasing trees with degree function $\phi(\omega)$ under the uniform model.
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Polynomial varieties: class of increasing tree with

$$\phi(\omega) = \phi_d \omega^d + \cdots + \phi_0,$$

where $d \geq 2$ and $\phi_d, \phi_0 \neq 0$. 

For mean value and variance of the number of subtrees,

$$E(X_{n,k}) \sim \text{Var}(X_{n,k}) \sim d^{d-1} n^{k-2},$$

where $k \to \infty$ as $n \to \infty$. 

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Theorem

(i) (Normal range) Let $k = o\left(\sqrt{n}\right)$ and $k \rightarrow \infty$ as $n \rightarrow \infty$. Then,

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\frac{X_{n,k} - \mu_{n,k}}{\sqrt{nd/((d-1)k^2)}} \xrightarrow{d} \mathcal{N}(0, 1).
$$

(ii) (Poisson range) Let $k \sim c\sqrt{n}$ as $n \rightarrow \infty$. Then,

$$
X_{n,k} \xrightarrow{d} \text{Poisson}\left(d/((d-1)c^2)\right).
$$

(iii) (Degenerate range) Let $k < n$ and $\sqrt{n} = o(k)$ as $n \rightarrow \infty$. Then,

$$
X_{n,k} \xrightarrow{L_1} 0.
$$
Theorem

(i) **(Normal range)** Let \( k = o \left( \sqrt{n/\ln n} \right) \) and \( k \to \infty \) as \( n \to \infty \). Then,

\[
\frac{X_{n,k} - \mu_{n,k}}{\sqrt{n/(k^2 \ln k)}} \xrightarrow{d} \mathcal{N}(0, 1).
\]

(ii) **(Poisson range)** Let \( k \sim c \sqrt{n/\ln n} \) as \( n \to \infty \). Then,

\[
X_{n,k} \xrightarrow{d} \text{Poisson}(2c^{-2}).
\]

(iii) **(Degenerate range)** Let \( k < n \) and \( \sqrt{n/\ln n} = o(k) \) as \( n \to \infty \). Then,

\[
X_{n,k} \xrightarrow{L_1} 0.
\]
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- Parameters of genealogical trees under different random models
- Universality of the phase change for the number of subtrees
  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)
- Phase change results for the number of nodes with out-degree $k$
  Important in computer science.
  A phase change from normal to degenerate is expected (no Poisson range).