Minicourse 3:
Limiting Distributions in Combinatorics

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Standard Reference

Philippe Flajolet and Robert Sedgewick,
Analytic Combinatorics,
Cambridge University Press, to appear 2008.
(http://algo.inria.fr/flajolet/Publications/books.html)

+ special reference for last part:

M. Drmota, B. Gittenberger and T. Klausner,
Extended admissible functions and Gaussian limiting distributions,
Math. Comput. 74 (2005), 1953–1966.

Sums of independent random variables and powers of generating functions

Coin tossing

- $\mathbb{P}\{ct = \text{head}\} = \mathbb{P}\{ct = \text{tail}\} = \frac{1}{2}$.
- random variable $\xi = \mathbb{I}_{\{ct=\text{tail}\}} = \begin{cases} 1 & \text{if tail} \\ 0 & \text{if head} \end{cases}$
- n independent runs: $\xi_1, \xi_2, \ldots, \xi_n$, $\boxed{\mathbb{P}\{\xi_j = 1\} = \mathbb{P}\{\xi_j = 0\} = \frac{1}{2}}$.
- $\boxed{X_n = \xi_1 + \xi_2 + \cdots + \xi_n}$ ... the number of tails within n runs

$$\boxed{\mathbb{P}\{X_n = k\} = \frac{\binom{n}{k}}{2^n}}$$

Sums of independent random variables and powers of generating functions

Counting generating function

$a_n = 2^n$ ... total number of possible n-runs

$a_{n,k} = \binom{n}{k}$ ... the number of n-runs with k tails

$$A_n(u) = \sum_{k\geq 0} a_{n,k} u^k = \sum_{k\geq 0} \binom{n}{k} u^k = (1+u)^n \text{ ... counting gen. func.}$$

$$A_n(1) = \sum_{k\geq 0} a_{n,k} = a_n = (1+1)^n = 2^n$$

Sums of independent random variables and powers of generating functions

Probability generating function

$$\begin{aligned}\mathbb{E}\, u^{X_n} &= \sum_{k\geq 0} \mathbb{P}\{X_n = k\} \cdot u^k \\ &= \sum_{k\geq 0} \frac{1}{2^n}\binom{n}{k} \cdot u^k \\ &= \frac{(1+u)^n}{2^n} = \frac{A_n(u)}{A_n(1)}\end{aligned}$$

$$\mathbb{P}\{X_n = k\} = \frac{a_{n,k}}{a_n} \quad \Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{A_n(u)}{A_n(1)}}$$

Sums of independent random variables and powers of generating functions

Powers of probability generating functions

$$\mathbb{E}\, u^{\xi} = \frac{1}{2} + \frac{1}{2}u = \frac{1+u}{2}$$

$$\Longrightarrow \quad \begin{aligned} \mathbb{E}\, u^{X_n} &= \mathbb{E}\, u^{\xi_1 + \cdots + \xi_n} \\ &= \mathbb{E}\left(u^{\xi_1} \cdots u^{\xi_n}\right) \\ &= \mathbb{E}\left(u^{\xi_1}\right) \cdots \mathbb{E}\left(u^{\xi_n}\right) \qquad \xi_j \text{ independent !!!} \\ &= \left(\frac{1+u}{2}\right)^n \end{aligned}$$

Sums of independent random variables and powers of generating functions

General fact

$\boxed{X_n = \xi_1 + \xi_2 + \cdots + \xi_n}$, where the r.v.'s ξ_j are **iid***

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \left(\mathbb{E}\, u^{\xi_1}\right)^n}$$

* **Notation.** "iid" ... independently and identically distributed

Sums of independent random variables and powers of generating functions

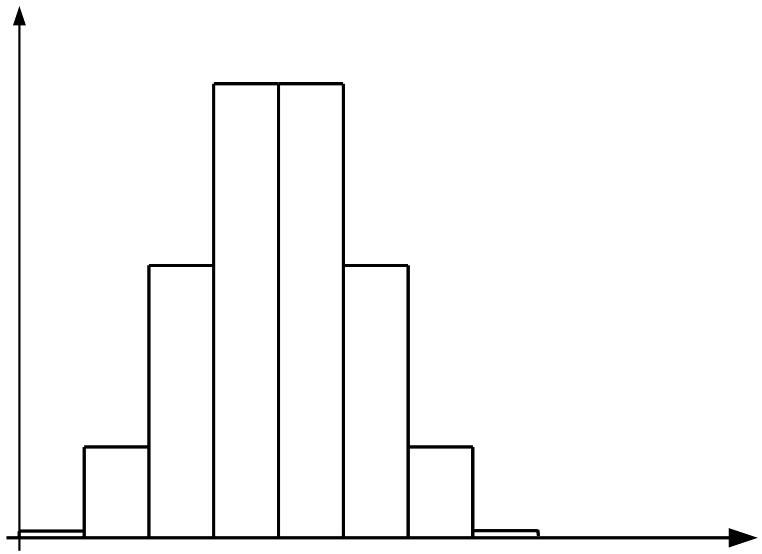
Relation to moment generating function $m_Z(v) = \mathbb{E}\, e^{vZ}$

$\mathbb{E}\,(Z^r)$ ... r-th moment of Z

$$\Longrightarrow \quad \sum_{r\geq 0} \mathbb{E}\,(Z^r)\frac{v^r}{r!} = \mathbb{E}\left(\sum_{r\geq 0}\frac{Z^r v^r}{r!}\right) = \mathbb{E}\, e^{vZ} = \mathbb{E}\, u^Z \quad \text{with } \boxed{u = e^v}.$$

A central limit theorem

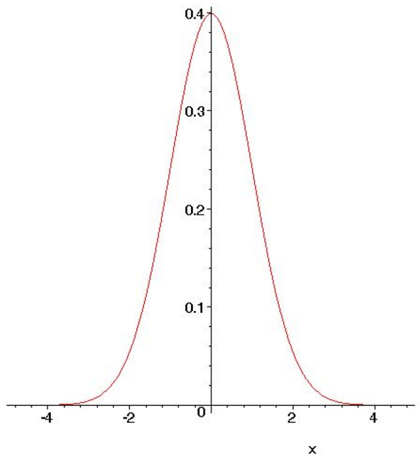
Binomial coefficients



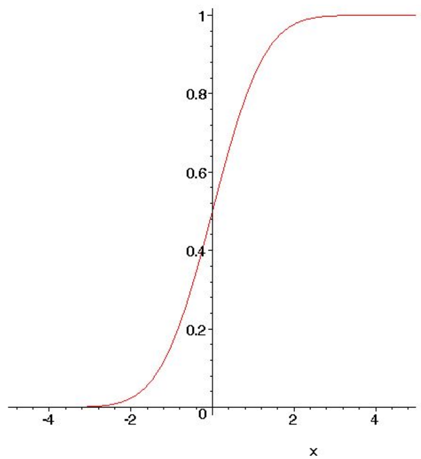
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{2^n}{\sqrt{\pi n/2}} \exp\left(-\frac{(k-\frac{n}{2})^2}{n/2}\right) + O(2^n/n)$$

A central limit theorem

Standard normal distribution



density: $f(t) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}.$



normal distribution function: $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$

A central limit theorem

Normally distributed random variable

Definition

A random variable Z *has* **standard nomal distribution** $N(0,1)$ *if*

$$\boxed{\mathbb{P}\{Z \leq x\} = \Phi(x)}.$$

A random variable Z *is* **normally distributed** *(or* **Gaussian***) with mean* μ *and variance* σ^2 *if its distribution function is given by*

$$\boxed{\mathbb{P}\{Z \leq x\} = \Phi\left(\frac{x-\mu}{\sigma}\right)},$$

Notation. $\boxed{\mathcal{L}(Z) = N(\mu, \sigma^2)}.$

A central limit theorem

Moment generating function of $N(\mu, \sigma^2)$:

$$\boxed{m_Z(v) = \mathbb{E}\, e^{vZ} = e^{\mu v - \frac{1}{2}\sigma^2 v^2}}.$$

Characteristic function of $N(\mu, \sigma^2)$:

$$\boxed{\varphi_Z(t) = \mathbb{E}\, e^{itZ} = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}}.$$

Standard normal distribution: $\mu = 0$, $\sigma^2 = 1$

$$\boxed{\mathbb{E}\, e^{vZ} = e^{\frac{1}{2}v^2}}, \qquad \boxed{\mathbb{E}\, e^{itZ} = e^{-\frac{1}{2}t^2}}$$

A central limit theorem

Definition *We say, that a sequence of random variables* X_n *satisfies* **a central limit theorem** *with (scaling) mean* μ_n *and (scaling) variance* σ_n^2 *if*

$$\boxed{\mathbb{P}\{X_n \le \mu_n + x \cdot \sigma_n\} = \Phi(x) + o(1)}$$

as $n \to \infty$.

Example. $X_n =$ number of tails in n runs of coin tossing:

$$\mathbb{P}\{X_n \le n/2 + x \cdot \sqrt{n/4}\} = \sum_{k \le n/2 + x \cdot \sqrt{n/4}} \frac{1}{2^n}\binom{n}{k}$$

$$\sim \sum_{k \le n/2 + x \cdot \sqrt{n/4}} \frac{1}{\sqrt{\pi n/2}} \exp\left(-\frac{(k - \frac{n}{2})^2}{n/2}\right) \sim \Phi(x).$$

X_n satisfies a central limit theorem with mean $\frac{n}{2}$ and variance $\frac{n}{4}$.

Central Limit Theorem

Definition *Weak convergence:*

$$\boxed{X_n \xrightarrow{\text{d}} X} \quad :\Longleftrightarrow \quad \boxed{\lim_{n\to\infty} \mathbb{P}\{X_n \le x\} = \mathbb{P}\{X \le x\}}$$

for all points of continuity
of $F_X(x) = \mathbb{P}\{X \le x\}$

Reformulation:

X_n satisfies **a central limit theorem** with (scaling) mean μ_n and (scaling) variance σ_n^2 is the same as

$$\boxed{\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{\text{d}} N(0,1)}.$$

A central limit theorem

Weak convergence via moment generating functions

$$\boxed{\lim_{n\to\infty} \mathbb{E}\, e^{vX_n} = \mathbb{E}\, e^{vX} \quad (v \in \mathbb{R}) \quad \Longrightarrow \quad X_n \xrightarrow{\mathsf{d}} X}$$

Moreover, we have convergence of all moments: $\mathbb{E}\,(X_n^r) \to \mathbb{E}\,(X^r)$.

Recall: $\mathbb{E}\, e^{vX_n} = \mathbb{E}\,((e^v)^{X_n}) = \mathbb{E}\, u^{X_n}$ for $u = e^v$.

A central limit theorem

Weak convergence via characteristic functions (Levy's Criterion)

$$\boxed{\lim_{n\to\infty} \mathbb{E}\, e^{itX_n} = \mathbb{E}\, e^{itX} \quad (t \in \mathbb{R}) \quad \Longleftrightarrow \quad X_n \xrightarrow{\mathsf{d}} X}$$

Moreover, if for all $t \in \mathbb{R}$

$$\psi(t) := \lim_{n\to\infty} \mathbb{E}\, e^{itX_n}$$

exists and $\psi(t)$ is continous at $t = 0$ then $\psi(t)$ is the characteristic function of a random variable X for which we have $X_n \xrightarrow{\mathsf{d}} X$.

Central Limit Theorem

Theorem

$\xi_1, \xi_2, \ldots$ iid, $\mathbb{E}\,\xi_i^2 < \infty$, $X_n = \xi_1 + \xi_2 + \ldots + \xi_n$

$$\Longrightarrow \quad \boxed{\frac{X_n - \mathbb{E}\,X_n}{\sqrt{\mathbb{V}\,X_n}} \xrightarrow{\text{d}} N(0,1)}$$

Remark. $\Longleftrightarrow \ \mathbb{P}\{X_n \le \mathbb{E}\,X_n + x\sqrt{\mathbb{V}\,X_n}\} = \Phi(x) + o(1).$

Proof

$\mu = \mathbb{E}\,\xi_i$, $\sigma^2 = \mathbb{V}\,\xi_i = \mathbb{E}\,(\xi_i^2) - (\mathbb{E}\,\xi_i)^2 \qquad \Longrightarrow \mathbb{E}\,X_n = n\mu$, $\mathbb{V}\,X_n = n\sigma^2$.

Central Limit Theorem

$$\varphi_{\xi_i}(t) = \mathbb{E}\, e^{it\xi_i} = e^{it\mu - \frac{1}{2}\sigma^2 t^2\,(1+o(1))} \qquad (t \to 0)$$

$$\varphi_{X_n}(t) = \varphi_{\xi_i}(t)^n$$

$$Z_n := (X_n - \mu n)/\sqrt{\sigma^2 n}$$

$$\begin{aligned}
\Longrightarrow \boxed{\varphi_{Z_n}(t)} &= \mathbb{E}\, e^{itZ_n} \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \mathbb{E}\, \left(e^{(it/(\sqrt{n}\sigma))(\xi_1+\cdots+\xi_n)}\right) \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot \boxed{\left(\mathbb{E}\, e^{(it/(\sqrt{n}\sigma)\xi_1}\right)^n} \\
&= e^{-it\sqrt{n}\mu/\sigma} \cdot e^{it\sqrt{n}\mu/\sigma - \frac{1}{2}t^2\,(1+o(1))} \\
&= e^{-\frac{1}{2}t^2\,(1+o(1))} \to \boxed{e^{-\frac{1}{2}t^2}}.
\end{aligned}$$

+ Levy's criterion.

A central limit theorem

Quasi-Power Theorem (Hwang)

Let X_n be a sequence of random variables with the property that

$$\boxed{\mathbb{E}\, u^{X_n} = A(u) \cdot B(u)^{\lambda_n} \cdot \left(1 + O\left(\frac{1}{\phi_n}\right)\right)}$$

holds uniformly in a complex neighborhood of $u = 1$, $\boxed{\lambda_n \to \infty}$ and $\boxed{\phi_n \to \infty}$, and $A(u)$ and $B(u)$ are analytic functions in a neighborhood of $u = 1$ with $A(1) = B(1) = 1$. Set

$$\mu = B'(1) \quad \text{and} \quad \sigma^2 = B''(1) + B'(1) - B'(1)^2.$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, X_n = \mu\lambda_n + O\left(1 + \lambda_n/\phi_n\right)}, \quad \boxed{\mathbb{V}\, X_n = \sigma^2\lambda_n + O\left(1 + \lambda_n/\phi_n\right)},$$

$$\boxed{\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathbb{V}\, X_n}} \xrightarrow{\mathrm{d}} N(0,1)} \quad (\sigma^2 \neq 0).$$

Bivariate generating functions

Bivariate counting generating function

$$A(x,u) = \sum_{n,k\geq 0} \binom{n}{k} u^k x^n = \sum_{n\geq 0} (1+u)^n x^n = \frac{1}{1-x(1+u)}.$$

Observation: this is a **rational function**!

Bivariate generating functions

Rational functions

$P(x,u), Q(x,u)$ polynomials:

$$A(x,u) = \sum_{n,k\geq 0} a_{n,k}\, u^k\, x^n = \frac{P(x,u)}{Q(x,u)}$$

Assumption: factorization of denominator

$$Q(x,u) = \prod_{j=1}^{r} \left(1 - \frac{x}{\rho_j(u)}\right)$$

with

$$\boxed{|\rho_1(u)| < \max_{2\leq j\leq r} |\rho_j(u)|} \qquad \text{for } |u-1| < \varepsilon.$$

Bivariate generating functions

Central limit theorem for rational functions

Suppose that $A(x,u) = \sum a_{n,k}\, u^k\, x^n$ with $a_{n,k} \ge 0$ is **rational** and satisfies the assumptions from above.

Let X_n be a sequence of random variables with

$$\mathbb{P}\{X_n = k\} = \frac{a_{n,k}}{a_n}$$

with $a_n = \sum_k a_{n,k}$.

Then X_n satisfies a **central limit theorem** with

$$\mu_n = -n\frac{\rho_1'(1)}{\rho_1(1)} \quad \text{and} \quad \sigma_n^2 = n\left(-\frac{\rho_1''(1)}{\rho_1(1)} - \frac{\rho_1'(1)}{\rho_1(1)} + \frac{\rho_1'(1)^2}{\rho_1(1)^2}\right).$$

Bivariate generating functions

Proof

Partial fraction decomposition:

$$A(x,u) = \frac{C_1(u)}{1 - x/\rho_1(u)} + \cdots + \frac{C_r(u)}{1 - x/\rho_r(u)}$$

$$\Longrightarrow \quad A_n(u) = \sum_{k\ge 0} a_{n,k}\, u^k = C_1(u)\rho_1(u)^{-n} + \cdots + C_r(u)\rho_r(u)^{-n} \sim C_1(u)\rho_1(u)^{-n}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{A_n(u)}{A_n(1)} \sim \frac{C_1(u)}{C_1(1)} \left(\frac{\rho_1(1)}{\rho_1(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem**.

Bivariate generating functions

Integer compositions

$3 = 1 + 1 + 1 = 2 + 1 = 1 + 2 = 3$ ... 4 compositions of 3.

$a_n =$ number of compositions of n, $A(x) = \sum a_n x^n$:

$$\boxed{A(x) = 1 + A(x)(x + x^2 + x^3 + \cdots) = 1 + A(x)\frac{x}{1-x}}.$$

$$\Longrightarrow \quad A(x) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x}$$

$$\Longrightarrow \quad \boxed{a_n = 2^{n-1}}$$

Bivariate generating functions

Integer compositions

$a_{n,k}$ = number of integer composition of n with k summands

$A(x,u) = \sum a_{n,k} u^k x^n$:

$$\boxed{A(x,u) = 1 + uA(x,u)(x + x^2 + x^3 + \cdots) = 1 + A(x,u)\frac{xu}{1-x}}.$$

$$\Longrightarrow \quad A(x,u) = \frac{1}{1 - \frac{xu}{1-x}} = \frac{1-x}{1-x(1+u)}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{n}{2}$ and $\sigma^2 = \frac{n}{4}$.

Bivariate generating functions

Systems of linear equations

Suppose, that several generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ldots, A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **linear system of equations**.

Then all generating functions $A_j(x,u)$ are rational and a **central limit theorem** for corresponding random variables is **expected**.

Bivariate generating functions

Meromorphic functions

The function $A(x,u)$ is meromorphic in x when u is considered as a parameter and there exists a dominant root $\rho_1(u)$ such that (locally)

$$\boxed{A(x,u) = \frac{C(x,u)}{1 - \frac{x}{\rho_1(u)}}}$$

$$\Longrightarrow \quad A_n(u) \sim C(\rho_1(u),u)\cdot \rho_1(u)^{-n}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} \sim \frac{C(\rho_1(u),u)}{C(\rho_1(1),1)} \left(\frac{\rho_1(1)}{\rho_1(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem**.

Bivariate generating functions

Number of cycles in permutations

$p_{n,k}$ = number of permutations of $\{1, 2, \dots, n\}$ with k cycles

$$\widehat{P}(x,u) = \sum_{n,k \geq 0} p_{n,k} \cdot u^k \cdot \frac{x^n}{n!} = e^{u \cdot \log \frac{1}{1-x}} = \frac{1}{(1-x)^u}$$

Remark: $p_{n,k} = (-1)^{n-k} s_{n,k}$, where $s_{n,k}$ are the **Stirling number of the first kind**.

Excursion: Singularity Analysis

Lemma 1 *Suppose that*

$$\boxed{y(x) = (1 - x/x_0)^{-\alpha}}.$$

Then

$$\boxed{y_n = (-1)^n \binom{-\alpha}{n} x_0^{-n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} x_0^{-n} + \mathcal{O}\left(n^{\alpha-2}\right) x_0^{-n}.}$$

Remark: This asymptotic expansion is uniform in α if α varies in a compact region of the complex plane.

Excursion: Singularity Analysis

Lemma 2 (Flajolet and Odlyzko) *Let*

$$y(x) = \sum_{n \ge 0} y_n \, x^n$$

be analytic in a region

$$\Delta = \{x : |x| < x_0 + \eta, \, |\operatorname{arg}(x - x_0)| > \delta\},$$

$x_0 > 0$, $\eta > 0$, $0 < \delta < \pi/2$.

Suppose that for some real α

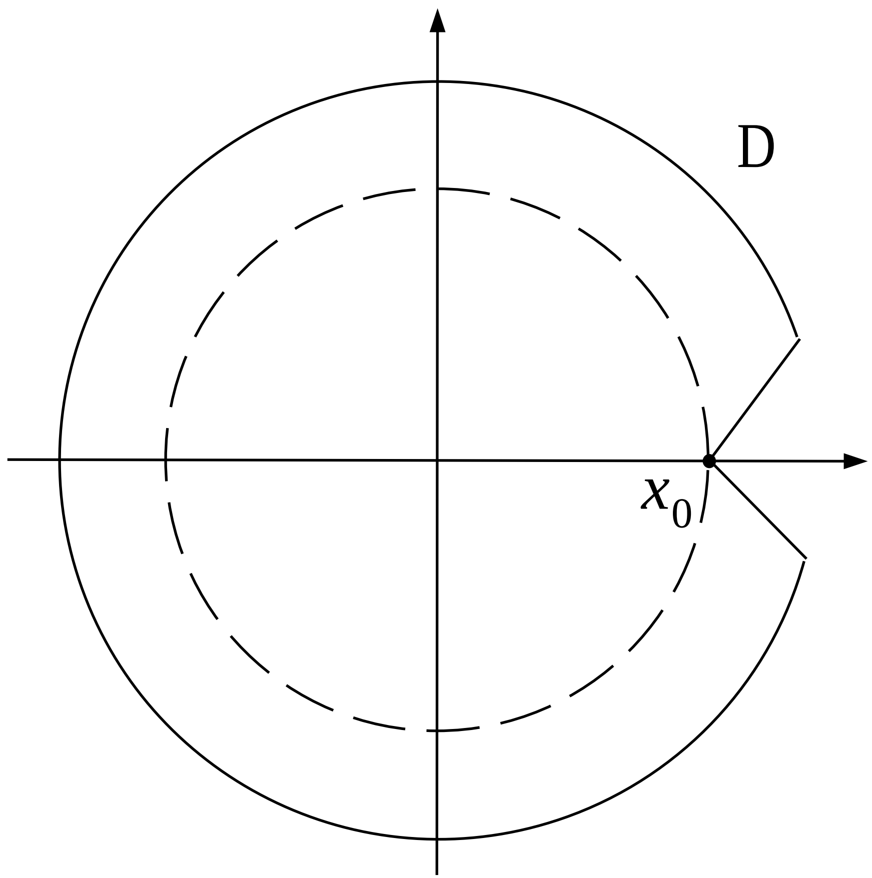
$$\boxed{y(x) = \mathcal{O}\left((1 - x/x_0)^{-\alpha}\right) \qquad (x \in \Delta).}$$

Then

$$\boxed{y_n = \mathcal{O}\left(x_0^{-n} n^{\alpha - 1}\right).}$$

Excursion: Singularity Analysis

Δ-region



Bivariate generating functions

Number of cycles in permutations (continued)

$$\widehat{P}(x,u) = e^{u \log \frac{1}{1-x}} = \frac{1}{(1-x)^u}$$

$$\begin{aligned} \Longrightarrow \quad p_n(u) &= \sum_{k\geq 0} p_{n,k} u^k \\ &\sim n! \frac{n^{u-1}}{\Gamma(u)} \\ &= n! \frac{e^{(u-1)\log n}}{\Gamma(u)} \end{aligned}$$

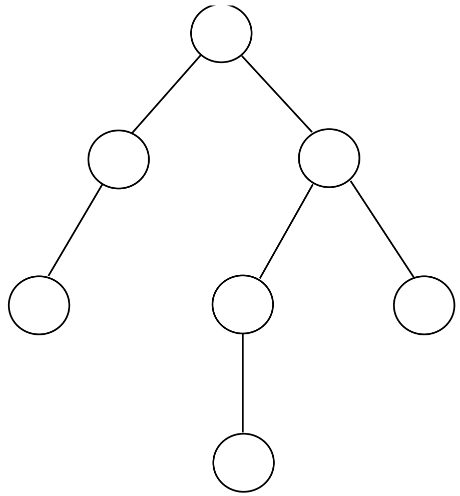
$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} \sim \frac{1}{\Gamma(u)} (e^{u-1})^{\log n}}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \log n$ and $\sigma_n^2 = \log n$.

Generalization: Exp-Log-Schemes: $F(x,u) = e^{h(u) \log \frac{1}{1-x} + R(x,u)}$.

Bivariate generating functions

Catalan trees g_n = number of Catalan trees of size n.



$$\boxed{G(x) = x(1 + G(x) + G(x)^2 + \cdots) = \frac{x}{1 - G(x)}}$$

$$G(x) = \frac{1 - \sqrt{1-4x}}{2} \quad \Longrightarrow \quad \boxed{g_n = \frac{1}{n}\binom{2n-2}{n-1}}.$$

(Catalan numbers)

Bivariate generating functions

Catalan trees with singularity analysis

$$G(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$$

$$\Longrightarrow \quad g_n \sim -\frac{1}{2} \cdot \frac{4^n n^{-3/2}}{\Gamma(-\frac{1}{2})} = \frac{4^{n-1}}{\sqrt{\pi} \cdot n^{3/2}}$$

Bivariate generating functions

Number of leaves of Catalan trees

$g_{n,k}$ = number of Catalan trees of size n with k leaves.

$$\boxed{G(x,u) = xu + x(G(x,u) + G(x,u)^2 + \cdots = xu + \frac{xG(x,u)}{1-G(x,u)}}$$

$$\Longrightarrow \quad G(x,u) = \frac{1}{2}\left(1 + (u-1)x - \sqrt{1-2(u+1)x+(u-1)^2x^2}\right)$$

$$\Longrightarrow \quad \boxed{G(x,u) = g(x,u) - h(x,u)\sqrt{1-\frac{x}{\rho(u)}}}$$

for certain analytic function $g(x,u), h(x,u)$, and $\rho(u)$.

Bivariate generating functions

Application of singularity analysis

Considering u as a parameter we get

$$G_n(u) = \sum_{k \geq 0} g_{n,k} u^k \sim \frac{h(\rho(u), u) \cdot \rho(u)^{-n} \cdot n^{-3/2}}{2\sqrt{\pi}}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\, u^{X_n} = \frac{G_n(u)}{G_n(1)} \sim \frac{h(\rho(u), u)}{h(\rho(1), 1)} \left(\frac{\rho(1)}{\rho(u)}\right)^n}$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{n}{2}$ and $\sigma_n^2 = \frac{n}{8}$

Bivariate generating functions

Cayley trees

$T_{n,k}$ = number of Cayley trees of size n with k leaves

$$T(x,u) = \sum_{n,k\geq 0} T_{n,k}\, u^k\, \frac{x^n}{n!}$$

$$\Longrightarrow \quad \boxed{T(x,u) = xe^{T(x,u)} + x(u-1)}$$

$$\Longrightarrow \quad ?????$$

Functional equations

Catalan trees: $G(x,u) = xu + xG(x,u)/(1 - G(x,u))$

Cayley trees: $T(x,u) = xe^{T(x,u)} + x(u-1)$

Recursive structure leads to functional equation for gen. func.:

$$\boxed{\boxed{A(x,u) = \Phi(x,u,A(x,u))}}$$

Functional equations

Linear functional equation: $\Phi(x,u,a) = \Phi_0(x,u) + a\Phi_1(x,u)$

$$\Longrightarrow \quad A(x,u) = \frac{\Phi_0(x,u)}{1-\Phi_1(x,u)}$$

Usually techniques similar to those used for rational resp. meromorphic functions work and prove a **central limit theorem**.

Functional equations

Non-linear functional equations: $\Phi_{aa}(x, u, a) \neq 0$.

Suppose that $\boxed{A(x, u) = \Phi(x, u, A(x, u))}$, where $\Phi(x, u, a)$ has a power series expansion at $(0, 0, 0)$ with non-negative coefficients and $\Phi_{aa}(x, u, a) \neq 0$.

Let $x_0 > 0$, $a_0 > 0$ (inside the region of convergence) satisfy the system of equations:

$$\boxed{a_0 = \Phi(x_0, 1, a_0), \quad 1 = \Phi_a(x_0, 1, a_0)}.$$

Then there exists analytic function $g(x, u), h(x, u)$, and $\rho(u)$ such that locally

$$\boxed{A(x, u) = g(x, u) - h(x, u)\sqrt{1 - \frac{x}{\rho(u)}}}.$$

Functional equations

Idea of the Proof.

Set $F(x,u,a) = \Phi(x,u,a) - a$. Then we have

$$\begin{aligned} F(x_0,1,a_0) &= 0 \\ F_a(x_0,1,a_0) &= 0 \\ F_x(x_0,1,a_0) &\neq 0 \\ F_{aa}(x_0,1,a_0) &\neq 0. \end{aligned}$$

Weierstrass preparation theorem implies that there exist analytic functions $H(x,u,a)$, $p(x,u)$, $q(x,u)$ with $H(x_0,1,a_0) \neq 0$, $p(x_0,1) = q(x_0,1) = 0$ and

$$F(x,u,a) = H(x,u,a)\Big((a-a_0)^2 + p(x,u)(a-a_0) + q(x,u)\Big).$$

Functional equations

$$F(x,u,a)=0 \quad \Longleftrightarrow \quad (a-a_0)^2+p(x,u)(a-a_0)+q(x,u)=0.$$

Consequently

$$\begin{aligned} A(x,u) &= a_0-\frac{p(x,u)}{2}\pm\sqrt{\frac{p(x,u)^2}{4}-q(x,u)} \\ &= \boxed{g(x,u)-h(x,u)\sqrt{1-\frac{x}{\rho(u)}}}, \end{aligned}$$

where we write

$$\frac{p(x,u)^2}{4}-q(x,u)=K(x,u)(x-\rho(u))$$

which is again granted by the Weierstrass preparation theorem and we set

$$g(x,u)=a_0-\frac{p(x,u)}{2} \quad \text{and} \quad h(x,u)=\sqrt{-K(x,u)\rho(u)}.$$

Functional equations

A central limit theorem for functional equations

Suppose that $\boxed{A(x,u) = \Phi(x,u,A(x,u))}$, where $\Phi(x,u,a)$ has a power series expansion at $(0,0,0)$ with non-negative coefficients and $\Phi_{aa}(x,u,a) \neq 0$ (+ *minor* technical conditions). Set

$$\mu = \frac{x_0 \Phi_x(x_0,1,a_0)}{\Phi(x_0,1,a_0)} \quad \text{and} \quad \sigma^2 = \text{“}long\ formula\text{''}.$$

Then then random variable X_n defined by $\mathbb{P}\{X_n = k\} = a_{n,k}/a_n$ satisfies a **central limit theorem** with

$$\mu_n = n\mu \quad \text{and} \quad \sigma_n^2 = n\sigma^2.$$

Functional equations

Number of leaves in Cayley trees $(T(x) = xe^{T(x)})$

$$T(x,u) = xe^{T(x,u)} + x(u-1)$$

$$x_0 = \frac{1}{e}, \quad t_0 = T(x_0) = 1.$$

$\Longrightarrow$ **central limit theorem** with $\mu_n = \frac{1}{e}\,n$ and $\sigma^2 = \frac{e-2}{e^2}\,n$.

Functional equations

Systems of functional equations

Suppose, that several generating functions

$$A_1(x,u) = \sum_{n,k} a_{1;n,k} u^k x^n, \ldots, A_r(x,u) = \sum_{n,k} a_{r;n,k} u^k x^n$$

satisfy a **system of non-linear equations**.

Then (under *suitable conditions*) all generating functions $A_j(x,u)$ (usually) have a squareroot singularity and a **central limit theorem** for corresponding random variables is **expected**.

Non-normal limit theorems

Example 1

$a_{n,k}$ = number of words "$aa\cdots abb\cdots b$" of length n with k letters b.
$= 1$ for $0 \le k \le n$.

$$A(x,u) = \frac{1}{1-x} \cdot \frac{1}{1-xu}$$

and

$$\boxed{\frac{X_n}{n+1} \xrightarrow{\mathrm{d}} U}$$

(U ... uniform distribution on $[0,1]$)

Non-normal limit theorems

Why is there NO central limit theorem?

$A(x, u)$ is a rational function BUT there is **no single root** $\rho_1(u)$ that dominates for u in a neighbourhood of 1.

Furthermore, for $u = 1$ there is a double pole, for $u \neq 1$ two single poles.

Non-normal limit theorems

Example 2

$f_{n,k}$ = number of forests with n nodes of k Cayley trees

X_n = number of trees in a random forest with n nodes.

$$F(x,u) = e^{uT(x)} = \sum_{k\ge 0} u^k \frac{T(x)^k}{k!}$$

Discrete limit distribution: $\boxed{\lim_{n\to\infty} \mathbb{P}\{X_n = k\} = \frac{e^{-1}}{(k-1)!}}$.

Non-normal limit theorems

Expected value (Ex 2)

$$\frac{\partial}{\partial u} F(x,u)\Big|_{u=1} = T(x)e^{T(x)}$$

$$T(x) = xe^{T(x)}, \quad T(x) = 1 - \sqrt{2}\sqrt{1-ex} + \cdots, \quad [x^n]e^{T(x)} = (n+1)^n$$

$$\Longrightarrow \quad T(x)e^{T(x)} = e - 2e\sqrt{2}\sqrt{1-ex} + \ldots$$

$$\Longrightarrow \quad \mathbb{E}\, X_n \sim \frac{2en!e^n n^{-3/2}(2\pi)^{-1/2}}{(n+1)^n} = 2.$$

Non-normal limit theorems

Limiting probabilities (Ex 2)

Similarly

$$\mathbb{P}\{X_n = k\} = \frac{n![x^n]\frac{T(x)^k}{k!}}{n^{n-1}}.$$

$$\frac{T(x)^k}{k!} = \frac{1}{k!} - \frac{\sqrt{2}}{(k-1)!}\sqrt{1-ex} + \ldots$$

$$\implies \quad \lim_{n\to\infty} \mathbb{P}\{X_n = k\} = \frac{e^{-1}}{(k-1)!} \qquad (k \geq 1).$$

Non-normal limit theorems

Example 3

$r_{n,k} =$ number of mappings on $\{1, \ldots n\}$ with k cyclic points; $r_n = n^n$.

$X_n =$ number of cyclic points in random mappings on $\{1, 2 \ldots n\}$.

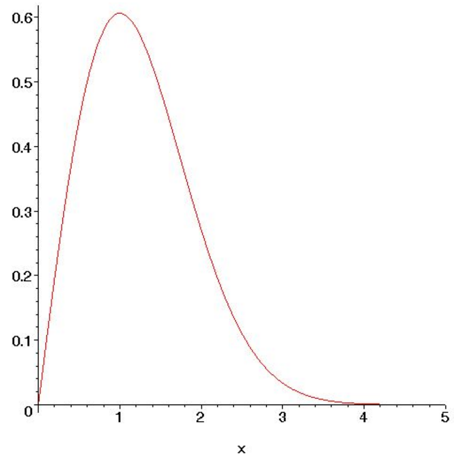
$$\boxed{R(x,u) = \sum_{n,k \geq 0} r_{n,k}\, u^k \frac{x^n}{n!} = \frac{1}{1 - uT(x)}}.$$

Rayleigh limiting distribution

$$\boxed{\frac{X_n}{\sqrt{n}} \xrightarrow{\text{d}} \mathcal{R}}$$

Non-normal limit theorems

Rayleigh distribution



density: $f(x) = xe^{-\frac{1}{2}x^2}$, $x \geq 0$.

distribution function $F(x) = 1 - e^{-\frac{1}{2}x^2}$, $x \geq 0$.

moments: $\mathbb{E}\left(\mathcal{R}^r\right) = 2^{r/2}\Gamma\left(\frac{r}{2} + 1\right)$.

Method of moments

Theorem

Z_n and Z random variables such that

$$\boxed{\lim_{n \longrightarrow \infty} \mathbb{E}\left(Z_n^r\right) = \mathbb{E}\left(Z^r\right)}$$

for all r and the moments $\mathbb{E}\left(Z^r\right)$ uniquely define the distribution of Z (for example the moment generating function $\mathbb{E}e^{vZ}$ exists around $v = 0$) then

$$\boxed{Z_n \xrightarrow{\mathrm{d}} Z}.$$

Method of moments

Moments and generating functions

$$A_n(u) = \sum_{k\geq 0} a_{n,k} u^k, \quad \mathbb{P}\{X_n = k\} = \frac{a_{n,k}}{A_n(1)}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}\Big(X_n(X_n-1)\cdots(X_n-r+1)\Big) = \frac{1}{A_n(1)} \left.\frac{\partial^r A_n(u)}{\partial u^r}\right|_{u=1}.}$$

Remark:

$$\left.\frac{\partial^r}{\partial u^r} A(x,u)\right|_{u=1} = \sum_{n\geq 1} A_n(1)\cdot \mathbb{E}\Big(X_n(X_n-1)\cdots(X_n-r+1)\Big)\cdot x^n.$$

Method of moments

Example 3 (continued)

$$R(x,u) = \frac{1}{1-uT(x)}$$
$$T(x) = 1-\sqrt{2}\sqrt{1-ex}+\cdots$$

$$\Longrightarrow \quad \frac{\partial^r}{\partial u^r} R(x,u)\Big|_{u=1} = \frac{r!T(x)^r}{(1-T(x))^{r+1}} \sim \frac{r!}{2^{\frac{r+1}{2}}(1-ex)^{\frac{r+1}{2}}}$$

$$\Longrightarrow \quad \frac{n!}{n^n}\cdot \mathbb{E}\Big(X_n(X_n-1)\cdots(X_n-r+1)\Big) \sim \frac{r!}{2^{\frac{r+1}{2}}} \frac{n^{\frac{r-1}{2}}e^n}{\Gamma\frac{r+1}{2}}$$

$$\Longrightarrow \quad \mathbb{E}\Big(X_n(X_n-1)\cdots(X_n-r+1)\Big) \sim n^{r/2}2^{r/2}\Gamma\left(\frac{r}{2}+1\right)$$

$$\Longrightarrow \quad \frac{X_n}{\sqrt{n}} \xrightarrow{\mathsf{d}} \mathcal{R}.$$

Admissible functions and centr. limit ths.

Hayman admissible functions

$$\boxed{f(z) = \sum_{n \geq 0} f_n z^n}$$

$$a(z) := \frac{z\, f'(z)}{f(z)} \qquad b(z) := z\, a'(z).$$

If $f(z)$ is **Hayman-admissible** and r_n is defined by $\boxed{a(r_n) = n}$ then

$$\boxed{f_n \sim \frac{f(r_n) r_n^{-n}}{\sqrt{2\pi b(r_n)}}}.$$

Admissible functions and centr. limit ths.

A recursively defined class of admissible functions

- $P(z)$ polynomial $\Longrightarrow$ $\boxed{e^{P(z)}}$ is admissible (if is has only non-negative coefficients).
- $f(z)$ admissible $\Longrightarrow$ $\boxed{e^{f(z)}}$ is admissible
- $P(z)$ non-negative polynomial, $f(z), g(z)$ admissible $\Longrightarrow$ $\boxed{P(z)f(z)}$, $\boxed{P(f(z))}$, $\boxed{f(z)g(z)}$ admissible.

Examples: $f(z) = e^{z+\frac{z^2}{2}}$, $f(z) = e^{e^z-1}$, ...

Admissible functions and centr. limit ths.

Recursively defined EXTENDED admissible functions

RULE 1

- $P(z,u)$ polynomial $\Longrightarrow$ $\boxed{f(z,u) = e^{P(z,u)}}$ is e-admissible (if is has only non-negative coefficients and positive coefficients at least in a cone)

- $f(z)$ admissible, $g(u)$ analytic for $|u| < 1 + \varepsilon$, $g(1) > 0$, $g'(1) + g''(1) - g'(1)^2/g(1) > 0 \Longrightarrow$ $\boxed{e^{f(z)g(u)}}$ is e-admissible.

Admissible functions and centr. limit ths.

RULE 2

Suppose that $f(z,u)$ and $g(z,u)$ are e-admissible, $h(z)$ is admissible and $P(z,u)$ is a polynomial with non-negative coefficients. $\Longrightarrow$

- $\boxed{f(z,u)g(z,u)}$ is e-admissible
- $\boxed{h(z)f(z,u)}$ is e-admissible
- $\boxed{P(z,u)f(z,u)}$ is e-admissible
- $\boxed{e^{f(z,u)}}$ is e-admissible
- $\boxed{e^{P(z,u)h(z)}}$ is e-admissible if P depends at least on u.
- $\boxed{e^{P(z,u)+h(z)}}$ is e-admissible if P depends on u and if h is entire
- $\boxed{P(z,u)+f(z,u)}$ is e-admissible

Admissible functions and centr. limit ths.

Theorem

$$f(z,u) = \sum_{n,k\geq 0} f_{n,k} u^k z^n \quad \text{e-admissible}, \quad \mathbb{P}\{X_n = k\} = \frac{f_{nk}}{f_n}.$$

$$\Longrightarrow \quad \boxed{\frac{X_n - \bar{a}(r_n,1)}{\sqrt{|B(r_n,1)|/b(r_n,1)}} \xrightarrow{\text{d}} \mathcal{N}(0,1)},$$

where $a(z,u) = zf_z(z,u)/f(z,u)$, $\boxed{a(r_n,1) = n}$, $\bar{a}(z,u) = uf_u(z,u)/f(z,u)$, $b(z,u) = za_z(z,u)$, $c(z,u) = ua_u(z,u) = z\bar{a}_z(z,u)$, $\bar{b}(z,u) = u\bar{a}_u(z,u)$, and

$$|B(z,u)| = \det \begin{pmatrix} b(z,u) & c(z,u) \\ c(z,u) & \bar{b}(z,u) \end{pmatrix}.$$

Admissible functions and centr. limit ths.

Example 1: Stirling numbers of the second kind

$$\boxed{S(z,u) = \sum_{n,k\geq 0} S_{n,k}\cdot u^k \cdot \frac{x^n}{n!} = e^{u(e^z-1)}}$$

$$[e^z - 1 \text{ admissible} \Longrightarrow S(z,u) \text{ e-admissible}]$$

Stirling numbers of the second kind satisfy a **central limit theorem** with $\mu_n = n/\log n$ and $\sigma_n^2 = n/(\log n)^2$.

Admissible functions and centr. limit ths.

Example 2: Permutations with bounded cycle length

$p_{\ell;n,k}$ = number of permutation of $\{1,\ldots,n\}$ with k cycles $\leq \ell$.

$$\boxed{P_\ell(z,u) = \sum_{n,k\geq 0} p_{\ell,n,k} \cdot u^k \cdot \frac{x^n}{n!} = e^{u\left(x+\frac{x^2}{2}+\cdots+\frac{x^\ell}{\ell}\right)}}.$$

We get a **central limit theorem** with $\mu_n = \dfrac{n}{\ell}$ and $\sigma_n^2 = \dfrac{n^{1-\frac{1}{\ell}}}{\ell^2(\ell-1)}$. $(\ell \geq 2)$

Thanks for your attention!