may be adopted instead of the orthogonal lattice). While we could have proceeded directly to discrete shapes, it is felt that a more solid and comprehensive understanding of the essence of shapes and shape-related issues including invariance, transformations and characterization can be achieved by starting with continuous shapes as connected sets of points in *continuous* spaces. Indeed, most discrete shapes can be understood as the result of sampling some continuous shape according to a pre-defined quantization scheme (see Section 3.1.4). Since the choice of such a scheme defines many of the properties of the respectively obtained discrete shapes and possibly affects the properties of continuous shapes (such as invariance to rotation) in particular ways, the continuous approach provides a more unified and uniform treatment of shapes and their properties.

4.2.1 Continuous Shapes and their Types

As discussed above, connectivity is herein considered to be *the* essential feature characterizing shapes. As will soon become clear, connectivity corresponds to a well-defined, although not trivial, mathematical concept. Informally speaking, it indicates that any two points inside a given set can be reached through at least a path fully contained in that set. Figure 4.2 presents several examples of planar (i.e., 2D) sets of points that satisfy the shape definition in the previous section. The reader should have no difficulty verifying that all these shapes correspond to connected sets.

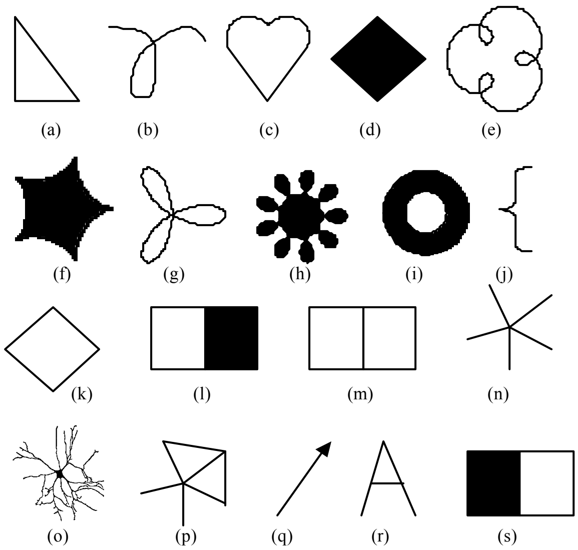


Figure 4.2: *Examples of planar continuous shapes. (The neural cell in (o) has been reprinted from Journal of Neuroscience Methods, 27, T. G. Smith Jr., W. B. Marks, G. D. Lange, W. H. Sheriff Jr. and E. A. Neale, A Fractal Analysis of Cell Images, 173-180, Copyright (1989), with permission from Elsevier Science.)*

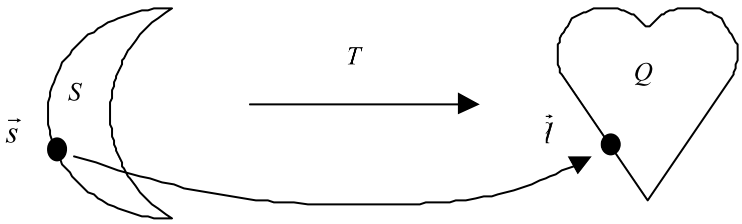


Figure 4.8: *The set of points composing shape S, including $\vec{s}$, are mapped by the morphic transformation T into a new shape $Q = T(S)$.*

Morphic transformations are particularly interesting because they allow us to relate different shapes. As a matter of fact, to identify the transformation taking one shape into another corresponds to learning almost everything about the relationship between both of these shapes. The transformation itself, as well as its properties (such as local magnification and phase, to be discussed in Section 4.9), often provides important insights about the physical processes relating distinct versions of a shape. For instance, the uniform distortion of a square into a rhombus, under the action of some parallel but opposing forces applied at two opposite vertices, can be verified to be representable by an affine transformation. Observe, however, that the interpretation of the obtained transformation taking a shape into another shape can only be properly validated and understood when considered in the light of additional information about the possible processes acting over the shapes, and the physical properties of the latter. Morphic transformations are discussed in more depth in Section 4.9.

We conclude this section with an example of a morphic transformation, illustrated in Figure 4.9 in terms of its *x*- (a) and *y*- (b) scalar field components (refer to Section 2.4.1), that transform a single shape (c) into the composed shape in (d).

4.4 CHARACTERIZING 2D SHAPES IN TERMS OF FEATURES

Given a specific shape, or a composed shape, it is often necessary to quantify some of its properties. This task, which constitutes one of the basic steps in shape classification, is henceforth called *shape characterization*. The respective *characterization* of a shape S can be made in terms of a series of its respective measures and properties, which are commonly referred to as *features*. For instance, a shape can be characterized in terms of its area, the total arc length of its boundary (i.e., its perimeter), its number of holes, its involves considering the distances between several of the points in the shape. Chapter 8 provides further discussion on how to select features for shape classification.

We have seen in the above discussion that features, represented in terms of a feature vector, can be associated with shapes. It is important to note that feature vectors live in a respective *feature space*. For instance, a feature vector composed of M real scalar measures is a vector in R^M. Although it is possible to consider heterogeneous feature spaces, defined by feature vectors involving more than one type of scalar (such as real, Boolean, etc.), most of the situations in this book will be restricted to real feature spaces, i.e., R^M. This important concept is illustrated in Figure 4.11, where a shape has been measured with respect to its area and perimeter, allowing it to be mapped into the feature space (perimeter)×(area), which is clearly a subset of R^2. It is generally expected that similar shapes will be mapped into feature vectors (i.e., points in the feature space) that are close to each other. Such a property is often implied by the continuity of the transformation taking the shape into its respective features. In case this transformation is continuous and the shapes define a continuum of variations, the respectively obtained representation in the feature space produced by a continuous feature extraction transformation will also be connected. The concept that similar shapes have similar features is illustrated in Figure 4.12, which shows a series of feature points (b), respective to the normalized centroid size (see Section 4.8.3) and perimeter of the similar polygonal shapes (a). It is clear that the feature points are close each other, starting to form a *cluster* in the feature space. On the other hand, a discontinuous feature extraction transformation will tend to imply disconnected regions in the feature space.

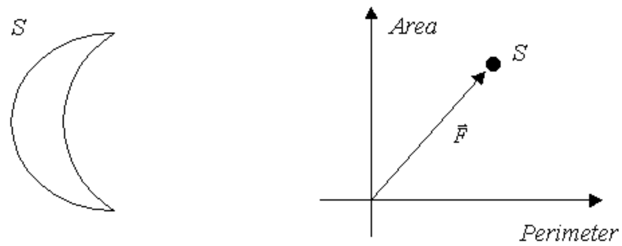


Figure 4.11: *The S shape mapped into the feature space defined by area and perimeter.*

preserved. This type of transformation has often been used in practice, for instance as a means of relating different species (see [Thompson, 1992]). Another important type of morphic transformations are defined by projections (see, for instance, the interesting book [Dubery and Willats, 1972]).

The main importance of morphic transformations in shape analysis stems from the fact that the shapes of interest often appear in a transformed fashion. For instance, a shape submitted to a shearing process (see Figure 4.32) will be related to the original shape by an affine transform. In case both such types of shapes, i.e., the original and transformed versions, are to be understood as equivalent, the selected features have to be invariant to the transformation in question. This is an especially important concept in shape analysis and classification, deserving special attention. For instance, in case the original shape (a) and its affine transformation (b) in Figure 4.32 are to be considered as equivalent, a suitable feature would be the number of vertices or the number of holes, which is invariant to the transformation in question. In case a shape and one of its rotated versions are to be considered as equivalent, suitable features would be the area, perimeter, number of vertices, etc. Therefore, an important issue in shape analysis and classification concerns the identification of the transformations underlying the several observed instances of the same shape. Often, such transformations are a direct consequence of the natural processes producing the shapes. Unfortunately, it is not always easy to identify such transformations, except in the case of simple transformations such as the affine and its specific cases. To cope with this problem one should consider several transformations and find that which better explains the observed shape variations. The remainder of this chapter presents the application of thin-plate splines as a reasonably general means to interpolate shape transformations. More specifically, given the original shape and its transformed version, both expressed in terms of landmark points, the thin-plate formulation allows us to obtain an interpolated approximation of the sought transformation.

4.9.6 Thin-Plate Splines

The concept of thin-plate splines was first applied to the analysis of plane shapes by Bookstein [Bookstein 1991]. In such approaches, a thin-plate is understood as a thin sheet of some stiff material (e.g., steel) with infinite extension. When specific control points along the plate are displaced, the plate undergoes a deformation in such a way as to minimize the total bending energy E implied by the transformation. This formulation can be immediately extended to planar shapes by using pairs of thin-plates, represented in terms of landmark points. We start by presenting and illustrating the traditional thin-plate and proceed by discussing its extension as a means to interpolate morphic transformations.

$$C = \begin{bmatrix} w_1 & w_2 & \cdots & w_n & a & b_x & b_y \end{bmatrix}^T$$

Now, the sought coefficients required to interpolate through the n control points can be easily obtained by solving the following simple matrix equation, provided the matrix M is not singular:

$$C = M^{-1}H \tag{4.20}$$

The bending energy E can be immediately obtained as

$$E = W^T T W \tag{4.21}$$

where W is the $n \times 1$ vector containing the n first rows of C.

The accompanying box provides a complete example of the above described thin-plate spline interpolation.

Example: ***1D Thin-Plate Spline Interpolation***

Obtain the thin-plate spline passing through the following five control points (–1, 0, 4); (0, 1, 5); (0, –1, 3); (1, 0, 4); and (0, 0, 2). These points are shown in Figure 4.39 (a).

Solution:

We have $n = 5$ and start by obtaining T, S_e, H, and M:

$$T = \begin{bmatrix} 0 & a & a & b & 0 \\ a & 0 & b & a & 0 \\ a & b & 0 & a & 0 \\ b & a & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad S_e = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad H = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 0 & \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{g} & 1 & -1 & 0 \\ \boldsymbol{a} & 0 & \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{g} & 1 & 0 & 1 \\ \boldsymbol{a} & \boldsymbol{b} & 0 & \boldsymbol{a} & \boldsymbol{g} & 1 & 0 & -1 \\ \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{a} & 0 & \boldsymbol{g} & 1 & 1 & 0 \\ \boldsymbol{g} & \boldsymbol{g} & \boldsymbol{g} & \boldsymbol{g} & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that values $\boldsymbol{a} = g\left(d_a = \sqrt{2}\right) \cong 0.6931$, $\boldsymbol{b} = g\left(d_b = 2\right) \cong 2.7726$ and $\gamma = g\left(d_\gamma = 1\right) \cong 0$ correspond to the respective distances between the *x*- and *y*-coordinates of the control points, as shown in the Figure 4.40. Now, by applying (4.20) we get

$$C = M^{-1}H = [0.4809 \quad 0.4809 \quad 0.4809 \quad 0.4809 \quad -1.9236 \quad 2 \quad 0 \quad 1]$$

and the bending energy is

$$E = W^T TW = 3.8472$$

The obtained interpolating spline has been used to transform a uniform grid, and the result is shown in Figure 4.39(b).

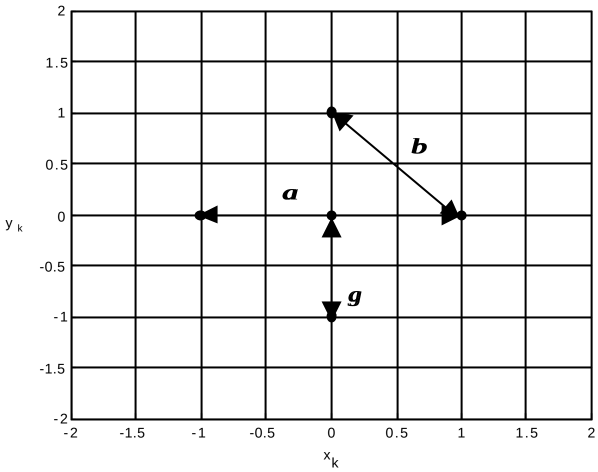
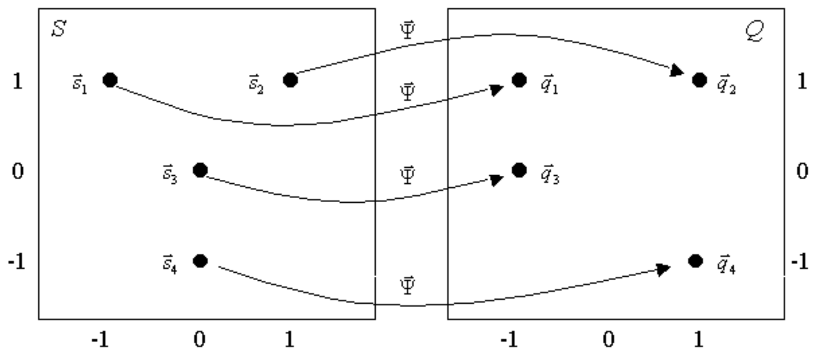


Figure 4.40: *The distances considered for calculating the elements of matrix M.*



$$\vec{s}_1 = (x_1, y_1) = (-1,1); \qquad \vec{q}_1 = (\tilde{x}_1, \tilde{y}_1) = (-1,1) = (\Psi_x(x_1, y_1), \Psi_y(x_1, y_1))$$
$$\vec{s}_2 = (x_2, y_2) = (1,1); \qquad \vec{q}_2 = (\tilde{x}_2, \tilde{y}_2) = (1,1) = (\Psi_x(x_2, y_2), \Psi_y(x_2, y_2))$$
$$\vec{s}_3 = (x_3, y_3) = (0,0); \qquad \vec{q}_3 = (\tilde{x}_3, \tilde{y}_3) = (-1,0) = (\Psi_x(x_3, y_3), \Psi_y(x_3, y_3))$$
$$\vec{s}_4 = (x_4, y_4) = (0,-1) \qquad \vec{q}_4 = (\tilde{x}_4, \tilde{y}_4) = (1,-1) = (\Psi_x(x_4, y_4), \Psi_y(x_4, y_4))$$

Figure 4.42: *Concepts and representations adopted in the pair of thin-plate splines approximation to shape analysis. S is the original shape and Q its transformed version or another shape to which S is to be compared.*

The sought interpolating thin-plate splines $\Psi_x(x, y)$ and $\Psi_y(x, y)$ can be obtained by applying the procedure described in Section 4.9.6.1 separately, or through the integrated approach presented in the following. First, matrices *T*, Q_e and *M* are obtained by the method described in Section 4.9.6.1 (items A, B and C, respectively). The following two additional matrices are then constructed:

(AA) The $(n+3)\times 2$ matrix H_2 containing the coordinates of the landmark points of shape Q is

$$H_2 = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_n & 0 & 0 & 0 \\ \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_n & 0 & 0 & 0 \end{bmatrix}^T$$

(BB) The $(n+3)\times 2$ matrix C_2 containing all the parameters in Equations (4.22) or (4.23):

$$C_2 = \begin{bmatrix} w_{x,1} & w_{x,2} & \cdots & w_{x,n} & a_x & b_{x,x} & b_{x,y} \\ w_{y,1} & w_{y,2} & \cdots & w_{y,n} & a_y & b_{y,x} & b_{y,y} \end{bmatrix}^T$$

The coefficients in Equations (4.22) and (4.23) can now be obtained by solving the following matrix equation:

$$C_2 = M^{-1} H_2 \tag{4.25}$$

and the bending energy E can be obtained as

$$E = Trace\{W^T TW\} \tag{4.26}$$

Figure 4.43 illustrates the application of the thin-plate splines to planar shapes. S is the original shape, and the shape Q its transformation or the shape to which it is being compared. The obtained interpolating thin-plate spline has been used to transform the orthogonal grid of Figure 4.43(a) into the deformed grid in (b). The bending energy E implied by this transformation is 1.4779. It should be borne in mind that the bending energy implied by the inverse morphic transformation, as implemented by the thin-plate formulation, is not equal to the bending energy implied by the respective direct transformation. In addition, observe that, as illustrated in Figure 4.44, excessive displacement of the control points can cause folding (or overlap) of the interpolating surface.

Figure 4.45 presents a more sophisticated example regarding the dynamic modification of a neural cell from the original configuration shown by circles into the new shape shown by asterisks. Observe that the thin-plate spline interpolation approach allows the comprehensive characterization of the spatial deformations related to the shape alterations, in this case the bending (identified by one asterisk) and growth (identified by two asterisks) of dendrites.

To probe further: ***Shapes***

Additional material on shape related concepts can be found in the literature of the most diverse areas, from complex variables to biological shape analysis. A good reference on transformations underlying projective drawing systems can be found in [Dubery and Willats, 1972]. A relatively old but still interesting book is [Thomson, 1992], a traditional approach to biological shape. See also M. Ghyka, The Geometry of Art and Life, Dover, NY, 1977, which provides a general perspective of shapes in art and life mostly in terms of proportions and T. Cook, The Curves of Life , Dover, NY, 1979, which concentrates on spiral shapes. More modern treatments of shapes include [Bookstein, 1991; Dryden and Mardia, 1998; Small, 1996; Otterloo, 1991].