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ORIGINAL ARTICLE

Research into teacher knowledge: a stimulus for development in mathematics teacher education practice

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Abstract In this paper, we document some developments in teacher education practice at one university, brought about by reflection on research into mathematics teacher knowledge. The authors are three members of the Cambridge-based research team who developed the Knowledge Quartet (KQ), a theory of mathematics teacher knowledge, with a focus on classroom situations in which this knowledge is applied. At the same time as being researchers, the authors were elementary mathematics teacher education instructors. They found that the KQ research brought about new awareness of the importance of some components of mathematics didactics, as well as providing new tools for undertaking some aspects of their teacher educator role. The paper explores some of these awarenesses and tools in detail.

Keywords Teacher knowledge · Mathematics teacher educator · Knowledge Quartet

1 Introduction

This paper is a contribution to a young field of research, which seeks to understand the ways in which mathematics teacher educators (MTEs) might grow in wisdom, competence and effectiveness in their work. Tzur (2001), an early contributor to the field, referred to the near absence of published MTE-development research as a "void in the literature" (p. 259). More recently, Goos (2009, p. 210) itemises evidence of the flourishing of mathematics teacher education as a distinctive field of enquiry, but adds that "research on the development of mathematics teacher educators is still in its infancy, with few published studies". It is to be expected that this will change over time: The 4th volume of the *International Handbook of Mathematics Teacher Education* (Jaworski and Wood 2008) is firmly established as a key reference in this new field, and a recent PME Discussion Group was devoted to MTEs' knowledge (Beswick and Chapman 2013). Meanwhile, the state of the art has similarities with the emergence of mathematics teaching as a research field, at first fuelled by action research, in the 1980s: until then, the research gaze was on students rather than teachers. Likewise, researchers into mathematics teaching, themselves typically MTEs, have only recently viewed themselves (or their work) as suitable objects of research, having previously attended to the knowledge and performance of their own 'students'.

The paper is a reflective account of an important phase in our own development as MTEs, in which we will describe and account for changes in our awareness and practice as a consequence of our activity as researchers in the field of mathematics teacher knowledge. The paper begins with a consideration of some theoretical perspectives on the learning of mathematics teacher educators, and a brief account of a research project on mathematics teacher knowledge, to which each of us made a major contribution. We then outline the substantive outcome of that research project, the 'Knowledge Quartet': a framework for the observation and analysis of mathematics teaching, with a focus on classroom application of teachers' mathematics-related knowledge. The remainder of the paper is devoted to reflection on, and discussion of, some ways in which engagement in this research project had a direct impact on our professional work with prospective teachers, thereby (we believe) contributing to our development as MTEs, and making us 'better' teacher educators. Despite having,

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between us, over 70 years' experience of preparing trainee teachers in elementary mathematics, we realised that this particular research activity had stimulated in us new awarenesses of the importance of some components of mathematics didactics, as well as providing new tools for undertaking some aspects of our teacher educator role. Details of these awarenesses and tools will be given later in the paper.

2 Theoretical framework

The professional objectives of teacher educators, including MTEs, are somewhat vicarious compared with those of teachers of mathematics. The latter group aim for their students to learn mathematics. Admittedly there is an absence of consensus about the purposes of learning mathematics (Ernest 2000), but the objective itself is rarely contested. However, consider MTEs; some aspects of what their students learn might be valued 'for its own sake'—for example an epistemological insight, or a new mathematical understanding. But the enterprise of mathematics teacher education is ultimately justified by the provision of new (or 'better') teachers of learners of mathematics, and its success is ultimately measured by outcomes once removed from their immediate field of influence. Bergsten and Grevholm (2008) write:

> The professional development of mathematics teacher educators must thus be seen in relation to the objectives of teacher education, which in turn is related to the objectives of the teaching profession. (Bergsten and Grevholm 2008, p.223.)

Perhaps it is for this reason that models of MTE activity (or knowledge, or learning) tend to encapsulate extant models of student and mathematics teacher activity/knowledge/learning within some superordinate structure, often presented in terms of a hierarchy of layers (of knowledge etc.), each layer being subsumed into the next (Goos 2009; Prestage and Perks 2001; Zaslavsky and Leiken 2004). In contrast, Bergsten and Grevholm (2008) propose a matrix model of mathematics teacher preparation activity which shows the teaching and learning of MTEs alongside that of other mathematics teacher education agents—students, student teachers, teachers, and university mathematicians. The model draws attention to the relationship between teaching and learning. In teaching *mathematics* a reflective MTE may also learn more mathematics. In teaching *how to teach mathematics* a reflective MTE may learn more about how to teach mathematics. This is undoubtedly true, but does not quite capture the MTE learning that we shall describe in this paper, since the provocation for our learning was not teaching, but research.

As we remarked, the objectives of mathematics teacher education are ultimately vicarious. It is hardly surprising, therefore, that the goals of what we are calling mathematics teacher educator 'development' are, as yet, unclear. In a Special Issue of the *Journal of Mathematics Teacher Education*, Brown and Coles (2010) address the topic in a neutral way, as "change", and ask what it might mean to say that a mathematics teacher educator has "changed". Reflecting on this question in the context of what we want to report in this paper, it is clear to us that we underwent a change in *awareness* concerning various aspects of mathematics teaching and mathematics teacher education. Mason (2008) relates awareness to attention, and to attitude, proposing that efforts to refresh and sustain effective teaching and teacher education can be "recast as educating awareness, sensitising attention, and enriching attitudes to mathematics, to learning mathematics, and to teaching mathematics" (p. 46). Our motivation for writing this paper is the realisation that important changes in our awareness came about because of our involvement in research. This 'realisation' is, of course, not so much a truth discovered as a belief reflecting the way we make sense of relatively recent changes now embedded in our MTE practice. It is of interest that the object of our research was not mathematics teacher education, but the relationship between mathematics teacher knowledge and mathematics teaching practice: we return to this observation towards the end of the paper.

In their survey of research in mathematics teacher education, Adler et al. (2005) pointed to the importance of the research activity of MTEs in helping them to understand and develop their own practice. However, there often exist institutional and cultural obstacles to MTEs developing their practice through research. The work of university departments of education is typically distributed across diverse programmes and agendas, including a leading role in the education and professional preparation of prospective teachers. There can be, in the UK at least, and probably elsewhere, a fuzzy divide between staff ('faculty') engaged in teacher preparation and those engaged in research. Thus, while teacher education is expected to be research-informed, this basis in scholarship most often rests on the research of academics other than those doing the 'training'. This state of affairs comes about for a number of reasons, and many faculty on both sides of the divide are very content with it. However, this paper exemplifies how mathematics teacher educators can benefit and learn from *their own* research activity, in ways that have direct relevance to their teacher education role. In the paper we reflect upon our own experience as education department faculty who have endeavoured to straddle the research–practice divide. We describe the integration of our research into our teaching, as we became aware of its significance for our work as MTEs.

We turn now to our research into the relationship between mathematics teacher knowledge and classroom practice.

3 The Knowledge Quartet

In 2002–2003, we undertook some empirical research into mathematics teachers' knowledge, in collaboration with two additional colleagues in Cambridge. Our approach to investigating the relationship between teacher knowledge and classroom practice was to observe and videotape novice teachers teaching. The participants were 12 graduate prospective ('trainee') elementary school teachers in our university faculty of education. We observed and videotaped two mathematics lessons taught by each participant. In the analysis of these videotaped lessons, we identified aspects of trainees' classroom actions that seemed to be informed by their *mathematics* subject matter knowledge or their *mathematical* pedagogical content knowledge (Shulman 1986). We realised later that most of these related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an 'invented' code, such as: 'choice of examples'; 'choice of representation'; 'adheres to textbook'; and 'decision about sequencing'. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team. This inductive process generated 20[1] agreed codes, which were subsequently grouped into four broad, superordinate categories, or 'dimensions'—hence the 'Quartet'. The four dimensions and the corresponding contributory codes are shown in Table 1.

3.1 Conceptualising the Knowledge Quartet

The concise conceptualisation of the Knowledge Quartet which now follows draws on the extensive range of data referred to above. As we observed earlier, the practical application of the Knowledge Quartet depends more on teachers and teacher educators understanding the broad characteristics of each of the four dimensions than on their recall of the contributory codes. For more extended accounts see Rowland, Huckstep and Thwaites (2005) and Rowland et al. (2009).

3.1.1 Foundation

Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures.

The first member of the KQ is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school, and at college/university, including initial teacher education, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge 'possessed', irrespective of whether it is being put to purposeful use. Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service and inservice). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

3.1.2 Transformation

Contributory codes: teacher demonstration; use of instructional materials; choice of representation; choice of examples.

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the KQ, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by "… the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (1987, p. 15, emphasis added). As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986 p. 9). Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature, such as the

[1] In 2002 there were, in fact, 18 codes: two more were subsequently added in the light of new data.

Table 1 The Knowledge Quartet: dimensions and contributory codes

Dimension	Contributory codes
Foundation	Awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology
Transformation	Choice and use of examples; choice and use of representation; use of instructional materials; teacher demonstration
Connection	Anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness
Contingency	Deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction

teachers' handbooks of textbook series or in the articles and 'resources' pages of professional journals. Increasingly, in the UK, teachers look to the Internet for 'bright ideas', and even for readymade lesson plans. Teachers' choice and *use of examples* has emerged as a rich vein for reflection and critique (Rowland 2008). This includes the use of examples to assist concept formation, to demonstrate procedures, and the selection of exercise examples for student activity.

3.1.3 Connection

> Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness.

The next category concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry. Indeed, a great deal of mathematics is held together by deductive reasoning. The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew et al. (1997): of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990), who also strenuously argued for the importance of connected knowledge for teaching.

Our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

3.1.4 Contingency

> Contributory codes: responding to students' ideas; deviation from agenda; teacher insight; (un)availability of resources.

Our final category concerns the teacher's response to classroom events that were not anticipated in the planning. In some cases it is difficult to see how they could have been planned for, although that is a matter for debate. In commonplace language this dimension of the KQ is about the ability to 'think on one's feet': it is about *contingent action.* Shulman (1987) proposes that most teaching begins from some form of 'text'—a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus—the teacher's intended actions—can be planned, the students' responses cannot.

Brown and Wragg (1993) suggested that 'responding' moves are the lynchpins of a lesson, important in the sequencing and structuring of a lesson, and observed that such interventions are some of the most difficult tactics for novice teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher. For further details, see Rowland, Jared and Thwaites (2011).

> A note concerning mathematical knowledge for teaching and the Knowledge Quartet

It is useful to keep in mind how the KQ differs from the well-known mathematical knowledge for teaching (MKT) egg-framework due to Deborah Ball and her colleagues at the University of Michigan, USA (Ball et al. 2008). The Michigan research team refers to MKT as a "practice-based theory of knowledge for teaching" (Ball and Bass 2003, p. 5). The same description could be applied to the Knowledge Quartet, but while parallels can be drawn between the methods and some of the outcomes, the two theories look very different. In particular, the theory that emerges from the Michigan studies aims to unpick and clarify the formerly somewhat elusive and theoretically undeveloped notions of 'subject matter knowledge' and 'pedagogical content knowledge'. In the Knowledge Quartet, however, the distinction between different *kinds* of mathematical knowledge is of lesser significance than the classification of the situations in which mathematical knowledge surfaces in teaching. In this sense, the two theories are complementary, so that each has useful perspectives to offer to the other.

4 Method

In writing this paper, our principal aim is to document and to account for some changes in our teacher educator practice that we deem to mark 'growth' or 'development', both in ourselves and in the pre-service teacher education programme in question. Our method is a form of narrative enquiry, with strong elements of autobiography. Our story began with the account of the Knowledge Quartet research, in the previous sections of this paper. It will continue in the following sections with the changes that we initiated in the pre-service teacher education programme. There are compelling reasons, made explicit later in this section, and in subsequent sections, for believing that the research described in the first part of the story motivated the changes documented in the second part.

In keeping with Connelly and Clandinin (1999), we perceive that our narrative research revolves around three foci, namely *field, field text* and *research text*. In this case, the field consists of multiple, interconnected sites—university lecture rooms, school classrooms, office spaces—associated with our 'dual' roles, as researchers and teacher educators. The field texts—in effect, the 'data'—are the narratives of our diverse professional activities in this field, some written some time ago in KQ research reports and the remainder written very recently in the compilation of this paper. Insofar as cross-checks with documentary evidence can be made, we believe that our narratives meet Denzin's (1989) criterion for 'historical truth', i.e. being in accordance with existing data on the events (notably extant research papers and teacher course documentation) that form the backdrop to our accounts. The boundary between this field text and our research text, in which the field text is embedded as a discrete and identifiable entity, is somewhat blurred. In the research text, the meaning and significance of our recorded experiences are further interpreted and constructed.

Before proceeding with the next part of our narrative—'what we have learned'—we introduce our argument for the connection between our engagement in the KQ research and the subsequent developments in our MTE practice. In keeping with Mason (2008, p. 46) we relate these developments to a process of "educating awareness". It is reasonable to suggest that the grounded theory methodology that underpinned our research was a significant factor in bringing about our new and enhanced awarenesses. Analysis of the video data entailed intensive scrutiny of the 24 lessons, which we watched, dissected, discussed and debated, over several months. The focus of our analytical attention was the application of mathematics teacher knowledge in acts of teaching. Because there were no pre-existing fine categories against which to analyse the classroom events, we were obliged to be attentive, and to respond to what we observed in creative acts of open coding. In this way, for example, a teacher's decision about how to represent (say) a set of numbers took on crucial significance and loomed large in our team discussions. Similarly, witnessing and itemising the range of options available, and the choices made about the representation of abstract mathematical entities, raised our awareness of the significance of such choices. The next awareness shift for us incubated over time, but then emerged in our MTE team discussions: awareness that we drew attention to the options for such representations in our existing 'methods' teaching with pre-service elementary teachers, but we did not draw out the consequences of different choices. Last, but not at all least, the acts of writing involved in compiling this paper were, without doubt, catalysts to purposeful, in-depth reflection on how the KQ research had brought about changes in ourselves and in our MTE practice. In his autobiography, Jean Piaget captured what we experienced: "I could not think without writing – but it had to be in a systematic fashion as if it were an article for publication". (1952, p. 61)

5 What we have learned from the Knowledge Quartet research

We now proceed to describe some of the ways in which the research outlined above brought about new awarenesses, and enabled new approaches, in our professional work as elementary mathematics educators. This will be organised into sections corresponding to specific issues, topics and approaches about which we became more sensitive and knowledgeable as a consequence of the research, namely: the role of 'theory' within initial teacher education; the role of representations and examples in mathematics teaching; the use of classroom video data within initial teacher education; and structuring review of, and reflection on, teaching.

5.1 The role of 'theory' within pre-service mathematics teacher education

A salutary finding of the longitudinal research project (Turner 2010, 2011) was that the beginning teachers did not draw on what we thought they had learned from our methods courses in the university to the extent that we might have hoped. The mathematical knowledge for teaching of beginning teachers might be expected to be mainly *propositional* (Shulman 1986), i.e. gained from their own mathematics education and from mathematics methods courses during teacher education programmes. Other forms of knowledge proposed by Shulman, i.e. *case study* or *strategic knowledge*, are likely to be more limited, as these require experience, which by definition beginning teachers do not have. Therefore, we might expect the practice of beginning teachers to draw significantly on propositional knowledge addressed during university courses, and later, with experience, to draw more often on case study and strategic knowledge. Research suggests that for beginning teachers to connect theory to practice, their learning needs to be situated in authentic experiences (Lave and Wenger 1991; Putnam and Borko 2000). Teaching placements are unlikely to provide sufficient authentic experiences for primary student teachers, for whom mathematics teaching is not the sole focus of the relatively short placements, to make such connections. It seems likely that teachers will only be able to make these connections through sustained working in classrooms alongside reflection on teaching and learning. Our research suggests that providing the KQ as a tool for reflecting on their teaching helps teachers to make links to propositional knowledge and to apply it within the context of their practice.

There were a number of instances in the longitudinal study (Turner 2010) where situations categorised under the *foundation* dimension indicated that, once in the classroom, trainees did not draw on propositional knowledge addressed during their graduate teacher education course. Although there was evidence that this was held as propositional knowledge, these beginning teachers were frequently unable to draw on this knowledge and activate it in their early teaching, as the following examples illustrate.

Amy. During her final school placement, in a lesson about counting with 4- to 5-year-old children, Amy asked her pupils to write 19 on their white boards. Several children wrote '1P', at least one wrote '99' and many wrote '91'. The trainee teacher focused on the reversal of the nine but did not address the problem of digit order. During the post-lesson interview the trainee teacher was asked why she thought children had reversed the digits. She answered:

> Because you say nine first, then you say the teen that's why often they write the nine first they often want to write nine first then write it from right to left instead of left to right.

Amy clearly knew about the problems children encounter in writing teen numbers (Anghileri 2007; Wigley 1997), but did not apply this knowledge in her practice.

Kate used a number line to help children complete addition calculations such as '8 + 8' and '3 + 4' by beginning at one of the numbers and then counting on the second number. This pre-supposed that children had reached the 'count on' stage in addition. However, observation of the children's independent use of the number lines suggested that some were still at the 'count all' stage (Carpenter and Moser 1984). Kate was asked if she remembered the stages children go through in learning addition:

> At first not knowing that you can just start at numbers, that you have to count the one, two, three … so you have to count three to get up to three before you can carry on.

Although she knew that some children would not be able to understand the addition strategy of starting with one number and then counting on the second number, this propositional knowledge was not drawn on in Kate's teaching.

We should not be surprised or disappointed when we find beginning teachers not drawing on this knowledge. Roth McDuffie, Drake and Herbel-Eisenman (2008) suggest that, to link theory to practice, beginning teachers need to work with real students, and to learn how to notice (Mason and Spence 1999; Star and Strickland 2008) without being directed by their mentors or MTEs. In their view, it is teachers' ability to focus on the mathematical thinking and learning of their pupils that is central. Discussion with Amy, structured by the KQ, helped her to focus on why her pupils had reversed the digits in the number 19, and to make connections with propositional learning from the university. Similarly, Kate was helped to relate the children's use of a number line when adding two numbers to her propositional knowledge about stages in learning about addition. At the early stages in their development as teachers of mathematics, they appeared to need the support of discussions with an MTE to make these connections. However, analysis of the same teachers' classroom practice over the next 2–3 years, using the KQ framework, suggested that these beginning teachers became more able to draw on propositional mathematical knowledge for teaching as they gained experience (Turner 2011, pp. 281–288). For propositional knowledge learned in the university to become activated in practice, beginning teachers need sustained experience of mathematics teaching, and to be supported in focused reflection on teaching which focuses on pupils' mathematical thinking and learning. The KQ was found to be a useful facilitator of such focused reflection.

5.2 The role of representations and examples in mathematics teaching

Despite our experience as teacher educators, the KQ research gave us a new appreciation and understanding of the importance of *examples* in mathematics teaching. When teachers teach mathematics they choose and use examples all the time—the relevant code was present in our coding of every lesson. In fact, our focus on examples was built on earlier work by the first author (e.g. Rowland 1998) and came at an interesting time from a national and international research perspective. While we were building an emergent theory of teacher-chosen examples (e.g. Rowland et al. 2003), Watson and Mason (2005) were developing a theory of learner-generated examples, applying and extending the ideas of Ference Marton on variation theory. Both of these perspectives were represented in a PME Research Forum (Bills et al. 2006) and in a special issue of *Educational Studies in Mathematics* (Bills and Watson 2008).

As a consequence of our own research, we realised and understood better the different purposes for which examples are used, and that the choice of examples is far from arbitrary—some examples 'work' better than others. These insights have had a significant effect on our practice in our role as mathematics teacher educators. So, whilst formerly we might have spoken in a general way about the importance of choosing examples with care, we are now able to offer our trainee teachers a more analytical account of the choice and use of examples in mathematics teaching and learning. In particular, we identify and exemplify three broad categories of examples that were commonplace in our data, but which, we argue, teachers would do well to avoid. We labelled these categories: examples which confuse the role of variables; examples intended to illustrate a particular procedure, for which another procedure would be more sensible; and randomly generated examples. For details, see e.g. Rowland et al. (2009).

By way of illustration, we exemplify the first of these categories (confusing the role of variables) here, with two excerpts from the 2002 classroom data.

Kirsty was reviewing the topic of Cartesian co-ordinates with a class of 10 to 11-year-old pupils. Kirsty began by asking the children for a definition of co-ordinates. One child volunteered that "the horizontal line is first and then the vertical line". Kirsty confirmed that this was essentially correct. She then moved on to assessing the pupils' understanding of this key convention by asking them to identify the co-ordinates of a number of points as she marked them on a co-ordinate grid, projected onto a screen at the front of the classroom. Before marking the first point, she reminded them that "the *x*-axis goes first". Kirsty's first example was the point (1, 1). It is interesting to speculate reasons for Kirsty's choice of this example, recognising that these 'reasons' might be of different types—pragmatic, pedagogical, affective and so on. In any case, the example would seem to be entirely ineffective in assessing what Kirsty wanted to determine: the children's grasp of the significance of the order of the two elements of the ordered pair.

Michael's lesson with a Year 4 class was about telling the time with analogue and digital clocks. One group was having difficulty with analogue quarter past, half past and quarter to. Michael intervened with this group, showing them first an analogue clock set at 6 o'clock. He then showed them a quarter past six and half past six. When asked to show half past seven on their clocks, one child put both hands on the seven. We can't be sure, but the child's inference from Michael's demonstration example (half past six) seems reasonably clear. Of the twelve possible examples available to exemplify half past, half past six is arguably the most unhelpful.

The role of *representations* in mathematics teaching has been extensively researched and theorised (e.g. Goldin 2002). Nevertheless, our research yielded further insights that we were able to bring to our work with trainee teachers. These include the importance of the mathematical appropriateness of representations used for pedagogical purposes. We had observed the trainees' propensity to choose representations on the basis of their superficial attractiveness at the expense of their mathematical relevance (Turner 2008). In addition, we are now better placed to emphasise the interplay between choice of representations and choice of examples. A case to which we often refer in this respect is Chloë's lesson on addition and subtraction (Rowland and Turner 2007). The objective of the lesson was for children to be made aware of 'compensation' strategies for adding and subtracting near multiples of ten (specifically, 9, 11, 19 and 21). This objective and the intended strategies are clarified in examples given in documentation (the *Framework*) published for the implementation of the National Numeracy Strategy in English primary schools in England (DfEE 1999). These include: "$58 + 21 = 79$ because it is the same as $58 + 20 + 1$; $70 - 11 = 59$ because it is the same as $70 - 10 - 1$". Chloë's chosen representation of the natural numbers to 100, and their base 10 structure in particular, was a large, vertically mounted 1–100 square. She modelled the subtraction procedures, moving a counter vertically and horizontally on the hundred square. For her first demonstration example, Chloë subtracted 19 from 70. It is reasonable to surmise that the choice of 70 was prompted by the example ($70 - 11$) in the Numeracy Strategy documentation. Because the representation in the *Framework* is *symbolic*, a minuend which is itself a multiple of ten makes the initial move (subtract a multiple of ten) straightforward. There is a complication which Chloë seems not to have anticipated when the representation is *spatial*—the hundred square. 70 is on the extreme right boundary of the 1 to 100 square. After moving up two squares to 50, there *is* no 'right one' square: it is therefore necessary to move down and to the extreme left of the next row, so the neat 'knights move' is obscured, and the procedure unnecessarily complicated. A moment's reflection reveals that 72 of the 100 available minuends would 'work' as intended, to demonstrate subtracting 19, in this representation.

5.3 Different uses of classroom video data within pre-service teacher education

The use of video in mathematics teacher education is well established (e.g. Borko et al. 2008; Maher 2008) and articulates well with case method teacher education pedagogy (Markovits and Smith 2008; Merseth 1996). In England, video resources of the kind developed by a government agency for National Numeracy Strategy training (Askew et al. 2004) have been much in evidence in primary teacher education More recently, the National Centre for Excellence in Teaching Mathematics has provided video resources online to promote current government priorities (efficient calculation) in primary mathematics (https://www.ncetm.org.uk/resources/40529). These video resources tend to feature exemplary 'best practice' examples of 'model' lessons given by experienced teachers, presumably with the intention that other teachers will emulate their example.

With the permission of the participants in our research, we use video clips from their lessons in a somewhat different way, and with a rather different purpose. When observing classroom mathematics teaching, novice elementary teachers have difficulty in analysing what they see in ways that could benefit their own teaching. Star and Strickland (2008), for example, found that prospective teachers attend to classroom environment and management issues when observing lesson videos, but have difficulty in noticing key features of the mathematics teaching. We describe in the next section how the KQ has been used as a tool to facilitate this mathematics content focus, but here we want to dwell on what pre-service teachers might learn from observing (usually on video) the kinds of exemplary teaching described above, as opposed to their novice peers' first attempts at classroom mathematics instruction.

In a study of case-based primary mathematics teacher preparation in Korea, Pang (2011) used videos of both pre-service and in-service teachers' lessons to stimulate the prospective teachers' attention to the mathematics-specific features of what they observed. Both types of video were found to be valuable, for different reasons. The expert teaching constituted a kind of 'existence proof', a realisation of the vision of good mathematics teaching that had been promoted in the university methods course, and (unlike much of what they saw in school-based placements) demonstrated to the trainee teachers that it was possible to realise the vision in practice. On the other hand, the not-so-good teaching of novices with whom they could immediately identify was especially beneficial in the analysis of how teachers draw on content and pedagogy in effective—and ineffective—instruction, including central and peripheral components of planning and teaching:

> [prospective teachers] differentiated effective mathematics lessons from seemingly good, but indeed unsuccessful, ones in terms of students' understanding. They were able to recognise that effective mathematics lessons were not related to splendid instructional materials or students' fun activities, but to the degree to which key mathematical content was meaningfully explored with students' thinking. The teachers claimed that this vision came from a vivid discussion of multiple cases in class, including unsuccessful and thought-provoking ones (Pang 2011, p. 787)

A study by Morris (2006) found that USA pre-service elementary teacher participants were more able to suggest improvements to observed mathematics instruction if the teaching was presented to them as not successful in promoting student learning. Under these conditions, the pre-service teachers' attention was found to shift from the teacher to the students. In the absence of guidance about the success of the observed instruction, the pre-service teachers' analysis appeared to be based on an assumption that "students learn what the teacher explains", and that that they kept their attention primarily on the teacher.

Using our research video data, we now deploy a range of authentic classroom scenarios to pose challenging mathematical and didactical problems, and to raise awareness and insight, in our university-based sessions with trainees. Many, if not most, of our video clips feature novice teachers, not 'experts', and as we observe them it is not hard even for trainee teachers to identify things that could be done differently, and maybe should be. We have written about some of these episodes elsewhere (e.g. Huckstep et al. 2006; Rowland 2010). These video stimuli promote lively and thoughtful discussions about what seemed to be successful and what 'went wrong', and why, and what these trainees would do themselves to avoid the errors made (in their judgement) so as to improve the instruction. By contrast, we propose that an expert teacher's lesson that 'goes well' is inspiring and motivating, but that the ingredients of its success can often be invisible to the novice trainee.

5.4 The use of the KQ to structure review of, and reflection on, teaching

Bergsten and Grevholm (2008) have described the KQ as a research-based theoretical tool which supports focused and systematic reflection on classroom practice. They suggest that use of the KQ in review discussions is a *linking practice*, helping teachers to connect their practical experience of teaching to the theoretical ideas addressed in their university courses. They consider that:

> Elements of mathematical knowledge in lesson episodes can be captured and understood, in discussions at post-observation meetings during practicum, when structured by the four dimensions of the knowledge quartet (p. 237).

Indeed, the KQ has been in use in recent years as a tool to analyse the teaching of elementary trainee teachers in our own pre-service programme, and to give detailed feedback on their *practicum* teaching, with a focus on the mathematical content of their lessons. Guidelines based on the framework (Rowland et al. 2009, pp. 35–37) were also developed to support university and school-based colleagues working with elementary trainee teachers who were not mathematics education experts. These guidelines were presented and very well received during mentor training sessions at the university, and continue to be made available to colleagues with responsibility for the supervision of school-based placements.

The use of the framework for supporting review of, and reflection on, mathematics teaching was the focus of the longitudinal study referred to earlier, between 2004 and 2008 (Turner 2010). In this study, the KQ was used as a tool to identify, analyse and chart developments in 12 beginning teachers' mathematical knowledge for teaching, and also to promote and support that development. As a tool for development, it was used to frame review discussions of mathematics teaching between teachers and the mathematics teacher educator/researcher. It was also used by the teachers to support individual reflection, helping them identify situations in which their mathematical knowledge for teaching was being put to use, and to frame their written *reflective* accounts.

In the early phases of the study, the lesson review meetings were intensive and took the form of a stimulated recall interview. During the first year of this 4 year study, the researcher [the second author] observed 11 lessons, following them up with stimulated recall interviews with the teacher in each case. She used a KQ analysis of the lessons, grouping her observations under the themes of the four dimensions, to determine questions to ask and comments to make as the teacher watched the videotape of their lesson. For example, a coding of *choice of representation* (CR) within **Transformation** in the KQ analysis suggested stopping the videotape to ask whether the trainee teacher (TT) thought the representation they had used in their explanation of a mathematical procedure was the most appropriate, or whether this might have caused some confusion.

TT	I mean one thing that I could have done here is I could have used the number line 'cus they do, they can do that and Alima certainly can do it on the number line if she was struggling with that then it would make sense to do the jumps, yer
Researcher	I was going to ask why you didn't you use an empty number line.
TT	It would have been a good point to introduce that.
Researcher	The way you are recording it there suggests what do you think?
TT	The column method, yer
Researcher	When you were doing that did you think 'I could use an empty number line'?
TT	No, no it's literally just now, I can see the number line makes so much more sense. They use it as well a lot so it would be absolutely logical.

The structure of these initial review meetings would be impossible to sustain across a large number of trainee teachers or with busy practising teachers. The methods employed in the second stage of the study were therefore more appropriate as a model for scaling up the adoption of the KQ for structuring post-lesson review meetings. During this stage, 18 lessons were observed and videotaped. Review meetings were based on a 'broad sweep' KQ analysis of detailed field notes made while observing the lesson. The researcher asked questions or commented on significant episodes which had been identified in the analysis, and the teachers made observations in relation to the codes and dimensions of the framework, with which they were now familiar. There was evidence from these review meetings that the structured reflections on their mathematics teaching influenced future practice:

Researcher	Have you thought about the knowledge quartet when you doing your maths planning or teaching?
TT	Yes definitely, like the things we talked about last time about picking the examples I have used that, like all the time, you know, picking examples or getting children to choose between two examples rather than saying just chose a number, just chose this, this or this more like that. I have used that a lot definitely (post-lesson reflective interview, phase two).

During the second phase of the study, teachers were expected to complete post-lesson reflective accounts of the observed lessons structured by the KQ. Eighteen of these were completed and analysed by the researcher.

The study also aimed to determine whether the KQ framework supported independent reflection on the mathematical content of teaching. Therefore, during the third phase teachers were not given feedback following their lessons, but were sent DVD copies of the lessons and asked to write reflective accounts independently, structured by the dimensions and codes of the KQ framework. During this phase, the researcher observed and analysed a further 18 lessons as well as analysing 18 analytical accounts of those lessons. A number of comments made by the teachers demonstrated that they found the framework useful when planning for, and reflecting on, their mathematics teaching. For example:

> I often find myself referring to it in my head when I am planning. …I think the most important effect is having the four headings, makes me more aware of what I am planning and teaching and why. You find yourself questioning yourself and justifying your decisions and choices, it makes you more purposeful klevein your choices, more precise. (Amy)

Bergsten and Grevholm (2008) discussed the importance of *practicum* (teaching practice) visits for MTE learning:

> Being present in all three milieus, that is course work at the home arena, and a taught lesson and follow-up discussion at the visiting arena, the mathematics teacher educator gets involved in [their] own teaching learning and reflecting activities as well as in those of the student teachers. (p.241)

The final phase of the study involved the researcher in observing four lessons, each followed by a post-lesson review meeting. Over the whole study the researcher observed and analysed 51 lessons using the KQ framework. She participated in 29 post-lesson review meetings which were structured by the KQ. The researcher also read and analysed 36 reflective accounts of the observed lessons. This research activity differed from 'regular' *practicum* visits undertaken in the role of MTE, in that the reflections on teaching of both the teacher and the researcher were structured by the KQ. Analysis was more rigorous and was focused on the mathematical content of the teaching. The teachers reflected on their teaching and on the learning of their pupils, and developed their understanding through discussions with the MTE and through writing their reflective accounts. The learning of the MTE was enhanced by observing and reflecting on teaching, through discussion with teachers about their teaching and learning, and through analysis of the teachers' reflective accounts. The MTE learning was also enhanced through reflection on children's learning in relation to the observed teaching and the reflective accounts of teaching. These contacts all presented opportunities for the teachers and the researcher to learn more about mathematics, about how children learn mathematics and about how to teach mathematics. The MTE also learned more about how to teach how to teach mathematics. Specifically, she learned how the KQ can be used effectively to frame lesson reviews so that they focus on the teachers' mathematical knowledge for teaching. She also learned how to use the KQ to help teachers to focus their independent reflection on the mathematical content of their teaching.

6 Conclusion

Teachers and teacher educators often approach their professional development through action research. This entails investigating one's own practice, adapting it and looking for evidence of the impact of this change. The development in our professional practice brought about by our research was a consequence of a very different process. We did not set out with the primary aim of developing our own practice. Rather, our focus was on the practice of trainee teachers as we tried to understand how their mathematical knowledge for teaching was revealed and applied in the act of teaching. However, in investigating the practice of trainee teachers, we developed a way of understanding mathematics teaching which supported our own professional development as teacher educators in a number of different ways.

Developments in our understanding of beginning teachers' mathematical knowledge for teaching, as revealed through KQ analysis of their practice, led to changes in the content of our methods courses, particularly in relation to the importance of examples and representations. We found that the mathematical knowledge for teaching that was 'learned' by trainees in our methods courses was not always available to beginning teachers in their practice. However, we discovered that teachers can be supported in applying this knowledge by provision of the KQ as a tool for focused reflection. We improved our teaching placement lesson reviews by using the KQ to focus discussion on the mathematical content of teaching, and began to induct school-based colleagues in the use of the KQ to support the mentoring of trainees. We also presented the KQ framework to trainees themselves, to support focused reflection on their mathematics teaching, so as to enable them to continue developing their mathematical knowledge for teaching during school placements and after their mathematics methods courses were completed. Furthermore, we developed new video resources for primary mathematics teacher education and new ways of using them. Finally, there was a bonus in terms of professional development from participating in the KQ research related to the development of understanding and cohesion within—and beyond—the elementary mathematics teaching team. Work using the KQ framework now involves colleagues from around the world. We continue to have intensive discussions about how we 'understand' individual codes, and this has contributed to further cohesion and cooperation, both within the team and within a much wider international KQ community (Weston, Kleve and Rowland, 2013).

These outcomes of our study illustrate the possibility of a symbiotic relationship between research into teaching and learning in classrooms and the professional development of teacher educators. The study also demonstrates how the roles of researcher and of teacher educator can be complementary and mutually supportive.

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