Basic families of functions and embeddings of free locally convex spaces

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Free abelian topological group

Let X be a completely regular topological space. The *(Markov) free abelian topological group on* X is an abelian topological group $A(X)$ which is algebraically generated by X and such that every continuous mapping f from X to an abelian topological group G lifts to a unique continuous homomorphism $\bar{f} : A(X) \to G$.

Free locally convex space

Let X be a completely regular topological space. The *free locally convex space on* X is a locally convex space $L(X)$ for which X forms a Hamel basis and such that every continuous mapping f from X to a locally convex space E lifts to a unique continuous linear operator $\bar{f} : L(X) \to E$.
The free locally convex space $L(X)$ always exists and is essentially unique.

$A(X) \hookrightarrow L(X)$

The group $A(X)$ is naturally embedded in $L(X)$ as a subgroup. Moreover, it is a topological subgroup in $L(X)$. Tkachenko announced this result in 1983; he observed that, taking any continuous pseudometric on X, extending it (by the Graev method) to a continuous seminorm on $L(X)$, and considering the restriction of this seminorm to $A(X)$, we obtain precisely the Graev extension of the pseudometric to the seminorm on $A(X)$. This assertion is true, but it needs be proved: it is not obvious that, if all coefficients are integer, then the minimum in the formula defining the norm is attained at an integer matrix. A complete proof of this fact was given by Uspenskiĭ(1990).

Tkachenko, Uspenskiĭ

The canonical homomorphism $i : A(X) \hookrightarrow L(X)$ is an embedding of $A(X)$ into the additive topological group of the LCS $L(X)$ as a closed additive topological subgroup.

Duality between $L_p(X)$ and $C_p(X)$

- Denote by $L_p(X)$ the free locally convex space $L(X)$ endowed with the weak topology. The canonical mapping $X \hookrightarrow L_p(X)$ is a topological embedding, and every continuous mapping f from X to a locally convex space E with the weak topology extends uniquely to a continuous linear operator $\bar{f} : L_p(X) \to E$.
- Denote by $C_p(X)$ the space of all continuous real-valued functions on X with the topology of pointwise (simple) convergence. Locally convex spaces $L_p(X)$ and $C_p(X)$ are in duality.

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- Denote by $C_p(X)$ the space of all continuous real-valued functions on X with the topology of pointwise (simple) convergence. Locally convex spaces $L_p(X)$ and $C_p(X)$ are in duality.

- Consequently, for every linear continuous surjection $T : C_p(X) \longrightarrow C_p(Y)$ the dual mapping $T^* : L_p(Y) \longrightarrow L_p(X)$ is an embedding of the locally convex spaces and vice versa, each embedding of the free locally convex spaces induces a linear continuous surjection of the spaces $C_p(X)$.
- For every linear open continuous surjection $T : C_p(X) \longrightarrow C_p(Y)$ the dual mapping $T^* : L_p(Y) \longrightarrow L_p(X)$ embeds $L_p(Y)$ into $L_p(X)$ as a closed linear subspace of the locally convex spaces and vice versa, each closed embedding of the free locally convex spaces induces a linear open continuous surjection of the spaces $C_p(X)$.

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Topological definitions

- A completely regular topological space X is called a *k-space* if $F \subset X$ is closed if and only if $F \cap K$ is closed in K for all compact subspaces $K \subset X$.
- A completely regular topological space X is called a *k_ω-space* if there exists what is called a *k_ω-decomposition* $X = \cup_{n\in\mathbb{N}} X_n$, where all X_n are compact, $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$, and a subset $F \subset X$ is closed if and only if all intersections $F \cap X_n$, $n \in \mathbb{N}$, are closed.
- The Lebesgue covering dimension $\dim X$ of a topological space X is defined to be the minimum value of n, such that every finite open cover α of X admits a finite open cover β of X which refines α and so that no point belongs to more than $n+1$ elements. If no such minimal n exists, the space is said to be of infinite covering dimension.

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L., Morris, Pestov, 1997

Let X and Y be k_ω-spaces. Let $h : L_p(X) \to L_p(Y)$ be an embedding of locally convex spaces. Then h also is an embedding of the locally convex space $L(X)$ into $L(Y)$.

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2. Theorem 1.2 (Pavlovskiĭ, 1980) Let X and Y be two metrizable compacta such that function spaces $C_p(X)$ and $C_p(Y)$ are linearly isomorphic (or equivalently, locally convex spaces $L_p(X)$ and $L_p(Y)$ are linearly isomorphic). Then X and Y have the same Lebesgue covering dimension: $\dim X = \dim Y$.

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One should compare this result with the famous Milutin's theorem: for any uncountable metrizable compact space X the Banach space $C(X)$ is linearly isomorphic to the Banach space $C[0,1]$.

Theorem 1.2 about $\dim X = \dim Y$ was subsequently extended by Pestov to arbitrary completely regular spaces X and Y. A further very impressive strengthening was obtained by Gul'ko in 1993: the same conclusion remains true for uniform homeomorphisms of function spaces.

Arkhangel'skiĭ's questions, 1990

1. Assume that $C_p(X)$ can be mapped by a linear continuous mapping onto $C_p(Y)$ for compact spaces X and Y. Is it true that $\dim Y \leq \dim X$?
2. Assume that $C_p(X)$ can be mapped by a linear continuous open mapping onto $C_p(Y)$ for compact spaces X and Y. Is it true that $\dim Y \leq \dim X$?

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1. Assume that $L(Y) \hookrightarrow L(X)$ for compact spaces X and Y. Is it true that $\dim Y \leq \dim X$?
2. Assume that $L(Y) \hookrightarrow L(X)$ as a closed subspace for compact spaces X and Y. Is it true that $\dim Y \leq \dim X$?
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Negative answers

- Theorem 2.1 (L., Morris, Pestov, 1997) For every finite-dimensional compactum Y the free locally convex space $L(Y)$ embeds as a linear topological subspace into the free locally convex space $L([0,1])$.
- Theorem 2.2 (L., Levin, Pestov, 1997) For every finite-dimensional compactum Y there exists a 2-dimensional compactum X such that $L(Y)$ embeds as a closed linear topological subspace into $L(X)$.

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Let us mention that in general, linear continuous surjections of $C_p(X)$ spaces fail to be open. Define a mapping $T : C_p[0,1] \longrightarrow C_p[0,1]$ as follows:

$$Tf(x) = 2f(x) - f(\frac{x+1}{2})$$

Then T is a linear, continuous and nonopen mapping onto. It could be shown that the inverse mapping T^{-1} is defined by $T^{-1}g(x) = \sum_{n=o}^{\infty} \frac{1}{2^{n+1}} g\left(\frac{x+2^n-1}{2^n}\right)$.
Denote $M = \{f(x) \in C[0,1] : f(0) = 0\}$

$$L = \{g(x) \in C[0,1] : \sum_{n=o}^{\infty} \frac{1}{2^n} g\left(\frac{2^n-1}{2^n}\right) = 0\}$$

Then M is a closed linear subspace of $C_p[0,1]$, L is a dense linear subspace of $C_p[0,1]$ and $T(M) = L$, $T(C_p[0,1] \setminus M) = C_p[0,1] \setminus L$, which means that T maps an open set onto a codense set.

Characterization for groups

Theorem 2.3 (L., Morris, Pestov, 1997) For a completely regular space X the free abelian topological group $A(X)$ embeds into $A([0,1]))$ as a topological subgroup iff X is a k_ω-space such that every compact subspace of X is metrizable and finite-dimensional.

Necessary condition for LCS

Theorem 2.4 (L., Morris, Pestov, 1997) Let X be a completely regular space such that the free locally convex space $L(X)$ embeds into $L([0,1])$ as a locally convex subspace. Then X is a metrizable compactum which can be represented as a countable union of finite-dimensional compact subspaces .

Surprising as it may seem, the free locally convex space $L(\mathbb{R})$ does not embed into $L([0,1])$, in spite of the existence of canonical embeddings $A(\mathbb{R}) \hookrightarrow L(\mathbb{R})$ and $A([0,1]) \hookrightarrow L([0,1])$ and a (non-canonical one) $A(\mathbb{R}) \hookrightarrow A([0,1])$. It is yet another illustration of the well-known fact that not every continuous homomorphism to the additive group of reals from a closed additive subgroup of an (even normable) LCS extends to a continuous linear functional on the whole space.

Last published result

Levin, 2011

Theorem 2.5 For every finite-dimensional compactum Y there exists a linear open continuous surjection $L : C_p[0,1] \longrightarrow C_p(Y)$, or equivalently, for every finite-dimensional compactum Y the free locally convex space $L(Y)$ embeds as a closed linear topological subspace into the free locally convex space $L([0,1])$.

Last unpublished result

Leiderman , Levin, 2009

There is a constant C such that for each natural n, there exists a continuous linear surjection $T_n : C_p[0,1] \longrightarrow C_p([0,1]^n)$ with $||T_n|| \leq 1$ and for every $f \in C_p([0,1]^n)$ there is $g \in C_p[0,1]$ such that $T_n(g) = f$ and $||g|| \leq C||f||$

Last unpublished and unconfirmed result

P. Gartside, 2009

Let X be a metrizable compactum which can be represented as a countable union of finite-dimensional compact subspaces. Then there is a continuous linear surjection of $C_p[0,1]$ onto $C_p(X)$.

The 13th Problem of Hilbert asked to prove that solution of the general equation of degree 7: $f^7 + xf^3 + yf^2 + zf + 1 = 0$ cannot be represented as a superposition of continuous functions of two variables.

The first Kolmogorov's result on Hilbert's Problem 13 stated that every continuous function of several variables can be represented as a superposition of continuous functions of three variables (1956). Vladimir Arnold, a 19 year old third-year student, in 1957 improved Kolmogorov's construction and showed that any continuous function of three variables can be represented as a superposition of continuous functions of two variables, thus proving that Hilbert's conjecture was incorrect. Shortly thereafter, following the rule of improving every result to its sharpest form, Kolmogorov found a new construction and proved a remarkable result: every continuous real valued function of n-variables from a segment can be expressed as a superposition of functions of just one variable, and addition.

Kolmogorov's Superposition Theorem

Every continuous function f defined on the n-dimensional cube can be represented by a superposition of the form $f(x_1, x_2, ..., x_n) = \sum_{i=1}^{2n+1} f_i(\sum_{j=1}^{n} \phi_{i,j}(x_j))$, where all the functions f_i and $\phi_{i,j}$ are defined and continuous on $\mathbb{R}$ and all the functions $\phi_{i,j}$ are independent of f.

Kolmogorov's Superposition Theorem suggests many other problems. It is quite natural to separate them into analytic and topological problems. The most natural question in the analytical direction is how smooth can the functions $\phi_{i,j}$ be chosen? An answer is the following: $\phi_{i,j}$'s can be selected to be increasing, and then can be taken in the class of Lipschitz-1 functions (Fridman). This property cannot be improved. Vitushkin and Henkin proved theorems from which it follows that a Kolmogorov type theorem must fail if the functions $\phi_{i,j}$ are assumed to be continuously differentiable. Thus, Hilbert's conjecture does hold if "continuous" is replaced by "continuously differentiable".

Let X be a compact metrizable space and let $\phi_i : X \longrightarrow [0,1]$ be continuous functions. Family $\{\phi_i : i = 1, ..., k\}$ is said to be a basic family if each function $f(x) \in C(X)$ is representable in the form: $f(x) = \sum_{i=1}^{k} g_i(\phi_i(x))$, $g_i(t) \in C[0,1]$

The following theorem is a generalization of the Kolmogorov's Superposition Theorem.

Ostrand's theorem

Let X an n-dimensional compact metrizable space. Then there exists a basic family $\{\phi_i : i = 1, ..., 2n+1\}$ of continuous functions on X .

Basic embeddings

Let X_i, $i = 1, 2, ..., n$ be compacta. A closed subset $Y \subset \prod_{i=1}^n X_i$ is said to be basic if for every $g \in C(Y)$ there exist $f_i \in C(X_i)$ such that
$g(y) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n)$ for every
$y = (x_1, x_2, ..., x_n) \in Y$.
Let $Y \subset \prod_{i=1}^n X_i$, and let $X = \oplus_{i=1}^n X_i$. Clearly, $C(X) \cong \prod_{i=1}^n C(X_i)$, and we can define $L : C(X) \longrightarrow C(Y)$ by

$$L(f)(y) = f_1(x_1) + f_2(x_2) + ... + f_n(x_n)$$

where $f = (f_1, f_2, ..., f_n) \in C(X)$ and $y = (x_1, x_2, ..., x_n) \in Y$. It is easy to see that L is a linear and continuous operator for both *sup*-norm and C_p-topologies on function spaces and that L is surjective whenever Y is basically embedded into $\prod_{i=1}^n X_i$. We call L the canonical linear surjection.

Using the Ostrand's theorem

According to Ostrand theorem (1965) every n-dimensional compactum Y can be basically embedded into $[0,1]^{2n+1}$. Denote by I the unit closed interval $[0,1]$. It is rather obvious that the space $C_p(I \oplus ... \oplus I)$ is linearly isomorphic to $C_p(I)$. Therefore, due to Ostrand's theorem, for any compactum Y we get a linear continuous surjection from $C_p[0,1]$ onto $C_p(Y)$, equivalently, an embedding $L(Y) \hookrightarrow L([0,1])$. This finishes the proof of Theorem 2.1. Observe that the free abelian topological groups on Y and on $[0,1]$ sit inside the corresponding free locally convex spaces in a right way, so we obtain a topological group embedding $A(Y) \hookrightarrow A([0,1])$.

Proof of Theorem 2.2

In order to achieve a closed linear embedding of $L(Y)$ into $L(X)$ we have to investigate more carefully the properties of the canonical linear surjection. The following result about the basic embedding into the product of two spaces plays a key role.

L.,Levin, Pestov, 1997

Let X_1 and X_2 be compacta and let $Y \subset X_1 \times X_2$ be a compact basic subset. Then the canonical linear surjection

$$L : C_p(X_1 \oplus X_2) \longrightarrow C_p(Y)$$

is open.

We use also the following theorem.

Sternfeld, 1993

Every n-dimensional compactum, $n \geq 2$, can be basically embedded into the product of $n+1$ 1-dimensional compacta.

Assume that Y is an n-dimensional compactum, $n > 2$. By Sternfeld's theorem there exists a basic embedding $Y \subset \prod_{i=1}^{n+1} Z_i$ with 1-dimensional compacta Z_i. Set $X_1 = Z_1 \times Z_2$ and $Y_1 = \prod_{i=3}^{n+1} Z_i$. It is known that for every compacta Z_1, Z_2 with $\dim Z_1 = n$ and $\dim Z_2 = 1$ we have $\dim(Z_1 \times Z_2) = n + 1$. Clearly, it implies that $\dim X_1 = 2$ and $\dim Y_1 = n - 1$. It is equally easy to check that the induced embedding $Y \subset X_1 \times Y_1$ is basic. Therefore, the canonical linear surjection $L : C_p(X_1 \oplus Y_1) \longrightarrow C_p(Y)$ is open. Replacing Y by Y_1 in the above construction and proceeding by induction on n, we construct 2-dimensional compacta $X_1, X_2, ..., X_{n-2}$ such that $C_p(\oplus_{i=1}^{n-2} X_i)$ admits an open continuous linear surjection onto $C_p(Y)$, and Theorem 2.2 follows. (The last compactum X_{n-2} may happen to be 1-dimensional).

Main open problem

Problem 1

Is the canonical linear surjection open for every basic embedding?

Levin was able to answer this question affirmatively in 2 important cases: for some Sternfeld-type basic emdeddings, and for some Kolmogorov-type basic embeddings.

Levin's theorems, 2011

1. let $Y \subset X_1 \times X_2 \times \ldots \times X_{n+1}$ be a Sternfeld-type basic embedding of an n-dimensional compactum Y into the product of 1-dimensional compacta $X_1, X_2, \ldots, X_{n+1}$. Then the canonical linear surjection

$$L : C_p(X_1 \oplus X_2 \oplus \ldots \oplus X_{n+1}) \longrightarrow C_p(Y)$$

is open.

2. let $Y \subset [0,1]^3$ be a Kolmogorov-type basic embedding of 1-dimensional compactum Y into the cube. Then the canonical linear surjection is open.

3. (Theorem 2.5) For every finite-dimensional compactum Y there exists a linear open continuous surjection $L : C_p[0,1] \longrightarrow C_p(Y)$, or equivalently, $L(Y)$ embeds as a closed linear topological subspace into $L([0,1])$.

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$$L : C_p(X_1 \oplus X_2 \oplus ... \oplus X_{n+1}) \longrightarrow C_p(Y)$$

is open.

2. let $Y \subset [0,1]^3$ be a Kolmogorov-type basic embedding of 1-dimensional compactum Y into the cube. Then the canonical linear surjection is open.

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3. (Theorem 2.5) For every finite-dimensional compactum Y there exists a linear open continuous surjection $L : C_p[0,1] \longrightarrow C_p(Y)$, or equivalently, $L(Y)$ embeds as a closed linear topological subspace into $L([0,1])$.

Proof of Theorem 2.5

Let Y be an n-dimensional compactum. By the first result there is a 1-dimensional compactum $Z = X_1 \oplus X_2 \oplus ... \oplus X_{n+1}$ for which $C_p(Z)$ admits an open continuous linear surjection onto $C_p(Y)$. Let $W =$ the disjoint union of 3 copies of $[0, 1]$. By the second result there is an open continuous linear surjection from $C_p(W)$ onto $C_p(Z)$. It remains to note that the spaces $C_p(W)$ and $C_p[0, 1]$ are linearly homeomorphic.

More open problems

Problem 2

Characterize compacta X admitting a linear open continuous surjection from $C_p[0,1]$ onto $C_p(X)$.

Problem 3

Is it true that every infinite compactum X admits a linear continuous surjection from $C_p(X)$ onto $C_p(X^2)$, or equivalently, the free locally convex space $L(X^2)$ embeds as a linear topological subspace into the free locally convex space $L(X)$?
For example, what happens if X be a pseudoarc?

Problem 4

The same question for the free abelian groups.
Is it true that every infinite compactum X the free abelian group $A(X^2)$ embeds as a subgroup into the free abelian group $A(X)$?
For example, what happens if X be a pseudoarc?

Old open problem

Find a characterization of $dim X$ in terms of the linear topological properies of $C_p(X)$ (equivalently, $L(X)$).

Thank you!