Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$.

András Zsák

Peterhouse, Cambridge

(Joint work with Thomas Schlumprecht.)

August 2014, Maresias, Brazil

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

If $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$ for some $A, B \in \mathcal{L}(\ell_p)$.

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

If $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$ for some $A, B \in \mathcal{L}(\ell_p)$.

It is natural to consider $\mathcal{L}(\ell_p \oplus \ell_q)$. Early results by Pietsch and Milman in 70s.

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

If $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$ for some $A, B \in \mathcal{L}(\ell_p)$.

It is natural to consider $\mathcal{L}(\ell_p \oplus \ell_q)$. Early results by Pietsch and Milman in 70s.

Pietsch [Operator Ideals, 1977]: $\mathcal{L}(\ell_p \oplus \ell_q)$ has exactly two maximal ideals, and all other proper, closed ideals are in one-to-one correspondence with closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

If $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$ for some $A, B \in \mathcal{L}(\ell_p)$.

It is natural to consider $\mathcal{L}(\ell_p \oplus \ell_q)$. Early results by Pietsch and Milman in 70s.

Pietsch [Operator Ideals, 1977]: $\mathcal{L}(\ell_p \oplus \ell_q)$ has exactly two maximal ideals, and all other proper, closed ideals are in one-to-one correspondence with closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Question (Pietsch): Are there infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$?

Some old results

Calkin [1941]: $\{0\} \subsetneq \mathcal{K}(\ell_2) \subsetneq \mathcal{L}(\ell_2)$.

Gohberg, Markus, Feldman [1960]: Same holds for ℓ_p, $1 \leq p < \infty$, and c_0.

If $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$ for some $A, B \in \mathcal{L}(\ell_p)$.

It is natural to consider $\mathcal{L}(\ell_p \oplus \ell_q)$. Early results by Pietsch and Milman in 70s.

Pietsch [Operator Ideals, 1977]: $\mathcal{L}(\ell_p \oplus \ell_q)$ has exactly two maximal ideals, and all other proper, closed ideals are in one-to-one correspondence with closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Question (Pietsch): Are there infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$?

Answer (Schlumprecht, Z): Yes for $1 < p < q < \infty$.

Some definitions

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T\colon X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon\|x\|$.

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T\colon X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon \|x\|$.

T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $E \subset X$ with $\dim E \geq n$, there exists $x \in E$ such that $\|Tx\| < \varepsilon \|x\|$.

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T: X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon\|x\|$.

T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $E \subset X$ with $\dim E \geq n$, there exists $x \in E$ such that $\|Tx\| < \varepsilon\|x\|$.

$$\{0\} \subsetneq \mathcal{K}(X, Y) \subset \mathcal{FS}(X, Y) \subset \mathcal{S}(X, Y) \subset \mathcal{L}(X, Y) \ .$$

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T\colon X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon \|x\|$.

T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $E \subset X$ with $\dim E \geq n$, there exists $x \in E$ such that $\|Tx\| < \varepsilon \|x\|$.

$$\{0\} \subsetneq \mathcal{K}(X, Y) \subset \mathcal{FS}(X, Y) \subset \mathcal{S}(X, Y) \subset \mathcal{L}(X, Y) \ .$$

Fix $T\colon U \to V$. For any X, Y we let $\mathcal{J}^T = \mathcal{J}^T(X, Y)$ be the closed ideal in $\mathcal{L}(X, Y)$ generated by operators fectoring through T.

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T : X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon\|x\|$.

T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $E \subset X$ with $\dim E \geq n$, there exists $x \in E$ such that $\|Tx\| < \varepsilon\|x\|$.

$$\{0\} \subsetneq \mathcal{K}(X, Y) \subset \mathcal{FS}(X, Y) \subset \mathcal{S}(X, Y) \subset \mathcal{L}(X, Y) \ .$$

Fix $T : U \to V$. For any X, Y we let $\mathcal{J}^T = \mathcal{J}^T(X, Y)$ be the closed ideal in $\mathcal{L}(X, Y)$ generated by operators fectoring through T.

$$\mathcal{J}^T(X, Y) = \overline{\mathrm{span}}\{ATB : A \in \mathcal{L}(V, Y),\ B \in \mathcal{L}(X, U)\} \ .$$

Some definitions

An *ideal* in $\mathcal{L}(X, Y)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ such that $ATB \in \mathcal{J}$ whenever $T \in \mathcal{J}$ and $A \in \mathcal{L}(Y)$, $B \in \mathcal{L}(X)$.

$T\colon X \to Y$ is *strictly singular* if for all $Z \subset X$ with $\dim Z = \infty$ and for all $\varepsilon > 0$, there exists $x \in Z$ such that $\|Tx\| < \varepsilon \|x\|$.

T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $E \subset X$ with $\dim E \geq n$, there exists $x \in E$ such that $\|Tx\| < \varepsilon \|x\|$.

$$\{0\} \subsetneq \mathcal{K}(X, Y) \subset \mathcal{FS}(X, Y) \subset \mathcal{S}(X, Y) \subset \mathcal{L}(X, Y) \ .$$

Fix $T\colon U \to V$. For any X, Y we let $\mathcal{J}^T = \mathcal{J}^T(X, Y)$ be the closed ideal in $\mathcal{L}(X, Y)$ generated by operators fectoring through T.

$$\mathcal{J}^T(X, Y) = \overline{\mathrm{span}}\{ATB : \ A \in \mathcal{L}(V, Y), \ B \in \mathcal{L}(X, U)\} \ .$$

If $U = V$ and $T = \mathrm{Id}_U$, then $\mathcal{J}^U = \mathcal{J}^{\mathrm{Id}_U}$.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$,

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$, $T_{12} \in \mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$,

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$, $T_{12} \in \mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$, $T_{12} \in \mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$, $T_{12} \in \mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$, $T_{12} \in \mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$.

Two maximal ideals:

$$\{T : \ T_{22} \in \mathcal{K}(\ell_q)\} = \mathcal{J}^{\ell_p}$$

$$\{T : \ T_{11} \in \mathcal{K}(\ell_p)\} = \mathcal{J}^{\ell_q}$$

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$, $T_{12} \in \mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$, $T_{12} \in \mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$.

Two maximal ideals:

$$\{T : \; T_{22} \in \mathcal{K}(\ell_q)\} = \mathcal{J}^{\ell_p}$$
$$\{T : \; T_{11} \in \mathcal{K}(\ell_p)\} = \mathcal{J}^{\ell_q}$$

Other closed, proper ideals:
$\{T : \; T_{jj} \in \mathcal{K}, \; T_{21} \in \mathcal{J}\}$ where $\mathcal{J}$ is a closed ideal in $\mathcal{L}(\ell_p, \ell_q)$.

Closed ideals in $\mathcal{L}(\ell_p \oplus \ell_q)$

Recall: if $T \in \mathcal{L}(\ell_p) \setminus \mathcal{K}(\ell_p)$, then $\mathrm{Id}_{\ell_p} = ATB$.

$T \in \mathcal{L}(\ell_p \oplus \ell_q)$ is a matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, where $T_{11} \in \mathcal{L}(\ell_p)$, $T_{22} \in \mathcal{L}(\ell_q)$, $T_{12} \in \mathcal{L}(\ell_q, \ell_p) = \mathcal{K}(\ell_q, \ell_p)$, $T_{12} \in \mathcal{L}(\ell_p, \ell_q) = \mathcal{S}(\ell_p, \ell_q)$.

Two maximal ideals:

$$\{T : \ T_{22} \in \mathcal{K}(\ell_q)\} = \mathcal{J}^{\ell_p}$$
$$\{T : \ T_{11} \in \mathcal{K}(\ell_p)\} = \mathcal{J}^{\ell_q}$$

Other closed, proper ideals:
$\{T : \ T_{jj} \in \mathcal{K}, \ T_{21} \in \mathcal{J}\}$ where $\mathcal{J}$ is a closed ideal in $\mathcal{L}(\ell_p, \ell_q)$.

E.g., $\mathcal{J} = \mathcal{L}(\ell_p, \ell_q)$ corresponds to $\mathcal{J}^{\ell_p} \cap \mathcal{J}^{\ell_2}$.

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map.

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)
Since for a flat vector x we have $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$, it follows that $I_{p,q} \in \mathcal{FS}$.

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)
Since for a flat vector x we have $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$, it follows that $I_{p,q} \in \mathcal{FS}$.

The formal inclusion map $\ell_p \sim \Big(\bigoplus \ell_2^n\Big)_{\ell_p} \to \Big(\bigoplus \ell_2^n\Big)_{\ell_q} \sim \ell_q$ is not finitely strictly singular.

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \leq p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)
Since for a flat vector x we have $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$, it follows that $I_{p,q} \in \mathcal{FS}$.

The formal inclusion map $\ell_p \sim \left(\bigoplus \ell_2^n\right)_{\ell_p} \to \left(\bigoplus \ell_2^n\right)_{\ell_q} \sim \ell_q$ is not finitely strictly singular. So

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q) \ .$$

Closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

Let $1 \le p < q < \infty$.

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{L}(\ell_p, \ell_q) \ ,$$

where $I_{p,q} \colon \ell_p \to \ell_q$ is the formal inclusion map. Milman [1970] proved the following:

If $E \subset c_0$ with $\dim E = n$, then there exists $x = (x_i) \in E$, $x \neq 0$, such that $|x_i| = \|x\|_\infty$ for at least n values of i. ('flat' vectors)
Since for a flat vector x we have $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$, it follows that $I_{p,q} \in \mathcal{FS}$.

The formal inclusion map $\ell_p \sim \left(\bigoplus \ell_2^n \right)_{\ell_p} \to \left(\bigoplus \ell_2^n \right)_{\ell_q} \sim \ell_q$ is not finitely strictly singular. So

$$\{0\} \subsetneq \mathcal{K}(\ell_p, \ell_q) \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q) \ .$$

We have at least 2 closed ideals.

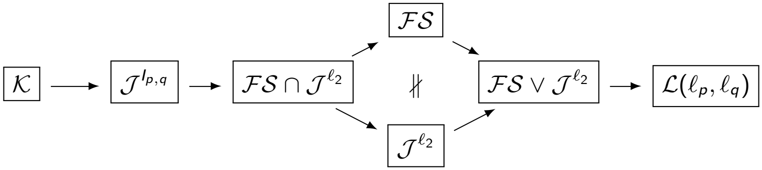
Recent results

Let $1 < p < 2 < q < \infty$.

Recent results

Let $1 < p < 2 < q < \infty$.

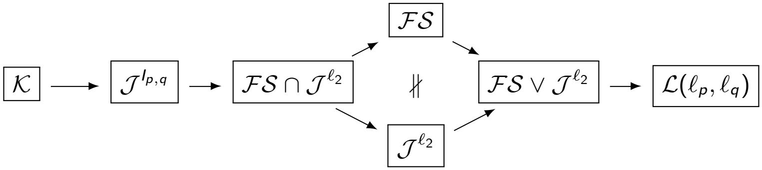
Sari, Schlumprecht, Tomczak-Jaegerman, Troitsky [2007]



Recent results

Let $1 < p < 2 < q < \infty$.

Sari, Schlumprecht, Tomczak-Jaegerman, Troitsky [2007]

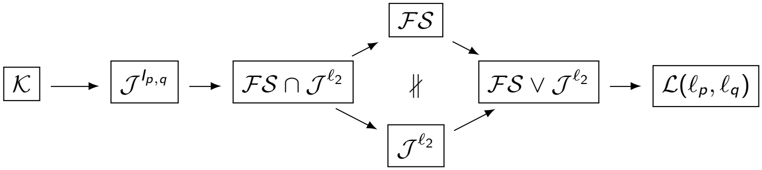


So we have 4 closed ideals.

Recent results

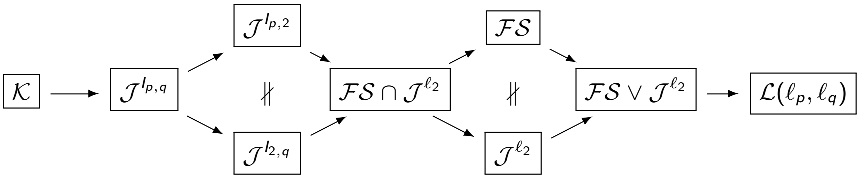
Let $1 < p < 2 < q < \infty$.

Sari, Schlumprecht, Tomczak-Jaegerman, Troitsky [2007]



So we have 4 closed ideals.

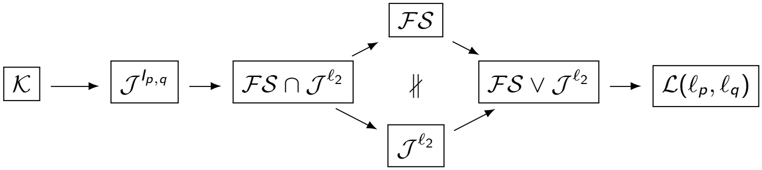
Schlumprecht [2011]:



Recent results

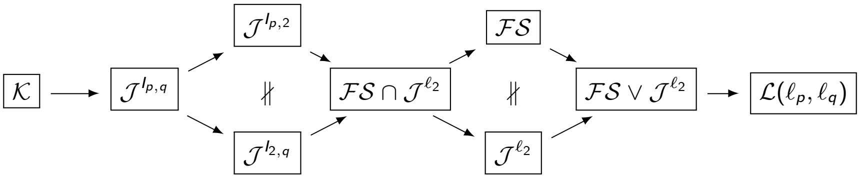
Let $1 < p < 2 < q < \infty$.

Sari, Schlumprecht, Tomczak-Jaegerman, Troitsky [2007]



So we have 4 closed ideals.

Schlumprecht [2011]:



So we have $4 + 2 = 7$ closed ideals.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U \colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^{n}$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P \colon \ell_p \to W$.
We let $I_{W,Z} \colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.
We let $I_{W,Z}\colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.
This yields the closed ideal $\mathcal{J}^{I_{W,Z}}$ of $\mathcal{L}(\ell_p, \ell_q)$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.
We let $I_{W,Z}\colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.
This yields the closed ideal $\mathcal{J}^{I_{W,Z}}$ of $\mathcal{L}(\ell_p, \ell_q)$. This contains $T = U \circ I_{W,Z} \circ P$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.
We let $I_{W,Z}\colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.
This yields the closed ideal $\mathcal{J}^{I_{W,Z}}$ of $\mathcal{L}(\ell_p, \ell_q)$. This contains $T = U \circ I_{W,Z} \circ P$.

Consider F_n, $\left(f_j^{(n)}\right)_{j=1}^n$, $Y = \left(\bigoplus F_n\right)_{\ell_p}$, $I_{Y,Z}$, $\mathcal{J}^{I_{Y,Z}}$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.
We let $I_{W,Z}\colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.
This yields the closed ideal $\mathcal{J}^{I_{W,Z}}$ of $\mathcal{L}(\ell_p, \ell_q)$. This contains $T = U \circ I_{W,Z} \circ P$.

Consider F_n, $\left(f_j^{(n)}\right)_{j=1}^n$, $Y = \left(\bigoplus F_n\right)_{\ell_p}$, $I_{Y,Z}$, $\mathcal{J}^{I_{Y,Z}}$.

Assume that $\left(f_j^{(n)}\right)_{j=1}^n$ dominates $\left(g_j^{(n)}\right)_{j=1}^n$ for all n. Then $I_{Y,W}$ is continuous, and $I_{Y,Z} = I_{W,Z} \circ I_{Y,W}$, and so $\mathcal{J}^{I_{Y,Z}} \subset \mathcal{J}^{I_{W,Z}}$.

Infinitely many closed ideals in $\mathcal{L}(\ell_p, \ell_q)$

WLOG $1 < p < 2$ and $p < q < \infty$.

Set $Z = \left(\bigoplus \ell_2^n\right)_{\ell_q}$ and fix an isomorphism $U\colon Z \to \ell_q$.

Consider G_n, $\dim G_n = n$, with normalized, 1-unconditional basis $\left(g_j^{(n)}\right)_{j=1}^n$.
We will have $G_n \subset \ell_p^{k_n}$ uniformly complemented.
Have $W = \left(\bigoplus G_n\right)_{\ell_p}$ with a projection $P\colon \ell_p \to W$.
We let $I_{W,Z}\colon W \to Z$ be the formal inclusion given by $I_{W,Z}\left(g_j^{(n)}\right) = e_j^{(n)}$.
This yields the closed ideal $\mathcal{J}^{I_{W,Z}}$ of $\mathcal{L}(\ell_p, \ell_q)$. This contains $T = U \circ I_{W,Z} \circ P$.

Consider F_n, $\left(f_j^{(n)}\right)_{j=1}^n$, $Y = \left(\bigoplus F_n\right)_{\ell_p}$, $I_{Y,Z}$, $\mathcal{J}^{I_{Y,Z}}$.

Assume that $\left(f_j^{(n)}\right)_{j=1}^n$ dominates $\left(g_j^{(n)}\right)_{j=1}^n$ for all n. Then $I_{Y,W}$ is continuous, and $I_{Y,Z} = I_{W,Z} \circ I_{Y,W}$, and so $\mathcal{J}^{I_{Y,Z}} \subset \mathcal{J}^{I_{W,Z}}$.

Under some conditions we show that $T \notin \mathcal{J}^{I_{Y,Z}}$.

Rosenthal's $X_{p,w}$ spaces

Fix $1 < p < 2$. Let (w_n) be a decreasing sequence in $(0, 1]$. Fix n. We let G_n be the span of a sequence $g_j^{(n)}$, $1 \le j \le n$, of independent symmetric, 3-valued random variables in L_p, where $\|g_j^{(n)}\|_{L_p} = 1$ and $w_n = \|g_j^{(n)}\|_{L_2}^{-1}$.

Rosenthal's $X_{p,w}$ spaces

Fix $1 < p < 2$. Let (w_n) be a decreasing sequence in $(0,1]$. Fix n. We let G_n be the span of a sequence $g_j^{(n)}$, $1 \le j \le n$, of independent symmetric, 3-valued random variables in L_p, where $\|g_j^{(n)}\|_{L_p} = 1$ and $w_n = \|g_j^{(n)}\|_{L_2}^{-1}$.

Then G_n^* is isomorphic to $\left(\mathbb{R}^n, \|\cdot\|_{p',w_n}\right)$, where

$$\|(a_j)_{j=1}^n\|_{p',w_n} = \left(\sum |a_j|^{p'}\right)^{\frac{1}{p'}} \vee w_n \left(\sum |a_j|^2\right)^{\frac{1}{2}} .$$

Rosenthal's $X_{p,w}$ spaces

Fix $1 < p < 2$. Let (w_n) be a decreasing sequence in $(0, 1]$. Fix n. We let G_n be the span of a sequence $g_j^{(n)}$, $1 \leq j \leq n$, of independent symmetric, 3-valued random variables in L_p, where $\|g_j^{(n)}\|_{L_p} = 1$ and $w_n = \|g_j^{(n)}\|_{L_2}^{-1}$.

Then G_n^* is isomorphic to $\left(\mathbb{R}^n, \|\cdot\|_{p',w_n}\right)$, where

$$\|(a_j)_{j=1}^n\|_{p',w_n} = \left(\sum |a_j|^{p'}\right)^{\frac{1}{p'}} \vee w_n \left(\sum |a_j|^2\right)^{\frac{1}{2}} .$$

The spaces F_n are defined using a different sequence (v_n). The condition we need is as follows.

Rosenthal's $X_{p,w}$ spaces

Fix $1 < p < 2$. Let (w_n) be a decreasing sequence in $(0, 1]$. Fix n. We let G_n be the span of a sequence $g_j^{(n)}$, $1 \leq j \leq n$, of independent symmetric, 3-valued random variables in L_p, where $\|g_j^{(n)}\|_{L_p} = 1$ and $w_n = \|g_j^{(n)}\|_{L_2}^{-1}$.

Then G_n^* is isomorphic to $\left(\mathbb{R}^n, \|\cdot\|_{p',w_n}\right)$, where

$$\|(a_j)_{j=1}^n\|_{p',w_n} = \left(\sum |a_j|^{p'}\right)^{\frac{1}{p'}} \vee w_n \left(\sum |a_j|^2\right)^{\frac{1}{2}} .$$

The spaces F_n are defined using a different sequence (v_n). The condition we need is as follows.

$$\frac{v_{\sqrt{cn}}}{w_n} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all } c \in (0, 1) .$$