A hierarchy of separable commutative Calkin algebras

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(joint work with Pavlos Motakis and Daniele Puglisi)

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- Let A be a Banach algebra. Does there exist a Banach space X such that the Calkin algebra of X is isomorphic, as a Banach algebra, to A ?
- The Calkin algebra of X is defined to be the space $Cal(X) = {}^{\mathcal{L}(X)}\!/\!_{\mathcal{K}(X)}$, where $\mathcal{L}(X)$ denotes the space of all bounded linear operators defined on X and $\mathcal{K}(X)$ denotes the spaces of all compact operators defined on X.
- We denote by $[T]$ the equivalence class of $T \in \mathcal{L}(X)$ in ${}^{\mathcal{L}(X)}\!/\!_{\mathcal{K}(X)}$.
- $Cal(X)$ endowed with the operation $[T] \circ [S] = [T \circ S]$ becomes a Banach algebra.

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Case $A = \mathbb{R}$

- S. Argyros and R. Haydon in 2011 constructed a Banach space X_{AH} that satisfies the “scalar plus compact” property.

  S. A. Argyros and R. Haydon: *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011) 1-54.

  Hence, $Cal(X_{AH})$ is one-dimensional.

Case $A = \ell_1(\mathbb{N}_0)$

- M. Tarbard in 2013 constructed a Banach space X_∞ such that $Cal(X_\infty)$ is isometric as a Banach algebra with the convolution algebra $\ell_1(\mathbb{N}_0)$.

Matthew Tarbard: *Operators on Banach Spaces of Bourgain-Delbaen Type*, arXiv:1309.7469 (2013).

Case $A = C(K)$ for K countable compact metric space

Theorem (P. Motakis - Daniele Puglisi - D.Z)

Let K be a countable compact metric space with finite Cantor-Bendixson index.

Then there exists a $\mathcal{L}_\infty$ space X such that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

[P.Motakis, D. Puglisi, D. Z] *A hierarchy of separable commutative Calkin algebras* , arXiv: 1407.8073.

Case $A = C(K)$ for K countable compact metric space

- The basic ingredients of our method are the following:
- The Argyros Haydon space X_{AH}.
- The Argyros-Haydon sum of a sequence of separable Banach spaces $(X_n)_n$, $(\sum_n \oplus X_n)_{AH}$, introduced in

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Finite sums of Argyros-Haydon spaces

- The space X_{AH} is a separable $\mathcal{L}^\infty$ space with dual isomorphic to ℓ_1.
- The construction of X_{AH} is a generalized modification of the Bourgain-Delbaen method which depends on a pair of sequences of natural numbers $(m_j, n_j)_j$ that satisfy certain growth conditions.
- For $L \subset \mathbb{N}$ infinite, we denote by $X_{AH}(L)$ the space constructed using the subsequence $(m_j, n_j)_{j \in L}$.
- For every $L \subset \mathbb{N}$ infinite, the space $X_{AH(L)}$ shares the same properties with X_{AH}.
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Finite sums of Banach spaces

- An observation.
- Let $(X_i)_{i=1}^n$ a finite sequence of Banach spaces and assume that every bounded linear operator

$$T : X_i \to X_j$$

is compact for every $i \neq j$.

Setting $X = (X_1 \oplus \ldots \oplus X_n)_\infty$ it follows that $Cal(X)$ is isometric with

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- Let $k \in \mathbb{N}$, $L_1, \ldots, L_k$ pairwise disjoint infinite subsets of $\mathbb{N}$ and
$$X = (X_1 \oplus \cdots \oplus X_k)_\infty ,$$
where $X_i = X_{AH(L_i)}$ for $i = 1, \ldots k$.

- Since L_i are pairwise disjoint we have that every $T : X_{AH(L_i)} \to X_{AH(L_j)}$ is compact for every $i \neq j$.

- Since $\mathcal{C}al(X_i)$ is one dimensional, by the above observation we obtain that $\mathcal{C}al(X)$ is k-dimensional.

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AH-sums of separable Banach spaces

- We will now see the Calkin algebra of the space

$$X = (\sum_n \oplus X_n)_{BD},$$

where $X_n = X_{AH}(L_n)$ for a sequence $(L_n)_n$ of pairwise disjoint infinite subsets of natural numbers.

- For a sequence $(X_n)_n$ of separable Banach spaces, the space $X = (\sum_n \oplus X_n)_{BD}$ is called a Bourgain Delbaen $\mathcal{L}^\infty$ sum of $(X_n)_n$ and is defined as a subspace of $(\sum \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty)_\infty$.

- The sets Δ_n are finite,pairwise disjoint and defined using the Bourgain-Delbaen method.

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- In particular, we define linear extension operators

$$i_n : \Big(\sum_{k\le n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty\Big)_\infty \to (\sum \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty)_\infty$$

such that

- $\sup_n \|i_n\| < \infty$.
- $(\sum_n \oplus X_n)_{BD} = \overline{\cup_n Y_n}$, where $Y_n = i_n\Big[\Big(\sum_{k\le n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty\Big)_\infty\Big]$.

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The Definition of BD-sums of Banach spaces

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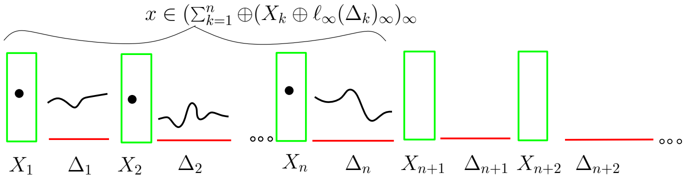
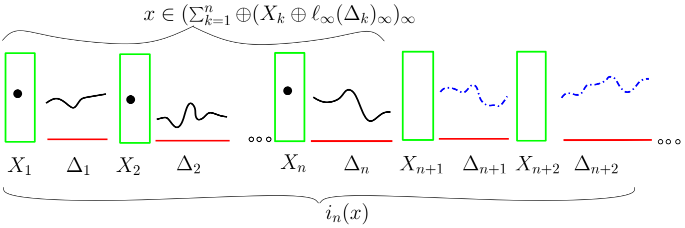
The BD-method in AH sums of Banach spaces

- Let x be a vector in $(\sum_{k\leq n}\oplus(X_k\oplus\ell_\infty(\Delta_n))_\infty)_\infty$



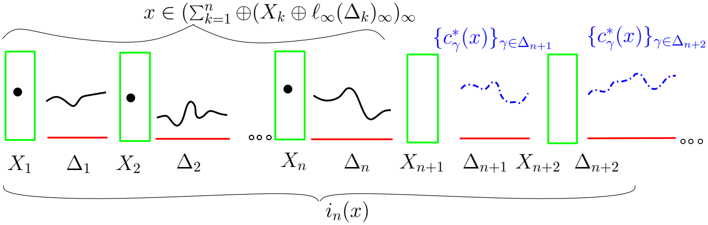
The BD- method in AH-sums of Banach spaces

- The vector $i_n(x)$, i.e. x is extended by assigning to it new values in $\ell_\infty(\cup_{k>n}\Delta_k)$.



The $\mathcal{L}^\infty$ structure of AH-sums

- The finite sets Δ_n are defined recursively and for each $\gamma \in \Delta_{n+1}$ we assign a linear functional $c_\gamma^* : (\sum_{k=1}^n \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty \to \mathbb{R}$ such that $i_n(x)(\gamma) = c_\gamma^*(x)$.



- This implies that for a fixed $n \in \mathbb{N}$, taking $x_k \in X_k$ with $x_k \in \bigcap_{\gamma \in \cup_{i=1}^n \Delta_i} Ker c_\gamma^*$ then the extended vector $i_k(x_k)$ does not have non zero values upon Δ_i for every $1 \leq i \leq n$.

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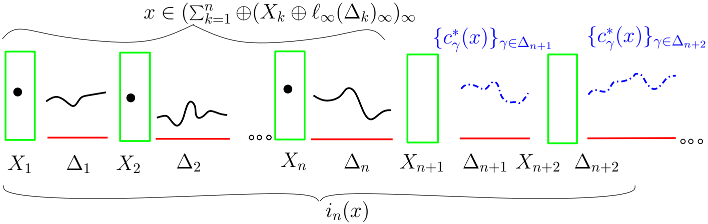

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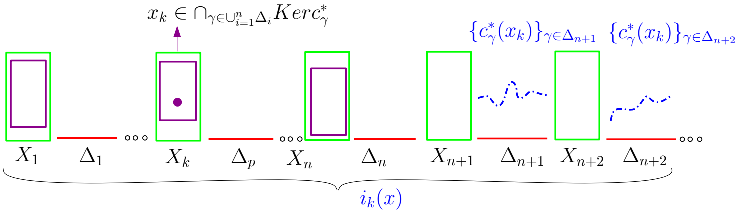

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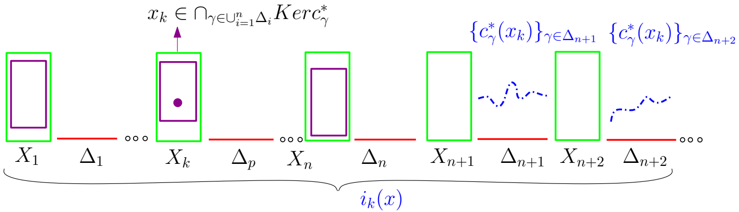
$$\|i_1(x_1) + \ldots + i_n(x_n)\| \simeq \max_{1 \le k \le n} \|x_k\|.$$

- Since Δ_i are finite, the above implies that

$$\langle [I_k] : k = 1, \ldots, n \rangle \simeq c_0(n).$$



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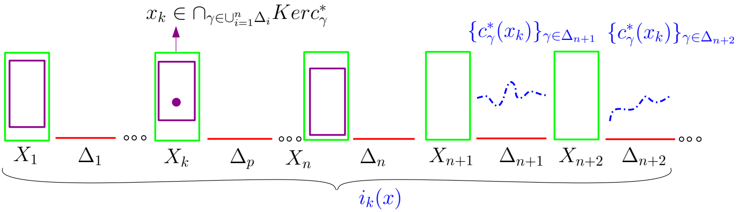

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AH-sums of separable Banach spaces

- Without taking any further assumptions for the separable X_n, the space $X = (\sum_n \oplus X_n)_{BD}$ satisfies the following basic properties:

- $X = \sum_n \oplus i_n[X_n \oplus \ell_\infty(\Delta_n)]$.

- Each X_n is isometric with $i_n[X_n]$ and complemented in X via projection I_n.

- An operator K defined on $X = (\sum_n \oplus X_n)_{BD}$ is called horizontally compact operator if

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- The dual X^* is isomorphic with $(\sum_n \oplus (X_n^* \oplus \ell_1(\Delta_n))_1)_1$.
- By considering some specific sequences $(X_n)_n$ of separable Banach spaces, the space X has the "scalar-plus-horizontally compact" property, i.e. every operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + K$ where $\lambda \in \mathbb{R}$ and K a horizontally compact operator.
- For example, if X_n has the Schur property for every $n \in \mathbb{N}$, or ℓ_1 does not embed isomorphically in X_n^* for every $n \in \mathbb{N}$, then the above holds.

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- For example, if X_n has the Schur property for every $n \in \mathbb{N}$, or ℓ_1 does not embed isomorphically in X_n^* for every $n \in \mathbb{N}$, then the above holds.

AH sums of separable Banach spaces

- For every $L \subset \mathbb{N}$ infinite the space $X = (\sum_n \oplus X_n)_{AH(L)}$ has the following additional properties:
- The dual X^* is isomorphic with $(\sum_n \oplus (X_n^* \oplus \ell_1(\Delta_n))_1)_1$.
- By considering some specific sequences $(X_n)_n$ of separable Banach spaces, the space X has the "scalar-plus-horizontally compact" property, i.e. every operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + K$ where $\lambda \in \mathbb{R}$ and K a horizontally compact operator.
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Calkin algebras of AH-sums

Proposition

Let L, L_n pairwise disjoints infinite subsets of $\mathbb{N}$ and $X_{AHsum} = (\sum \oplus X_{AH}(L_n))_{AH(L)}$. The space X_{AHsum} has the "scalar-plus-horizontally compact" property.

- Observe also that since $X_{AH}^* \simeq \ell_1$, the space X_{AHsum} is $\mathcal{L}^\infty$ space.
- Moreover, Since every operator $T : X_{AH(L_n)} \to X_{AH(L_m)}$ is compact, we conclude that the space

$$\mathcal{L}(X_{AHsum}) = \overline{\langle I, (I_n)_n, \mathcal{K}(X_{AHsum}) \rangle},$$

where I denotes the identity map upon X_{AHsum} and for each n, I_n is the projection defined on X_{AHsum} with image isometric with $X_{AH(L_n)}$.

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AH-sums of AH-spaces

- Hence $\mathcal{C}al(X_{AHsum}) = \mathcal{L}(X_{AHsum})/\mathcal{K}(X_{AHsum}) = \overline{\langle [I], ([I_n])_n \rangle}$.
- Using the $\mathcal{L}^\infty$ structure of the BD-sum $(\sum \oplus X_{AH}(L_n))_{AH(L)}$ described earlier we obtain that

$$\langle [I_k] : \quad k = 1, \ldots, n \rangle \simeq^{C_n} c_0(n).$$

- Using the $\mathcal{L}^\infty$ structure of the spaces $X_{AH(L_n)}$, we have that $(C_n)_n$ is uniformly bounded and by the above we conclude that the Calkin algebra of X_{AHsum} is isomorphic to c.

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The Generalization of $X_{AHsum} = (\sum_n \oplus X_{AH(L_n)})_{AH(L)}$

- We generalize the above concept using well founded trees $\mathcal{T}$ with a unique root such that every non maximal node of $\mathcal{T}$ has infinitely countable immediate successors.
- For such a tree $\mathcal{T}$ and $L \subset \mathbb{N}$ infinite we construct Banach spaces $X_{(\mathcal{T},L)}$ using induction on the order of $\mathcal{T}$.

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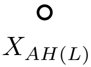
The definition of the spaces $X_{(\mathcal{T},L)}$

For $\mathcal{T}$ is a singleton and $L \subset \mathbb{N}$ we define $X_{(\mathcal{T},L)}$ to be the space $X_{AH}(L)$.

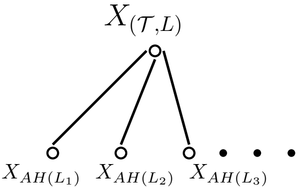
Tree of rank zero:



The definition of the spaces $X_{(\mathcal{T},L)}$

For a tree of order one we define $X_{(\mathcal{T},L)} = \left(\sum \oplus X_{(\mathcal{T}_n,L_n)}\right)_{AH(L_0)}$.

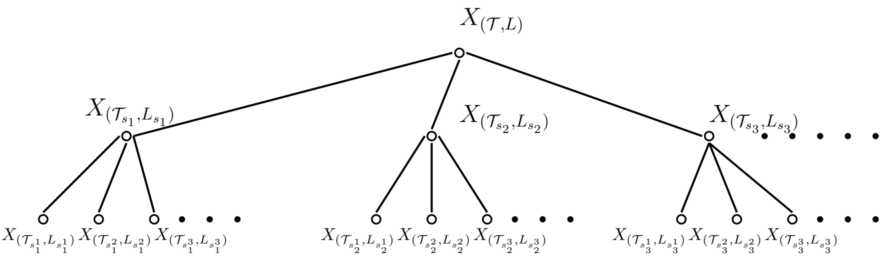
Tree of rank 1:



The definition of the spaces $X_{(\mathcal{T},L)}$

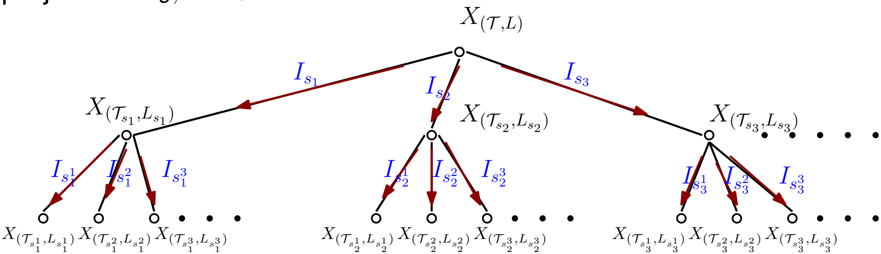
For a tree of order two we define $X_{(\mathcal{T},L)} = \left(\sum \oplus X_{(\mathcal{T}_n,L_n)}\right)_{AH(L_0)}$ etc...

Tree of rank 2:



Properties of the spaces $X_{(\mathcal{T},L)}$

There space $X_{(\mathcal{T},L)}$ is accompanied by a set of norm-one projections $I_s, s \in \mathcal{T}$.



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Proposition

- *For every tree $\mathcal{T}$ and $L \subset \mathbb{N}$ infinite, the space $X_{(\mathcal{T},L)}$ is $\mathcal{L}^\infty$ and if $o(\mathcal{T}) > 0$ it has the "scalar-plus-horizontally compact" property.*

- Note that $o(\mathcal{T}) = 0$, the space $X_{(\mathcal{T},L)}$ has the "scalar plus compact" property as it coincides with the space $X_{AH(L)}$.

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Operators defined on $X_{(\mathcal{T},L)}$

- For an operator S defined on $X_{\mathcal{T},L}$ we denote by S_t the induced operator

$$I_t \circ S \circ I_t$$

which can considered upon $X_{(\mathcal{T}_t,L_t)}$.

- Every $S \in \mathcal{L}(X_{(\mathcal{T},L)})$ corresponds to a unique family $(\lambda_t)_{t\in\mathcal{T}}$ of scalars chosen to satisfy:
- If t is maximal (and hence $X_{(\mathcal{T}_t,L_t)} = X_{AH(L_t)}$), $S_t - \lambda_t I_t$ is compact, while
- If t non maximal, $S_t - \lambda_t I_t$ is horizontally compact.

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The Calkin algebras of $X_{(\mathcal{T},L)}$

- The functional $f_S : \mathcal{T} \to \mathbb{R}$ that assigns to each $t \in \mathcal{T}$ the scalar λ_t, is continuous.
- We define $\bar{\Phi}_{(\mathcal{T},L)} : \mathcal{L}(X_{(\mathcal{T},L)}) \to C(\mathcal{T})$ by the rule

$$S \to f_S.$$

- The induced operator

$$\Phi_{(\mathcal{T},L)} : \mathcal{L}(X_{(\mathcal{T},L)}) / \mathcal{K}(X_{(\mathcal{T},L)}) = Cal(X_{(\mathcal{T},L)}) \to C(\mathcal{T})$$

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Proposition

Let $\mathcal{T}$ be a tree of finite rank and L be an infinite subset of the natural numbers. Then the map $\Phi_{\mathcal{T},L} : Cal(X_{(\mathcal{T},L)}) \to C(\mathcal{T})$ is bounded below.

- Hence, $\mathcal{C}al(X_{(\mathcal{T},L)}) \simeq C(\mathcal{T})$ as a Banach algebra, if $o(\mathcal{T}) < \omega$.

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The main result

Theorem (P. Motakis - Daniele Puglisi - D.Z)

Let K be a countable compact metric space with finite Cantor-Bendixson index.

Then there exists a $\mathcal{L}_\infty$ space X such that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

- By Sierpinski Mazurkiewichz K is homeomorphic to a countable ordinal number of the form $\omega^k \cdot n$, $k, n \in \mathbb{N}$.
- $X = \left(\sum_{i=1}^{n} \oplus X_{(\mathcal{T}, L_i)} \right)_\infty$, where $\mathcal{T} = \omega^k$ and $(L_i)_i$ pairwise disjoint.

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Can it be extended?

- Question: is the above theorem true for every countable compact metric space?
- Question: is the map $\Phi_{(\mathcal{T},L)} : Cal(X_{(\mathcal{T},L)}) \to C(\mathcal{T})$ always onto?

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Indications for affirmative answers

- The dual of $Cal(X_{(\mathcal{T},L)})$ is separable and has the Schur property.

- The Calkin algebra of $X_{(\mathcal{T},L)}$ is commutative as a Banach algebra and as a Banach space it is c_0 saturated and has the Dunford-Pettis property.

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Thank you!