Disjoint sequences in Banach lattices

Pedro Tradacete

Mathematics Department, UC3M

Based on joint work with J. Flores, F. Hernández, E. Semenov, E. Spinu, V. Troitsky

First Brazilian Workshop in Geometry of Banach Spaces
25-29 August 2014, Maresias

Disjointly homogeneous Banach lattices:

Definition

E is disjointly homogeneous (DH) $\Leftrightarrow$ $\forall\,(x_n)$, (y_n) normalized disjoint in E, $\exists\,(n_k)$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big\|\sum_{k=1}^{\infty} a_k y_{n_k}\Big\|$$

Disjointly homogeneous Banach lattices:

Definition

E is disjointly homogeneous (DH) $\Leftrightarrow$ $\forall\,(x_n)$, (y_n) normalized disjoint in E, $\exists\,(n_k)$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big\|\sum_{k=1}^{\infty} a_k y_{n_k}\Big\|$$

Examples: L_p, Lorentz spaces $L_{p,q}$, $\Lambda(W,p)$, ...

Disjointly homogeneous Banach lattices:

Definition

E is disjointly homogeneous (DH) $\Leftrightarrow \forall\,(x_n), (y_n)$ normalized disjoint in E, $\exists\,(n_k)$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big\|\sum_{k=1}^{\infty} a_k y_{n_k}\Big\|$$

Examples: L_p, Lorentz spaces $L_{p,q}$, $\Lambda(W,p)$, ...

Definition

E is p-disjointly homogeneous (p-DH) if every normalized disjoint sequence (x_n) in E has a subsequence such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p} \big(\sup_k |a_k| \text{ in case } p=\infty\big)$$

Applications of DH Banach lattices

Theorem

E DH with finite cotype and unconditional basis.

$$T \in \mathcal{SS}(E) \Rightarrow T^2 \in \mathcal{K}(E)$$

Applications of DH Banach lattices

Theorem

E DH with finite cotype and unconditional basis.

$$T \in \mathcal{SS}(E) \Rightarrow T^2 \in \mathcal{K}(E)$$

Theorem

E 1-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{DP}(E)$

Applications of DH Banach lattices

Theorem

E DH with finite cotype and unconditional basis.

$$T \in \mathcal{SS}(E) \Rightarrow T^2 \in \mathcal{K}(E)$$

Theorem

E 1-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{DP}(E)$

Theorem

E 2-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{K}(E)$

Applications of DH Banach lattices

Theorem

E DH with finite cotype and unconditional basis.

$$T \in \mathcal{SS}(E) \Rightarrow T^2 \in \mathcal{K}(E)$$

Theorem

E 1-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{DP}(E)$

Theorem

E 2-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{K}(E)$

Theorem

E discrete with a disjoint basis and DH.

$$T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{K}(E)$$

Duality

Question: Is the property DH stable by duality?

Duality

Question: Is the property DH stable by duality?

Known-facts:

- E ∞-DH $\Rightarrow$ E^* 1-DH.

Duality

Question: Is the property DH stable by duality?

Known-facts:

- E ∞-DH $\Rightarrow$ E^* 1-DH.
- $L_{p,1}$ is 1-DH but $L_{p,1}^* = L_{p',\infty}$ is not DH.

Duality

Question: Is the property DH stable by duality?

Known-facts:

- E ∞-DH $\Rightarrow$ E^* 1-DH.
- $L_{p,1}$ is 1-DH but $L_{p,1}^* = L_{p',\infty}$ is not DH.
- Maybe for E reflexive?

Duality

Question: Is the property DH stable by duality?

Known-facts:

- E ∞-DH $\Rightarrow$ E^* 1-DH.
- $L_{p,1}$ is 1-DH but $L^*_{p,1} = L_{p',\infty}$ is not DH.
- Maybe for E reflexive?

We will see that in general the answer is negative

Positive results

Definition

A Banach lattice E has property $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^* f_n^*\| \nrightarrow 0$.

Positive results

Definition

A Banach lattice E has property $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^* f_n^*\| \nrightarrow 0$.

Theorem

Let E be a reflexive Banach lattice with property $\mathfrak{P}$.

$$E^* \ DH \ \Rightarrow \ E \ DH$$

Positive results

Definition

A Banach lattice E has property $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^* f_n^*\| \nrightarrow 0$.

Theorem

Let E be a reflexive Banach lattice with property $\mathfrak{P}$.

$$E^* \; DH \;\Rightarrow\; E \; DH$$

$$E^* \; p - DH \;\Rightarrow\; E \; q - DH \; \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

Positive results

Definition

A Banach lattice E has property $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^* f_n^*\| \nrightarrow 0$.

Theorem

Let E be a reflexive Banach lattice with property $\mathfrak{P}$.

$$E^* \ DH \ \Rightarrow \ E \ DH$$

$$E^* \ p-DH \ \Rightarrow \ E \ q-DH \ \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

Corollary

Let E be a reflexive Banach lattice satisfying an upper p-estimate.

$$E^* \ q-DH \ \Rightarrow \ E \ p-DH \ \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

Orlicz spaces

Theorem

An Orlicz space $L_\varphi(0,1)$ is p-DH $\Leftrightarrow E_\varphi^\infty \cong \{t^p\}$.

$$E_\varphi^\infty = \bigcap_{s>0} \overline{\left\{\frac{\varphi(r\cdot)}{\varphi(r)} : r \geq s\right\}}.$$

Orlicz spaces

Theorem

An Orlicz space $L_\varphi(0,1)$ *is p-DH* $\Leftrightarrow E_\varphi^\infty \cong \{t^p\}$.

$$E_\varphi^\infty = \bigcap_{s>0} \overline{\left\{ \frac{\varphi(r\cdot)}{\varphi(r)} : r \geq s \right\}}.$$

$$\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p \Rightarrow E_\varphi^\infty \cong \{t^p\}$$

Orlicz spaces

Theorem

An Orlicz space $L_\varphi(0,1)$ is p-DH $\Leftrightarrow E^\infty_\varphi \cong \{t^p\}$.

$$E^\infty_\varphi = \bigcap_{s>0} \overline{\{\frac{\varphi(r\cdot)}{\varphi(r)} : r \geq s\}}.$$

$$\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p \Rightarrow E^\infty_\varphi \cong \{t^p\}$$

Remark: $L_\varphi(0,1)$ is p-DH $\Leftrightarrow L^*_\varphi(0,1)$ is q-DH $(\frac{1}{p}+\frac{1}{q}=1)$.

Orlicz spaces

Theorem

An Orlicz space $L_\varphi(0,1)$ is p-DH $\Leftrightarrow E_\varphi^\infty \cong \{t^p\}$.

$$E_\varphi^\infty = \bigcap_{s>0} \overline{\{\frac{\varphi(r\cdot)}{\varphi(r)} : r \geq s\}}.$$

$$\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p \Rightarrow E_\varphi^\infty \cong \{t^p\}$$

Remark: $L_\varphi(0,1)$ is p-DH $\Leftrightarrow L_\varphi^*(0,1)$ is q-DH $(\frac{1}{p}+\frac{1}{q}=1)$.

Theorem

A separable Orlicz space $L_\varphi(0,\infty)$ is p-DH $\Leftrightarrow C_\varphi(0,\infty) \cong \{t^p\}$.

$$C_\varphi(0,\infty) = \overline{\mathrm{conv}}\,\{F \in C(0,1) \mid \exists s>0,\ F(\cdot) = \frac{\varphi(s\cdot)}{\varphi(s)}\}.$$

Counterexemples

Example

Given $1 < p < \infty$ let

$$\varphi(t) = \begin{cases} t^p & t < 1 \\ t^p \log(1+t) & t \geq 1 \end{cases}$$

The Orlicz space $L^{\varphi}(0,\infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.

Counterexemples

Example

Given $1 < p < \infty$ let

$$\varphi(t) = \begin{cases} t^p & t < 1 \\ t^p \log(1+t) & t \geq 1 \end{cases}$$

The Orlicz space $L^{\varphi}(0,\infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.

Theorem (Knaust-Odell)

Let E be an atomic Banach lattice. If E is p-DH and E^* is p'-DH, then there is $C > 0$ such that every disjoint sequence in E has a subsequence C-equivalent to the basis of ℓ_p.

Counterexemples

Example

Given $1 < p < \infty$ let

$$\varphi(t) = \begin{cases} t^p & t < 1 \\ t^p \log(1+t) & t \geq 1 \end{cases}$$

The Orlicz space $L^\varphi(0,\infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.

Theorem (Knaust-Odell)

Let E be an atomic Banach lattice. If E is p-DH and E^* is p'-DH, then there is $C > 0$ such that every disjoint sequence in E has a subsequence C-equivalent to the basis of ℓ_p.

Theorem (Johnson-Odell)

There is a p-DH atomic Banach lattice with no uniform constant on the equivalence.

Projections onto disjoint sequences

Question: Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

Projections onto disjoint sequences

Question: Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

It does provided:

1. E contains infinitely many atoms (in particular, discrete),

Projections onto disjoint sequences

Question: Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

It does provided:

1. E contains infinitely many atoms (in particular, discrete),
2. E is non-atomic and contains certain complemented unconditional basic sequences (Casazza-Kalton),

Projections onto disjoint sequences

Question: Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

It does provided:

1. E contains infinitely many atoms (in particular, discrete),
2. E is non-atomic and contains certain complemented unconditional basic sequences (Casazza-Kalton),
3. E is a rearrangement invariant space.

Projections onto disjoint sequences

Theorem

Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.

Projections onto disjoint sequences

Theorem

Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.

Theorem

If E is a separable non-reflexive DH Banach lattice, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Projections onto disjoint sequences

Theorem

Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.

Theorem

If E is a separable non-reflexive DH Banach lattice, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Theorem

Let E be reflexive Banach lattice containing a complemented disjoint sequence. If E and E^* are DH, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Projections onto disjoint sequences

Theorem

Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.

Theorem

If E is a separable non-reflexive DH Banach lattice, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Theorem

Let E be reflexive Banach lattice containing a complemented disjoint sequence. If E and E^* are DH, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Theorem

Let E be a p-DH Banach lattice which is p-convex with $1 < p < \infty$. Then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Disjoint sequences in Banach lattices

Pedro Tradacete

Mathematics Department, UC3M

Based on joint work with J. Flores, F. Hernández, E. Semenov, E. Spinu, V. Troitsky

First Brazilian Workshop in Geometry of Banach Spaces
25-29 August 2014, Maresias