Sequences in Large Spaces

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Outline

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1. Weakly Null Sequences

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2. Ramsey Theory of Fronts and Barriers

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12. Compact families on trees

Structure in weakly null sequences

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Theorem (Bessage-Pelczynski, 1958)

For every $\varepsilon > 0$, *every normalized weakly null sequence* (x_n) *contains an infinite* $(1+\varepsilon)$*-basic subsequence* (x_{n_i}).

Structure in weakly null sequences

Theorem (Bessage-Pelczynski, 1958)

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Theorem (Maurey-Rosenthal, 1977)

(1) *For every* $\varepsilon > 0$ *and every* $\alpha < \omega^\omega$*, every normalized weakly null sequence in* $\mathcal{C}(\alpha + 1)$ *has a* $(2+\varepsilon)$*-unconditional subsequence.*

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(2) *For every $\varepsilon > 0$ every normalized weakly null sequence in $\mathcal{C}(\omega^\omega + 1)$ has a $(4+\varepsilon)$-unconditional subsequence.*

(3) *There is a normalized weakly null sequence in $\mathcal{C}(\omega^{\omega^2} + 1)$ with no unconditional subsequence.*

Finite and partial unconditionality

Theorem (Odell 1993, Dodos-LopezAbad-Todorcevic, 2011)

Let k a positive integer and $\varepsilon > 0$. Suppose that for every $i < k$ we are given a normalized weakly null sequence $(x_n^i)_{n=0}^{\infty}$ in some Banach space X. Then, there exists an infinite set M of integers such that for every $\{n_0 < \cdots < n_{k-1}\} \subseteq M$ the k-sequence $(x_{n_i}^i)_{i<k}$ is $(1+\varepsilon)$-unconditional.

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Theorem (Arvanitakis 2006, Gasparis-Odell-Wahl, 2006 Todorcevic 2005)

Suppose that (x_n) is a normalized weakly-null sequence in $\ell_\infty(\Gamma)$ with the property that

$$\inf\{|x_n(\gamma)| : n \in \mathbb{N}, \gamma \in \Gamma\} > 0.$$

Then (x_n) contains an infinite unconditional basic subsequence.

Theorem (Elton 1978)

For every $0 < \varepsilon \leq 1$ *there is a constant* $C(\varepsilon) \geq 1$ *such that every normalized weakly null sequence* (x_n) *has an infinite subsequence* (x_{n_i}) *such that*

$$\|\sum_{i\in I} a_i x_{n_i}\| \leq C(\varepsilon)\|\sum_{j\in J} a_j x_{n_j}\|$$

for every pair $I \subseteq J$ *of subsets of* $\mathbb{N}$ *and every choice* $(a_j : j \in J)$ *of scalars such that* $\varepsilon \leq |a_j| \leq 1$ *for all* $j \in J$*.*

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Problem (Elton unconditionality constant problem)

Is $\sup_{0<\varepsilon\leq 1} C(\varepsilon) < \infty$*?*

Nash-Willimas's theory of fronts and barriers

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Theorem (Nash-Williams 1965)

Suppose $\mathcal{H} = \mathcal{H}_0 \cup \cdots \cup \mathcal{H}_l$ is a finite partition of a thin family $\mathcal{H}$ of finite subsets of $\mathbb{N}$. Then there is an infinite set $M \subseteq \mathbb{N}$ and $i < l$ such that $\mathcal{H} \restriction M \subseteq \mathcal{H}_i$, where

$$\mathcal{H} \restriction M = \{s \in \mathcal{H} : s \subseteq M\}.$$

Theorem (Pudlak-Rödl 1982)

For every front $\mathcal{B}$ on $\mathbb{N}$ and every mapping $f : \mathcal{B} \to \mathbb{N}$ there exist an infinite subset M of $\mathbb{N}$ and a mapping $\varphi : \mathcal{B} \restriction M \to [\mathbb{N}]^{<\infty}$ such that:

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(3) *for $s, t \in \mathcal{B} \restriction M$, $f(s) = f(t)$ iff $\varphi(s) = \varphi(t)$.*

Remark

There can be only one mapping $\varphi : \mathcal{B} \restriction M \to [\mathbb{N}]^{<\infty}$ satisfying the conditions (1), (2) and (3) from the Theorem on a given infinite subset M of $\mathbb{N}$.

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Suppose that a Banach space X has density $< \mathfrak{m}$ and that its dual X^* has an uncountable normalized w^*-null sequence. Then X has a quotient with a Schauder basis of length ω_1.

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Remark

Recall that $\mathfrak{m}$ is the Baire-category number of the class of compact Hausdorff spaces satisfying the **countable chain condition.**

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2. Given a family $\mathcal{F}$ of cardinality $< \mathfrak{b}$ the ideal

$$\mathcal{F}^{\perp} = \{x \in [S]^{\leq \aleph_0} : (\forall Y \in \mathcal{F})|x \cap Y| < \aleph_0\}$$

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2. It is also known that PID is consistent with GCH.
3. It is known that PID implies, for example, the Souslin Hypothesis.

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PID and Asplund spaces

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An **Asplund space**, or a **strong differentiability space** is a Banach space X with the property that every continuous convex function $f : U \to \mathbb{R}$ on an open convex domain $U \subseteq X$ is Fréchet differentiable in every point of a dense G_δ-subset of U.

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Remark

This is a well studied class of spaces with many pleasant properties such as the **projectional resolution of the identity** of its dual space, the **norm-fragmentability of the** w^***-topology** of the dual ball, **separability of the dual of every separable subspace**, etc.

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Assume PID. The following are equivalent:

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1. *Every non-separable Asplund space has an uncountable ε-biorthogonal system for every $\varepsilon > 0$.*
2. $\mathfrak{b} = \aleph_2$.

Weakly null sequences on Polish spaces

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Theorem (Mycielski 1964)

Suppose $M_n \subseteq X^{k_n} (n = 0, 1, \ldots)$ *is a sequence of subsets of the finite powers of some fixed Polish space* X *and suppose that* M_n *is a meager subset of* X^{k_n} *for all* n*. Then there is a perfect set* $P \subseteq X$ *such that* $[P]^{k_n} \cap M_n = \emptyset$ *for all* n*.*

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Theorem (Argyros-Dodos-Kanellopoulos 2008)

Suppose X *is a Polish space and that* $(f_a)_{a \in 2^{\mathbb{N}}}$ *is a bounded sequence in* $\ell_\infty(X)$ *such that* $(x, a) \mapsto f_a(x)$ *is a Borel function from* $X \times 2^{\mathbb{N}}$ *into* $\mathbb{R}$ *and that*

$$|\{a \in 2^{\mathbb{N}} : f_a(x) \neq 0\}| \leq \aleph_0 \text{ for all } x \in X.$$

Then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ *such that the sequence* $(f_a)_{a \in P}$ *is 1-unconditional.*

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Definition

Fix a rooted finitely branching tree U with no terminal nodes. A subtree T of U will be called a **strong subtree** if the levels of T are subsets of the levels of U and if for every $t \in T$ every immediate successor of t in U is extended by a unique immediate successor of t in T.

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Theorem (Halpern-Läuchli 1966)

For every sequence $U_0, ..., U_{d-1}$ of rooted finitelly branching trees with no terminal nodes and for every finite colouring of the level product $U_0 \otimes \cdots \otimes U_{d-1}$*, we can find for each $i < d$ a strong subtree T_i of U_i such that the T_i's share the same level set and such that the level product $T_0 \otimes \cdots \otimes T_{d-1}$ is monochromatic.*

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Theorem (Miliken 1981)

For every finite Borel colouring of the space $\mathcal{S}_\infty(U)$ of all strong subtrees of U there is a strong subtree T of U such that the set $\mathcal{S}_\infty(T)$ of strong subtrees of T is monochromatic.

A parametrized Ramsey theorem for perfect sets

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Theorem

For every countable Borel colouring of the product $[2^{\mathbb{N}}]^2 \times [\mathbb{N}]^\infty$ *with colours that are invariant under finite changes on the second coordinate, there is a perfect set* $P \subseteq 2^{\mathbb{N}}$ *and an infinite set* $M \subseteq \mathbb{N}$ *such that the product* $[P]^2 \times [M]^\infty$ *is monochromatic.*

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Example (Pol 1986)

Pol's compact set of Baire class-1 function is represented as

$$\mathbb{P} = 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\},$$

where the points of the Cantor tree $2^{<\mathbb{N}}$ are isolated, the nodes of a branch of this tree converge to the corresponding member of $2^{\mathbb{N}}$ and ∞ is the point that compactifies the rest of the space.

Theorem (Todorcevic 1999)

Suppose K is a separable compact set of Baire class-1 functions defined on some Polish space X. Let D be a countable dense subset of K, and let f be a point of K that is not G_δ in K. Then there is a homeomorphic embedding

$$\Phi : \mathbb{P} \to K$$

such that $\Phi(\infty) = f$ and $\Phi[2^{<\mathbb{N}}] \subseteq D$.

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Theorem (Argyros-Dodos-Kanellopoulos 2008)

Every infinite-dimensional dual Banach space has an infinite-dimensional quotient with a Schauder basis.

Unconditional sequences

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2. *When does an infinite normalized weakly null sequence in some normed space contains an infinite unconditional subsequence?*

Theorem (Johnson-Rosenthal 1972, Hagler-Johnson 1977)

If the dual X^* of some Banach space X contains an infinite unconditional basic sequence then X admits a quotient with an unconditional basis.

Free-set Property and Product-Ramsey Property

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Definition

We say that an index-set Γ has the **free-set property**, and write $\mathrm{FSP}(\Gamma)$, if every algebra $\mathcal{A}$ on Γ with no more than countably many operations has an infinite *free set*, an infinite subset X of Γ such that no $x \in X$ is in the sub algebra of $\mathcal{A}$ generated by $X \setminus \{x\}$.

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Definition

We say that Γ has the **product-Ramsey property**, and write $\mathrm{PRP}(\Gamma)$, if for every colouring

$$\chi : \Gamma^{<\omega} \to 2$$

of the set of all finite sequences of the index-set Γ into 2 colours there exists an infinite sequence (X_i) of 2-element subsets of Γ or, equivalently, an infinite sequence (X_i) of *infinite* subsets of Γ, such that χ is constant on $\prod_{i<n} X_i$ for all n.

Proposition

$\mathrm{PRP}(\Gamma)$ *implies* $\mathrm{FSP}(\Gamma)$ *but not vice versa.*

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The free-set property fails for index-sets of cardinalities $< \aleph_\omega$.

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Theorem (Koepke 1984, DiPrisco-Todorcevic 1999)

The following are equiconsistent:

1. $\mathrm{FSP}(\aleph_\omega)$.
2. $\mathrm{PRP}(\aleph_\omega)$.
3. *There is an index set supporting a non-principal countably complete ultrafilter.*

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Theorem (Shelah 1980)

$\mathrm{PRP}_2(\aleph_\omega)$ *is consistent relative to the existence of infinitely many compact cardinals.*

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Question

What is the equiconsistency result here?

Long weakly-null sequences

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Theorem (Dodos-LopezAbad-Todorcevic 2011)

If a normalized weakly null sequence $(x_i)_{i \in I}$ is indexed by a set I that has the free-set property then it contains an infinite unconditional basic subsequence.

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Theorem (Dodos-LopezAbad-Todorcevic 2011)

Suppose that $(x_\gamma)_{\gamma\in\Gamma}$ is a normalized and separated sequence in some Banach space X containing no ℓ_1. If the index-set Γ satisfies $\mathrm{PRP}_2(\Gamma)$ then there is an infinite sequence (β_n, γ_n) of pairs of elements of Γ such that the semi-normalized sequence $(x_{\beta_n} - x_{\gamma_n})$ is unconditional.

Corollary

It is consistent relative to the existence of a measurable cardinal that every normalized weakly null sequence of length at least $\aleph_\omega$ *has an infinite unconditional subsequence.*

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Question

Can $\aleph_\omega$ be lowered to some $\aleph_n$ in both or one of these corollaries?

Positional graphs and conditional weakly null sequences

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For a family $\mathbf{V} \subseteq [\Gamma]^{<\omega}$, we associate the corresponding **positional graph**

$$\mathcal{G}_n(\mathbf{V}) = (\mathbf{V},\ \Delta(n)^c),$$

where we put an edge between two finite $F, G \in \mathbf{V}$ if they are **not** in the $\Delta(n)$-position. Let $\mathcal{G}_n(\Gamma) = ([\Gamma]^{<\omega},\ \Delta(n)^c)$.

Question

For which Γ and $\mathbf{V} \subseteq [\Gamma]^{<\omega}$, the positional graph $\mathcal{G}_n(\mathbf{V})$ is **countably chromatic**?

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$\mathcal{G}_0(\omega_1)$ is countably chromatic but $\mathcal{G}_0(\omega_2)$ is not.

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For which Γ and $\mathbf{V} \subseteq [\Gamma]^{<\omega}$, the positional graph $\mathcal{G}_n(\mathbf{V})$ is **countably chromatic**?

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$\mathcal{G}_0(\omega_1)$ *is countably chromatic but* $\mathcal{G}_0(\omega_2)$ *is not.*

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We say that a family $\mathbf{V}$ of finite subsets of Γ is **dense** if for every infinite $A \subseteq \Gamma$ there is infinite $B \subseteq A$ such that $[B]^{<\omega} \subseteq \mathbf{V}$.

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Question

For which Γ there exist dense $\mathbf{V} \subseteq [\Gamma]^{<\omega}$ and an integer n such that the corresponding positional graph $\mathcal{G}_n(\mathbf{V})$ is countably chromatic?

Lemma (Key Fact 1)

If for some integer n there is a dense family $\mathbf{V} \subseteq [\Gamma]^{<\omega}$ such that $\mathcal{G}_n(\mathbf{V})$ is countably chromatic, then there is a normalized weakly null sequence indexed by Γ without infinite unconditional basic subsequence.

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(2) $c(F) = c(G)$ implies that $|F| = |G|$.

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This leads us to the collection of **special functionals**

$$\mathcal{F} = \{\sum_{i<k} |s_i|^{-1/2} \mathbf{1}_{s_i} : (s_i)_{i<k} \text{ is a finite } c\text{-special block-sequence}\}.$$

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Lemma

The weakly null sequence $(e_\gamma)_{\gamma\in\Gamma}$ contains no infinite unconditional basic subsequence.

Countably chromatic positional graphs

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Fix a positive integer n and for each $1 \leq k \leq n$ fix

$$\rho^k : [\omega_k]^2 \to \omega_{k-1}$$

such that

1. $\rho^k(\alpha_0, \alpha_2) \neq \rho^k(\alpha_1, \alpha_2)$,
2. $\rho^k(\alpha_0, \alpha_1) \neq \rho^k(\alpha_1, \alpha_2)$,
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Now, for each $0 \leq k \leq n$, we define

$$\varphi_k : [\omega_n]^{k+1} \to \omega_{n-k}$$

by letting

1. $\varphi_0 =$ the identity function on ω_n,
2. $\varphi_k(\alpha_0, \alpha_1, ..., \alpha_k) = \rho^{n-k+1}(\varphi_{k-1}(\alpha_0, ..., \alpha_{k-1}), \varphi_{k-1}(\alpha_1, ..., \alpha_k))$.

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Question

Is there for every non-negative integer n a reflexive space of density $\aleph_n$ without infinite unconditional basic sequence?

Subsymmetric sequences

A sequence (x_n) $(n < \omega)$ in some Banach space X is **subsymmetric** if there is a constant $C \geq 1$ such that for every pair F and G of finite subsets of ω of the same cardinality k and every sequence a_n $(n < k)$ of scalars,

$$\frac{1}{C} \parallel \sum_{n<k} a_n x_{F(n)} \parallel \leq \parallel \sum_{n<k} a_n x_{G(n)} \parallel \leq C \parallel \sum_{n<k} a_n x_{F(n)} \parallel .$$

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Theorem (Schreier, 1930)

There is an infinite weakly null sequence with no infinite subsymmetric subsequence

Subsymmetric sequences

A sequence (x_n) $(n < \omega)$ in some Banach space X is **subsymmetric** if there is a constant $C \geq 1$ such that for every pair F and G of finite subsets of ω of the same cardinality k and every sequence a_n $(n < k)$ of scalars,

$$\frac{1}{C} \| \sum_{n<k} a_n x_{F(n)} \| \leq \| \sum_{n<k} a_n x_{G(n)} \| \leq C \| \sum_{n<k} a_n x_{F(n)} \| .$$

Theorem (Schreier, 1930)

There is an infinite weakly null sequence with no infinite subsymmetric subsequence

Theorem (Tsirelson, 1972)

There is a reflexive infinite-dimensional Banach space X with no infinite subsymmetric sequence.

Theorem (LopezAbad-Todorcevic, 2013)

Let Γ *be an infinite cardinal. The following are equivalent:*

(1) $\Gamma \to (\omega)_2^{<\omega}$.

(2) *Every separated normalized sequence* (x_α) $(\alpha < \Gamma)$ *has a subsymmetric subsequence.*

(3) *There are no large compact and hereditary families on* Γ*.*

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Remark

Recall that a family $\mathcal{F}$ of subsets of Γ is **compact** if its pointwise closure consists only of finite subsets of Γ

Recall also that such an $\mathcal{F}$ is **hereditary** if it is closed under taking subsets.

We say that a family $\mathcal{F}$ of subsets of Γ is **large** if every infinite sunsets of Γ contains elements of $\mathcal{F}$ of arbitrary large finite cardinality.

Theorem (Argyros-Motakis, 2014)

There is a Banach space X of density $2^{\aleph_0}$ with no infinite subsymmetric sequence.

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What is the minimal cardinal Γ with the property that every Banach space X of density at least Γ must contain an infinite subsymmetric sequence?

Theorem (Brech-LopezAbad-Todorcevic, 2014)

There is a Banach space X of density bigger than the first ω-Mahlo cardinal with no infinite subsymmetric sequence.

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Definition

Let $\mathcal{B}$ and $\mathcal{C}$ be two families of subsets of some index-set Γ and let $\mathcal{H}$ be a family of subsets of ω.
We say that $\mathcal{C}$ is $(\mathcal{B}, \mathcal{H})$-**large** if every infinite sequence (t_k) of elements of $\mathcal{B}$ has an infinite subsequence $(t_k)_{k \in M}$ such that $\bigcup_{k \in v} t_k \in \mathcal{C}$ for all $v \in \mathcal{H} \upharpoonright M$.
We say that $\mathcal{C}$ is $\mathcal{B}$-**large** if it is $(\mathcal{B}, \mathcal{S})$-large, where $\mathcal{S}$ is the **Schreier family** on ω.

Definition

A sequence $(\mathcal{F}_n)$ $(n < \omega)$ of families of subsets of some index set Γ is a **CL-sequence** (sequence of **consecutively large** families) whenever $\mathcal{F}_0 = [\Gamma]^{\leq 1}$ and for every $n < \omega$:

1. $\mathcal{F}_n$ is compact and hereditary,
2. $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$,
3. $\mathcal{F}_{n+1}$ is $\mathcal{F}_n$-large.

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Theorem (Brech-LopezAbad-Todorcevic, 2014)

If some cardinal Γ supports a CL-sequence then there is a Banach space X of density Γ with no infinite subsymmetric sequence.

Constructing CL-sequences

Lemma (Key 1)

Suppose T is a tree such that

1. *There is a CL-sequence on chains of T,*
2. *There is a CL-sequence on the set of immediate successors of every node of T.*

Then there is a CL-sequence on T.

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Question

For which cardinals Γ do we have a tree T of cardinality Γ satisfying the two conditions?

Definition

Recall that a **C-sequence** on a cardinal Γ is a sequence $\overrightarrow{C} = (C_\gamma : \gamma \in \Gamma)$ such that C_γ is a closed and unbounded subset of γ for all $\gamma \in \Gamma$.

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A subset C of Γ is a **limit** of a subsequence $(C_{\gamma_n} : n < \omega)$ of $\overrightarrow{C}$ if every initial segment of C is an initial segment of all but finitely many C_{γ_n}'s.

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A C-sequence $\overrightarrow{C} = (C_\gamma : \gamma \in \Gamma)$ is **small** if there is a function $f : \Gamma \to \Gamma$ such that for every limit C of a subsequence of $\overrightarrow{C}$ we have that $\mathrm{otp}(C) < f(\min(C))$.

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If Γ supports a small C-sequence then then there is a tree T on Γ which has a CL-sequence on its chains.

If moreover, Γ is inaccessible then there is a CL-sequence on immediate successors of every node of T and therefore, there is a CL-sequence on Γ.

Define

$$\rho_0 : [\Gamma]^2 \to \mathcal{P}(\Gamma)^{<\omega}$$

recursively by

$$\rho_0(\alpha, \beta) = \langle C_\beta \cap \alpha \rangle^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$$

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Definition

To a given C-sequence C_α $(\alpha < \Gamma)$ one associates the notion of a **walk** from an ordinal $\beta < \Gamma$ to a smaller ordinal α :

$$\beta_0(\alpha) = \beta > \beta_1(\alpha) > \cdots > \beta_n(\alpha) = n,$$

where $n = |\rho_0(\alpha, \beta)|$ and where

$$\beta_{i+1}(\alpha) = \min(C_{\beta_i(\alpha)} \setminus \alpha).$$

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For $\alpha < \beta < \gamma < \Gamma$, we have

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3. *there is unique j such that* $\gamma_i(\alpha) = \gamma_i(\beta)$ *for all* $i \leq j$ *and* $\alpha \leq \gamma_{j+1}(\alpha) < \beta$.