Extracting bimonotone basic sequences from long weakly null sequences

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Abstract

This talk involves the following problem. Given a long weakly null normalized sequence of vectors in a Banach space, when can one find a long subsequence which is a bimonotone basic sequence? Some geometric technical tools to address the above problem are discussed. This in an on-going work.

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- It is *weakly null* if $f(x_\alpha) \to 0$ as $\alpha \to \mu$ for each $f \in \mathrm{X}^*$.
- In the typical case where $\mu = \kappa$, an uncountable regular cardinal (or merely having uncountable cofinality), the above can be rephrased as follows: there is $\beta < \kappa$ such that

$$f(x_\alpha) = 0, \quad \text{for } \beta < \alpha < \kappa.$$

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Unconditional sequences

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Unconditional sequences

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- That is, a subsequence $y_\gamma = x_{\alpha_\gamma}$, $0 \leq \gamma < \kappa$, which is 1-unconditional:

$$\left\| \sum_{\gamma \in \Gamma} a_\gamma y_\gamma \right\| \leq \left\| \sum_{\gamma \in \Lambda} a_\gamma y_\gamma \right\|$$

for $\Gamma \subset \Lambda \subset \kappa$ with Λ finite.

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  for $\Gamma \subset \Lambda \subset \kappa$ with Λ finite.
- The existence of such a subsequence (in different cases) is a long-standing problem.

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- On the other hand, it is known that under rather general assumptions a weakly null normalized long sequence admits a long subsequence which serves as a monotone basic sequence.
- Recall: Monotone means that the basis projections are norm-1.
- Therefore it is natural to ask if one can find subsequences having a property between monotonicity and unconditionality.

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Bimonotone sequences

- Note that in the definition of unconditionality it is clear that the norm of the basis projection P_α and its coprojection $Q_\alpha = I - P_\alpha$ given by

$$\sum_{\gamma<\kappa} a_\gamma e_\gamma \mapsto \sum_{\gamma<\alpha} a_\gamma e_\gamma, \quad \sum_{\gamma<\kappa} a_\gamma e_\gamma \mapsto \sum_{\alpha\leq\gamma<\kappa} a_\gamma e_\gamma$$

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- Note that in the 1-unconditional basis case there are vastly more canonical bimonotone projections (2^ω), compared to the bimonotone basis case (ω). Thus bimonotonicity is heuristically much closer to monotonicity than to unconditionality.

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Let X be a Banach space satisfying... (a strong Asplund type property). Suppose that $\{x_\alpha\}_{\alpha<\omega_1} \subset \mathrm{X}$ is a weakly null normalized transfinite sequence. Then there exists a subsequence $\{\alpha_\gamma\}_{\gamma<\omega_1}$ such that $\{x_{\alpha_\gamma}\}_{\gamma<\omega_1}$ forms a bimonotone basic sequence.

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- Also reasonable to ask whether less dispersed (than weakly null) sequence admits a bimonotone block basis:
  i.e. ω_1-many countable successive blocks of ordinals,

$$\{\beta_\theta^{(\gamma)}\}_{\theta<\eta^{(\gamma)}} \subset \omega_1, \quad 0 \leq \gamma < \omega_1$$

and a bimonotone basic sequence $\{z_\gamma\}_{\gamma<\omega_1} \subset \mathrm{X}$ such that

$$z_\gamma = \sum_{\theta<\eta^{(\gamma)}} a_\theta^{(\gamma)} x_{\beta_\theta^{(\gamma)}}, \quad 0 \leq \gamma < \omega_1.$$

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Philosophical preparations

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- For instance, one could ask if for a separable $\mathrm{Z} \subset \mathrm{X}$ the annihilator $\mathrm{Z}^{\perp}$ 1-norms a coseparable subspace Y (a kind of reverse 1-SCP).
- Refinement: If for a separable $\mathrm{Z} \subset \mathrm{X}$ the annihilator $\mathrm{Z}^{\perp}$ 1-norms a separable space $E \subset \mathrm{X}$, does there exist a coseparable $\mathrm{Y} \subset \mathrm{X}$ such that $E \subset \mathrm{Y}$ and $\mathrm{Z}^{\perp}$ 1-norms Y?

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(ii) X^{**} is WLD.

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3. Both X and X^* are Asplund.

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WLD

- Recall that a Banach space X is weakly Lindelöf determined (WLD) if there is an M-basis, i.e. a biorthogonal system $\{(x_\alpha, x_\alpha^*)\}_{\alpha<\mu} \subset \mathrm{X} \times \mathrm{X}^*$ such that

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such that additionally for each $f \in \mathrm{X}^*$

$$|\{\alpha \colon f(x_\alpha) \neq 0\}| \leq \aleph_0.$$

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Sketch of the proof of Lemma 1/4

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$$|||x|||_{k,\varepsilon}^2 = \|x\|^2 + \varepsilon \sum_{n=0}^{k} (a_n^*(x))^2, \quad \varepsilon > 0$$

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- That is, the Hahn-Banach extensions

$$HB_{k,\varepsilon} \colon ([a_n \colon n \leq k]^\perp)^* \to \mathrm{X}^{**},\ HB_{k,\varepsilon} \colon x^{**}|_{[a_n \colon n \leq k]^\perp} \mapsto x^{**},$$

  with

$$\|x^{**}\|_{\mathrm{X}^{**}} = \|x^{**}|_{[a_n \colon 1 \leq n \leq k]^\perp}\|_{([a_n \colon 1 \leq n \leq k]^\perp)^*},$$

  both under respective renorming $|||\cdot|||_{k,\varepsilon}$, are uniquely defined.

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- Therefore $HB_{k,\varepsilon}$ becomes a linear isometry $([a_n\colon 1 \leq n \leq k]^{\perp})^* \to \mathrm{X}^{**}$ (under the given equivalent norm). Its image is finite-codimensional.

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$$W := \bigcap_{k<\omega} \bigcap_{m<\omega} HB_{k,1/m}(\{x^{**}|_{[a_n \colon 1\leq n\leq k]^{\perp}} \colon x^{**} \in \mathrm{X}^{**}\}) \subset \mathrm{X}^{**}.$$

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- A moments reflection with $\varepsilon = 1/m \searrow 0$ and the Hahn-Banach Thm yields that $A^{\perp}$ 1-norms W.

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- Since X^{**} is WLD it satisfies (σ) and thus the above intersection is a coseparable subspace.
- By using the coseparability of W, let $(f_n)_{n<\omega} \subset \mathrm{X}^{***}$ be a sequence such that $\bigcap_n \ker f_n = W$.

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- Since X is WLD and Asplund it has a shrinking M-basis $\{(x_\alpha, x_\alpha^*)\}_{\alpha<\omega_1}$. Thus $\overline{[x_\alpha^* \colon \alpha]}^{\omega^*} = \mathrm{X}^{***}$.

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- The rest of the argument follows from condition (σ) of X.

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Thank you!