Boundaries, polyhedrality and LFC norms

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Boundaries

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- $B = \{\pm e_n^* \in \ell_1 \ : \ n \in \mathbb{N}\}$ is a **countable** boundary of $(c_0, \|\cdot\|_\infty)$.
- Boundaries can be highly irregular.

Polyhedral and LFC norms

Definition (Klee 60)

A norm $\|\cdot\|$ is **polyhedral** if, given any finite-dimensional subspace $Y \subseteq X$, there exist $f_1, \ldots, f_n \in S_{(X^*,\|\cdot\|)}$ such that

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Example

The natural norm on c_0 is both polyhedral and LFC-$(e_n^*)_{n\in\mathbb{N}}$.

The benefits of small boundaries

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- X is c_0**-saturated**.

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Definition

$A \subseteq X^*$ is **summable** (with respect to $(x_n)_{n\in\mathbb{N}}$) if $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for all $f \in A$.

Subsets of X^* that are still small, but a bit larger

Definition (Fonf, Pallares, S, Troyanski 2014)

- $E \subseteq X^*$ is w^***-locally relatively norm compact** (w^*-LRC) if, given $x \in E$, there exists w^*-open $U \ni x$ such that $E \cap U$ is relatively norm compact.

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- If E is relatively w^*-discrete then it is w^*-LRC.
- In particular, if $(x_\gamma)_{\gamma\in\Gamma}$ is a M-basis, then (x_γ^*) is w^*-LRC.
- By Baire Category, if $Y \subseteq X^*$ is any inf-dim subspace, then S_Y is never σ-w^*-LRC.

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A definition of happiness

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Corollary

If X has a LFC-H norm, where H is happy, then X admits norms as above.

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Examples

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- Spaces having σ-compact boundaries (Fonf, Hájek).
- $(C(K), \|\cdot\|_\infty)$, where K is σ-discrete (Hájek, Haydon 07).
- Certain spaces having a M-basis (some Orlicz 'sequence' spaces and preduals of Lorentz 'sequence' spaces $d(w, 1, A)$).

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Example

$C[0, \omega_1]$ admits both a polyhedral norm and a C^∞-smooth LFC norm. However, it admits no norm having LUR dual norm, so no boundary of $C[0, \omega_1]$ (in any norm) is happy.

A characterisation of isomorphic polyhedrality?

Theorem (Fonf 1981)

Let $(X, \|\cdot\|)$ be a polyhedral Banach space. Then the set

$$B = \{f \in B_{X^*} \ : \ f \text{ is } w^*\text{-strongly exposed}\}$$

is a **minimal** boundary of X, and $|B| = \text{dens}(X)$.

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Theorem

Let $(X, \|\cdot\|)$ be a weakly Lindelöf determined (WLD) polyhedral Banach space. Then the minimal boundary B can be written as

$$B = \bigcup_{n=1}^{\infty} B_n,$$

where each B_n is relatively norm- (equivalently w^*-) discrete. In particular, B is σ-w^*-LRC.

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- Does the above result apply in full generality?
- Is B contained in a happy set? If so, then given X WLD, X is isomorphically polyhedral if and only if it admits a norm having a boundary contained in a happy set.