A new Proof of Zippin's Embedding Theorem and Applications

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and then applying

Theorem (Davis, Figiel, Johnson, and Pełczyński, 1974)

A weakly compact operator from a Banach space X into a Banach space Z, which has a shrinking basis, factors through a reflexive space with a basis.

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Assume (P) is a property of (separable) Banach spaces. Does every Banach space X with property (P) embed into a Banach space Z with property (P) having a (certain) basis/FDD ?

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- (Johnson & Zheng 2008, 2011) Characterization of subspaces of spaces having an unconditional basis.

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Causey (2013 and 2014), yes (for all $\alpha < \omega_1$) but for Szlenk index of Z in embedding problem we have: $Sz(Z) = \omega^{\alpha+1}$, resp.

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Our goal:
A new proof of Zippin's Embedding Theorem, in which for a given space X, with X^* separable, or X separable and reflexive, the space Y, in which X embeds, inherits as many properties from X as possible.

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All known proofs of Zippin's Theorem (Zippin's original proof and a proof by Ghoussoub, Maurey, and Schachermayer, as well as proof by Bossard) start by embedding X into $Z = C(\Delta)$, $\Delta =$Cantor set (which has a basis), and then modifying Z until the modification has a shrinking basis but still contains X.

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We start with a Markushevich basis (e_i) of X (every separable space has such a basis) or more generally, a Finite Dimensional Markushevich Decomposition (FMD), and augment it *just enough to produce a space Z with a shrinking Finite Dimensional Decomposition (FDD), which contains X.*

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As we will see, several properties of X will be automatically inherited by Z and W.

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d) *if* X *is reflexive and has the* *w-Unconditional Tree Property* *then* (w_i) *is unconditional.*

w^*-**Unconditional Tree Property** (Johnson-Zheng):
Every w^*-null tree in S_{X^*}, (inf. countably branching, inf. countable height) has a branch which is unconditional.

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If $\dim(E_k) = 1$, for all $k \in \mathbb{N}$, say $E_k = \text{span}(e_k)$, then (e_k) is called a Markushevich basis.

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Markushevich, 1943: Every separable Banach space has a 1-norming Markushevich basis (e_j), which can be chosen to be shrinking if X^* is separable.
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Finite Dimensional Decompositions

An FMD (E_n) is called a Finite Dimensional Decomposition of X (FDD) if every $x \in X$ can be uniquely written as

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and an FDD (E_n) is called unconditional if above representation of every $x \in X$ converges unconditional, or, equivalently, if $u = \sup_{A \subset \mathbb{N}, \text{ finite}} \left\| P^E_A \right\| < \infty$.

Two simple, but Key Arguments

Assume that X^* is separable and that (E'_i) is a shrinking Finite Dimensional Markushevich Decomposition. (F'_i) its biorthogonal sequence.

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- *every, with respect to (F_j), skipped block sequence (x_n^*) in X^* ($F_n = span(F'_i : i_{n-1} < i \le i_n)$) is basic with projection constant at most 3.*
- *and, if X^* has the unconditional tree property for some constant C, every skipped block sequence (x_n^*) in X^* with respect to F_n is $2C$-unconditional.*

Lemma (Johnson, 1977)

Let $(\varepsilon_k) \subset (0,1)$. *There exists a strictly increasing* $(n_k)\subset\mathbb{N}$ *with:* *For every* $x^* \in B_{X^*}$ *there exists* $(j_k) \in \prod_{k=1}^{\infty}\{n_k, n_k+1, \dots n_{k+1}\}$ *with* $\|x^*|_{E_{j_k}}\|_{E^*_{j_k}} < \varepsilon_k$, *for* $k \in \mathbb{N}$.

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Remark

Since (E_n) *not necessarily FDD it could be that* $\|x^*|_{E_n}\|_{E^*_K} << \|P^E_n(x^*)\|_{X^*}$.

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Conclusion: the set

$$B^* = \left\{ x^* \in B_{X^*} : \begin{matrix} \exists \ (j_k) \in \prod_{k=1}^\infty \{n_k, n_k+1, \dots n_{k+1}\} \\ x^*|_{E_{j_k}} \equiv 0, \ k = 1, 2, \dots \end{matrix} \right\},$$

is $\frac{1}{2}$-norming the space X, so without loss of generality:

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We define: $Z_k = \oplus_{j=n_{k-1}+1}^{n_{k+1}-1} E_j$ (note the overlap!)
For $(z_k) \in c_{00}\big(\oplus_{k=1}^\infty Z_k\big)$ put:

$$\|(z_k)\|_Z = \sup_{(x_k^*) \in \mathbb{B}} \Big| \sum_{k=1}^\infty x_k^*(z_k) \Big|.$$

Z is then the completion of $c_{00}\big(\oplus_{k=1}^\infty Z_k\big)$ with respect to $\|\cdot\|$.

Properties of Z

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$$\begin{aligned}
\|I(x)\| &= \sup_{(x_k^*)\in\mathbb{B}} \sum_{k=1}^{\infty} x_k^*(P^E_{(n_{k-1},n_{k+1})}(x)) \\
&= \sup_{(x_k^*)\in\mathbb{B}} \sum_{k=1}^{\infty} x_k^*(x) \\
&= \sup_{(x_k^*)\in\mathbb{B}} \Big(\sum_{k=1}^{\infty} x_k^*\Big)(x) = \sup_{x^*\in B} |x^*(x)| = \|x\|.
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For $z = (z_k) \in c_{00}\big(\oplus_{k=1}^{\infty} Z_k\big)$, and $m \le n$ we have

$$\begin{aligned}
\left\|P^Z_{[m,n]}(z)\right\| &= \sup_{(x_k^*)\in\mathbb{B}} \Big| \sum_{k=m}^{n} x_k^*(z_k)\Big| \\
&\le \sup_{(x_k^*)\in\mathbb{B}} \left\|(x_k^*)_{k=m}^n\right\|_{Z^*} \|z\|_Z \\
&\le \sup_{(x_k^*)\in\mathbb{B}} \Big\| \sum_{k=m}^{n} x_k^*\Big\|_{X^*} \|z\|_Z \\
&\left[\sum y_k^* \in B^* \Rightarrow (y_k^*)\in\mathbb{B}, \text{ thus } \left\|(x^*)_{k=m}^n\right\|_{Z^*} \le \Big\| \sum_{j=m}^{n} x_k^*\Big\|_{X^*}\right] \\
&\le 3\|z\| \\
&\Big[(x_k^*) \text{ is a skipped block with respect to } (F_j)\Big]
\end{aligned}$$

3) The set $\mathbb{B}$ (seen as subset of B_{Z^*}) is 1-norming Z, w^* compact, and the map

$$\Psi : \mathbb{B} \to B, \quad (x_k^*) \mapsto \sum_{j=1}^{\infty} x_k^*,$$

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5) For every skipped block (z_k^*) in $\mathbb{B}$ (with respect to (Z_j^*)), $(\Psi(z_k^*))$ is an isometrically equivalent to a skipped block in X^* with respect to (F_k). And for every skipped block (x_k^*) in B with respect to F_j, $\big(\Phi_{\bar{j}}(x_k^*)\big)$ is an isometrically equivalent skipped block in $\mathbb{B}$ with respect to (Z_j^*).

6) Y Banach space, $N \in \mathbb{N}$, $T_k : Y \to Z_k$, for $k = 1, 2, \ldots N$.
We want to find an expression of the norm of
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For $\overline{x}^* = (x_k^*)_{k=1}^N \in \mathbb{B}_N$, let

$$T_{\overline{x}^*} : \mathrm{span}(x_k^* : 1 \le k \le N) \to Y^*, \quad \sum a_k x_k^* \mapsto \sum a_k x_k^* \circ T_k.$$

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We want to find an expression of the norm of
$T : Y \to Z, \quad y \mapsto (T_k(y) : k = 1, \dots N)$.
Example: $P_A^Z : Z \to Z$, $(z_j : j \in \mathbb{N}) \mapsto (z_j : j \in A)$, $A \subset \mathbb{N}$ fin.
Define

$$\mathbb{B}_N = \{(x_k^*) \in \mathbb{B} : x_{N+1}^* = x_{N+2}^* = \dots = 0\}$$

$$\equiv \left\{ (x_k^*)_{k=1}^N \subset X^* : \begin{array}{c} \exists (j_k) \in \prod_{k=1}^N \{n_k, n_k+1, \dots, n_{k+1}\} \\ \mathrm{rg}_E(x^*) \subset (j_{k-1}, j_k), k \in \mathbb{N}, \| \sum_{k=1}^N x_k^* \| \le 1 \end{array} \right\}$$

For $\overline{x}^* = (x_k^*)_{k=1}^N \in \mathbb{B}_N$, let

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Then $\|T\|_{L(Y,Z)} = \sup_{\overline{x}^* \in \mathbb{B}_N} \|T_{\overline{x}^*}\|$.
If $T = P_A^Z$, and thus $T_k = P_k^Z$, if $k \in A$, and $T_k = 0$ otherwise. Then

$$\|P_A^Z\| = \sup_{\overline{x} \in \mathbb{B}} \Big\| \sum_{k \in A} x_k^* \Big\|_{Z^*} = \sup_{\overline{x} \in \mathbb{B}} \Big\| \sum_{k \in A} x_k^* \Big\|_{X^*}.$$

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Problem (Finite Dimensional Part)

Assume that E is a finite dimensional space whose modulus of uniform convexity is $w(\cdot)$. Is there a constant C (could depend on $w(\cdot)$ but not on anything else) so that E is C-isomorphic to a subspace of finite dimensional space F whose modulus of uniform convexity is also $w(\cdot)$ (or a function $v(r)$ only depending on $w(\cdot)$), so that F has a basis whose constant is at most C?