An ℓ_1 preserving operator on $L_1(0, 1)$ which is not an isomorphism

Gideon Schechtman

Joint work with Bill Johnson, Amir Nasseri and Tomasz Tkocz

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main Results motivation Th 2 implies Th 1

• ℓ_1 preserving non-isomorphism

Theorem 1: There is an operator $T : L_1(0, 1) \rightarrow L_1(0, 1)$ that is an isomorphism when restricted to every subspace isomorphic to ℓ_1 but is not an isomorphism.

• Restricted invertible operator with large kernel

Theorem 2: There is an operator $T : L_1(0, 1) \rightarrow L_1(0, 1)$ such that for some $\varepsilon, \delta > 0$ $||Tf|| \ge \delta ||f||$ for all *f* with $|\text{supp}f| \le \varepsilon$ but Ker *T* is infinite dimensional.

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Although this was not our initial motivation, we found out that there is quite extensive literature on the subject.

An operator $T: X \to Y$ is called Tauberian if $T^{**-1}(Y) = X$.

The notion was termed and studied by Kalton and Wilansky [76].

A recent book by Gonzalez and Martinez-Abejon [2010] is recommended to anybody interested.

In particular the book deals extensively with Tauberian operators from L_1 spaces and basically contains the following:

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Theorem [G,M-A]: Let $T : L_1(0, 1) \rightarrow Y$. TFAE

0. T is Tauberian

- For all normalized disjoint sequence {x_i}, lim inf_{i→∞} || Tx_i|| > 0
- If {*x_i*} is equivalent to the unit vector basis of ℓ₁ then there is an *N* such that *T*<sub>[[*x_i*][∞]_{ℓ₁}, is an isomorphism.
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- 3. there is $\varepsilon, \delta > 0$ such that $||Tf|| > \varepsilon$ for all *f* with $|\text{supp}(f)| < \delta$

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If *R* is a reflexive subspace of $L_1 = L_1[0, 1]$, and $T : L_1 \rightarrow L_1/R$ is the quotient map then *T* satisfy 3 and so is Tauberian. In the book Gonzalez and Martinez-Abejon ask whether a Tauberian $T : L_1 \rightarrow L_1$ can have an infinite dimensional kernel. Our second theorem answers this positively:

Theorem: There is an operator $T : L_1(0, 1) \to L_1(0, 1)$ such that for some $\varepsilon, \delta > 0$ $||Tf|| \ge \delta ||f||$ for all *f* with $|\text{supp}(f)| \le \varepsilon$ but Ker *T* is infinite dimensional.

Recall that it's unknown whether there is a reflexive $R \subset L_1$ such that L_1/R embeds back into L_1 .

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Back to the characterization of Tauberian operators from L_1

Theorem: Let $T : L_1(0, 1) \rightarrow Y$. TFAE

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1. For all normalized disjoint sequence $\{x_i\}$,

 $\liminf_{i\to\infty}\|Tx_i\|>0$

2. If $\{x_i\}$ is equivalent to the unit vector basis of ℓ_1 then there is an *N* such that $T_{|[x_i]_{i=N}^{\infty}}$ is an isomorphism.

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Proof of 3 implies 2:

• Assume $\{x_i\}$ equivalent to the ℓ_1 basis and $T_{|[x_i]_{i=N}^{\infty}}$ is an not an isomorphism for any *N*.

• Passing to a block basis we may assume that there is a normalized $\{y_i\}$ equivalent to the ℓ_1 basis and $||Ty_i|| \rightarrow 0$ as fast as we want.

• Given $\lambda > 1$, passing to another block basis we can also assume $\{y_i\}$ is λ equivalent to the ℓ_1 basis.

• Then, $\{y_i\}$ is basically disjointly supported. and this contradicts 3.

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Recall our two main theorems:

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It is now clear that the second one implies the first.

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Proof of Th 2 implies Th 1:

Indeed, let $\{x_n, x_n^*\}$ be a Marcinkiewicz basis for Ker T. Let \bar{x}_n^* a norm preserving extension of x_n^* . Let $a_n > 0$ with $\sum a_n << 1$ and $S : L_1 \to \ell_1$ given by $S(x) = \sum a_n \bar{x}_n^*(x) e_i$.

Then $S + T : L_1 \rightarrow L_1 \oplus_1 \ell_1$ is the required operator.

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another characterization finite dimensional analogue proof of theorem 2 proof of finite dimensional analogue

another characterization

For the proof of Theorem 2 we'll need another simple characterization of Tauberian operators from L_1 spaces.

Lemma: Let μ be any measure on any measure space. $T: L_1(\mu) :\to Y$ is Tauberian iff there is an r > 0 and a natural number N such that if $\{x_n\}_{n=1}^N$ are disjoint unit vectors in $L_1(\mu)$ then $\max_{1 \le n \le N} ||Tx_n|| \ge r$.

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BGIKS Theorem

Theorem [Berinde,Gilbert,Indyk,Karloff,Strauss, 08]: For each *n* sufficiently large putting m = [3n/4], there is an operator $T : \ell_1^n \to \ell_1^m$ such that $\frac{1}{4} ||x||_1 \le ||Tx||_1 \le ||x||_1$

for all x with $\sharp supp(x) < n/400$.

More generally

Theorem [BGIKS, 08]: For each ε and m < n sufficiently large there is an operator $T : \ell_1^n \to \ell_1^m$ such that

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(1-\varepsilon)\|x\|_{1} \le \|Tx\|_{1} \le \|x\|_{1}
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for all x with $\sharp supp(x) \le \phi(n/m, \varepsilon)n$.

 $(\phi(t,\varepsilon) > 0 \text{ for all } t > 1, \varepsilon > 0.)$

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proof of Theorem 2

Denote the operator from the previous slide by T_n and note that it satisfies that $||T|| \le 1$ and if $x_1, x_2, \ldots, x_{400}$ are disjoint unit vectors in ℓ_1^n then $\max_{1 \le n \le 400} ||Tx_n||_1 \ge 1/4$. Also $\dim Ker(T) > n/4$

Then the ultraproduct $\overline{T} = (\prod T_n)_{\mathcal{U}} : (\prod \ell_1^n)_{\mathcal{U}} \to (\prod \ell_1^m)_{\mathcal{U}}$ satisfies $\|\overline{T}\| \le 1$ and if x_1, x_2, \dots, x_{400} are disjoint unit vectors in $(\prod \ell_1^n)_{\mathcal{U}}$ then $\max_{1 \le n \le 400} \|\overline{T}x_n\|_1 \ge 1/4$. Also dim $Ker(\overline{T}) = \infty$.

 $(\prod \ell_1^n)_{\mathcal{U}}$ and $(\prod \ell_1^m)_{\mathcal{U}}$ are huge L_1 spaces. Pick a separable subspace of $Ker\overline{T}$, let L be the closed sublattice generated by it and T the restriction of \overline{T} to L. T is then the required operator.

(*L* is an L_1 space which contain the kernel which is a reflexive subspace so can't be ℓ_1 so is $L_1(0, 1)$.)

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proof of Theorem 2

Denote the operator from the previous slide by T_n and note that it satisfies that $||T|| \le 1$ and if $x_1, x_2, \ldots, x_{400}$ are disjoint unit vectors in ℓ_1^n then $\max_{1 \le n \le 400} ||Tx_n||_1 \ge 1/4$. Also dim $Ker(T) \ge n/4$.

Then the ultraproduct $\overline{T} = (\prod T_n)_{\mathcal{U}} : (\prod \ell_1^n)_{\mathcal{U}} \to (\prod \ell_1^m)_{\mathcal{U}}$ satisfies $\|\overline{T}\| \le 1$ and if x_1, x_2, \dots, x_{400} are disjoint unit vectors in $(\prod \ell_1^n)_{\mathcal{U}}$ then $\max_{1 \le n \le 400} \|\overline{T}x_n\|_1 \ge 1/4$. Also dim*Ker* $(\overline{T}) = \infty$.

 $(\prod \ell_1^n)_{\mathcal{U}}$ and $(\prod \ell_1^m)_{\mathcal{U}}$ are huge L_1 spaces. Pick a separable subspace of *Ker* \overline{T} , let *L* be the closed sublattice generated by it and *T* the restriction of \overline{T} to *L*. *T* is then the required operator.

(*L* is an L_1 space which contain the kernel which is a reflexive subspace so can't be ℓ_1 so is $L_1(0, 1)$.)

expanders

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Here

 $\Gamma(S) = \{r \in R; \text{ such that there is an } s \in S \text{ with } (s, r) \in E\}.$

More generally: For each ε and m < n sufficiently large there is a bipartite graph G = (L, R, E) with |L| = n, |R| = m and left degree $d = \phi(n/m, \varepsilon)$ such that for all $S \subset L$ with $|S| \le \phi(n/m, \varepsilon)n$, $|\Gamma(S)| \ge (1 - \varepsilon)|S|$.

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proof of the BGIKS theorem

$$Te_i = rac{1}{d} \sum_{(i,j)\in E} e_j.$$

Clearly $||T|| \leq 1$.

To prove the lower bound take wlog $x = \sum_{i=1}^{n} a_i e_i$ with $|a_1| \ge |a_2| \ge \cdots \ge |a_k|$, $a_{k+1} = \cdots = a_n = 0$, and we want to evaluate ||Tx|| from below.

Order the edges $u_t = (i_t, j_t)$ in lexicographic order, t = 1, 2, ..., dn.

An edge u_t causes a collision if there is an earlier edge u_s with $j_s = j_t$.

Put $E' = \{$ all edges which do not cause a collision $\}$, and $E'' = E \setminus E'.$

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Lemma:
$$\sum_{(i,j)\in E''} |a_i| \leq \varepsilon d ||x||.$$

So





 $\geq \sum_{j=1}^m \sum_{(i,j)\in E} |a_i| - 2\sum_{j=1}^m \sum_{(i,j)\in E''} |a_i| \geq (1-2\varepsilon)d\|x\|.$

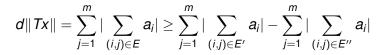
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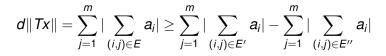
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For $S \subset L$ of cardinality $s = |S| \le n/10d$ and $T \subset R$ of cardinality t = |T| < 5ds/8 let $A_{S,T}$ be the event that all the edges from *S* go to *T*.

We want to show that the union of all the $A_{S,T}$ -s has probability less than 1.

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Gideon Schechtman Non-isomorphism ℓ_1 preserving operator on $L_1(0, 1)$

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