An ℓ_1 preserving operator on $L_1(0,1)$ which is not an isomorphism

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Joint work with
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Main results

- ℓ_1 preserving non-isomorphism

Theorem 1: There is an operator $T : L_1(0,1) \to L_1(0,1)$ that is an isomorphism when restricted to every subspace isomorphic to ℓ_1 but is not an isomorphism.

- Restricted invertible operator with large kernel

Theorem 2: There is an operator $T : L_1(0,1) \to L_1(0,1)$ such that for some $\varepsilon, \delta > 0$

$$\|Tf\| \geq \delta \|f\| \quad \text{for all } f \text{ with } |\mathrm{supp} f| \leq \varepsilon$$

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Although this was not our initial motivation, we found out that there is quite extensive literature on the subject.

An operator $T : X \to Y$ is called Tauberian if $T^{**-1}(Y) = X$.

The notion was termed and studied by Kalton and Wilansky [76].
A recent book by Gonzalez and Martinez-Abejon [2010] is recommended to anybody interested.
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0. T is Tauberian
1. For all normalized disjoint sequence $\{x_i\}$, $\liminf_{i\to\infty} \|Tx_i\| > 0$
2. If $\{x_i\}$ is equivalent to the unit vector basis of ℓ_1 then there is an N such that $T_{|[x_i]_{i=N}^{\infty}}$ is an isomorphism.
3. there is $\varepsilon, \delta > 0$ such that $\|Tf\| > \varepsilon$ for all f with $|\mathrm{supp}(f)| < \delta$

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If R is a reflexive subspace of $L_1 = L_1[0,1]$, and $T : L_1 \to L_1/R$ is the quotient map then T satisfy 3 and so is Tauberian. In the book Gonzalez and Martinez-Abejon ask whether a Tauberian $T : L_1 \to L_1$ can have an infinite dimensional kernel. Our second theorem answers this positively:

Theorem: There is an operator $T : L_1(0,1) \to L_1(0,1)$ such that for some $\varepsilon, \delta > 0$

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Recall that it's unknown whether there is a reflexive $R \subset L_1$ such that L_1/R embeds back into L_1.

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Th 2 implies Th 1

Back to the characterization of Tauberian operators from L_1

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Proof of 3 implies 2:

• Assume $\{x_i\}$ equivalent to the ℓ_1 basis and $T_{|[x_i]_{i=N}^{\infty}}$ is an not an isomorphism for any N.

• Passing to a block basis we may assume that there is a normalized $\{y_i\}$ equivalent to the ℓ_1 basis and $\|Ty_i\| \to 0$ as fast as we want.

• Given $\lambda > 1$, passing to another block basis we can also assume $\{y_i\}$ is λ equivalent to the ℓ_1 basis.

• Then, $\{y_i\}$ is basically disjointly supported. and this contradicts 3.

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Indeed, let $\{x_n, x_n^*\}$ be a Marcinkiewicz basis for Ker T.
Let $\bar{x}_n^*$ a norm preserving extension of x_n^*.
Let $a_n > 0$ with $\sum a_n << 1$ and $S : L_1 \to \ell_1$ given by

$$S(x) = \sum a_n \bar{x}_n^*(x) e_i.$$

Then $S + T : L_1 \to L_1 \oplus_1 \ell_1$ is the required operator.

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another characterization

For the proof of Theorem 2 we'll need another simple characterization of Tauberian operators from L_1 spaces.

Lemma: Let μ be any measure on any measure space. $T : L_1(\mu) :\to Y$ is Tauberian iff there is an $r > 0$ and a natural number N such that if $\{x_n\}_{n=1}^N$ are disjoint unit vectors in $L_1(\mu)$ then $\max_{1 \le n \le N} \|Tx_n\| \ge r$.

BGIKS Theorem

Theorem [Berinde,Gilbert,Indyk,Karloff,Strauss, 08]:
For each n sufficiently large putting $m = [3n/4]$, there is an operator $T : \ell_1^n \to \ell_1^m$ such that

$\frac{1}{4}\|x\|_1 \leq \|Tx\|_1 \leq \|x\|_1$

for all x with $\sharp\mathrm{supp}(x) \leq n/400$.

More generally

Theorem [BGIKS, 08]: For each ε and $m < n$ sufficiently large there is an operator $T : \ell_1^n \to \ell_1^m$ such that

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proof of Theorem 2

Denote the operator from the previous slide by T_n and note that it satisfies that $\|T\| \le 1$ and if $x_1, x_2, \dots, x_{400}$ are disjoint unit vectors in ℓ_1^n then $\max_{1\le n\le 400} \|Tx_n\|_1 \ge 1/4$.
Also $\dim Ker(T) \ge n/4$.

Then the ultraproduct $\bar{T} = (\prod T_n)_{\mathcal{U}} : (\prod \ell_1^n)_{\mathcal{U}} \to (\prod \ell_1^m)_{\mathcal{U}}$ satisfies $\|\bar{T}\| \le 1$ and if $x_1, x_2, \dots, x_{400}$ are disjoint unit vectors in $(\prod \ell_1^n)_{\mathcal{U}}$ then $\max_{1\le n\le 400} \|\bar{T}x_n\|_1 \ge 1/4$.
Also $\dim Ker(\bar{T}) = \infty$.

$(\prod \ell_1^n)_{\mathcal{U}}$ and $(\prod \ell_1^m)_{\mathcal{U}}$ are huge L_1 spaces. Pick a separable subspace of $Ker\bar{T}$, let L be the closed sublattice generated by it and T the restriction of $\bar{T}$ to L. T is then the required operator.

(L is an L_1 space which contain the kernel which is a reflexive subspace so can't be ℓ_1 so is $L_1(0,1)$.)

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Special expander [probably Pinsker 73]: For each n sufficiently large there is a bipartite graph $G = (L, R, E)$ with $|L| = n$, $|R| = m = [3n/4]$ and left degree $d = 32$ such that for all $S \subset L$ with $|S| \leq n/10d$, $|\Gamma(S)| \geq 5d|S|/8$.

Here
$\Gamma(S) = \{r \in R;$ such that there is an $s \in S$ with $(s, r) \in E\}$.

More generally: For each ε and $m < n$ sufficiently large there is a bipartite graph $G = (L, R, E)$ with
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proof of the BGIKS theorem

$$Te_i = \frac{1}{d} \sum_{(i,j) \in E} e_j.$$

Clearly $\|T\| \leq 1$.

To prove the lower bound take wlog $x = \sum_{i=1}^{n} a_i e_i$ with $|a_1| \geq |a_2| \geq \cdots \geq |a_k|$, $a_{k+1} = \cdots = a_n = 0$, and we want to evaluate $\|Tx\|$ from below.

Order the edges $u_t = (i_t, j_t)$ in lexicographic order, $t = 1, 2, \ldots, dn$.

An edge u_t causes a collision if there is an earlier edge u_s with $j_s = j_t$.

Put $E' = \{\text{all edges which do not cause a collision}\}$, and $E'' = E \setminus E'$.

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