Uniform classification of classical Banach spaces

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Ribe '76. The local structure is preserved: There exists $K = K(\phi)$ such that every finite dimensional subspace of X K-embeds into Y, and vice versa.

Johnson-Lindenstrauss-Schechtman '96
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Open for ℓ_1 (Lipschitz case too)

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We will give another.

Result

Theorem. Suppose $\phi : X \to Y$ is a uniform homeomorphism and Y is reflexive. Then there exists $K = K(\phi)$ such that for all n and all asymptotic spaces $(x_i)_{i=1}^n$ of X and all scalars $(a_i)_{i=1}^n$, we have

$$\|\sum_{i=1}^{n} a_i x_i\| \leq K \sup \|\sum_{i=1}^{n} a_i y_i\|$$

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If $Y = \ell_p$, then this means

$$\|\sum_{i=1}^n a_i x_i\| \le K(\sum_{i=1}^n |a_i|^p)^{1/p}.$$

Thus, X cannot contain ℓ_2 if $p > 2$.

Asymptotic structure

Maurey-Milman-Tomczak-Jaegermann '94 Let X be a Banach space with a normalized basis (or a minimal system) (u_i). Write $n < x < y$ if
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An n-dimensional space with basis $(e_i)_1^n$ is called an **asymptotic space** of X, write $(e_i)_1^n \in \{X\}_n$, if for all $\varepsilon > 0$

$$\forall m_1 \; \exists m_1 < x_1 \;\; \forall m_2 \; \exists m_2 < x_2 \;\; \ldots \forall m_n \; \exists m_n < x_n$$

such that the resulting blocks (called **permissible**) satisfy $(x_i)_1^n \overset{1+\varepsilon}{\sim} (e_i)_1^n$.

Asymptotic tree

$(e_i)_1^n \in \{X\}_n$ means that for all $\varepsilon > 0$ there exists a block tree of n-levels

$$T_n = \{x(k_1, k_2, \dots, k_j) : 1 \leq j \leq n\}$$

so that every branch $(x(k_1), x(k_1, k_2), \dots, x(k_1, \dots, k_n))$ is $(1+\varepsilon)$-equivalent to $(e_i)_1^n$.

Asymptotic-ℓ_p spaces

X is **asymptotic-ℓ_p** (**asymptotic-c_0** for $p = \infty$), if there exists $K \geq 1$ such that for all n and $(e_i)_1^n \in \{X\}_n$, $(e_i)_1^n \overset{K}{\sim}$ uvb ℓ_p^n.

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- L_p is not. Indeed, every C-unconditional $(x_i)_1^n \subset L_p$ is CK_p-equivalent to some asymptotic space of L_p.

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- Tsirelson space T is asymptotic-ℓ_1.
- T^* is asymptotic-c_0.

Envelope functions

Define the **upper envelope** function r_X on c_{00} by

$$r_X(a_1, \dots, a_n) = \sup_{(e_i)_1^n \in \{X\}_n} \| \sum_i^n a_i e_i \|$$

and the **lower envelope** g_X by

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- X is asymptotic-ℓ_p iff $g_X \simeq \|.\|_p \simeq r_X$.
- $r_X \simeq \|.\|_\infty$ implies X is asymptotic-c_0.

The upper envelope is invariant

Theorem. Suppose $\phi : X \to Y$ is uniform homeomorphism, and X and Y are reflexive. Then there exists $K = K(\phi)$ such that for all scalars $a = (a_i) \in c_{00}$, we have

$$\frac{1}{K} r_Y(a) \leq r_X(a) \leq K r_Y(a).$$

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Corollary. Suppose X is uniformly homeomorphic to a reflexive asymptotic-c_0 space. Then X is asymptotic-c_0.

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Example. T^*

The main technical theorem

Theorem. Suppose $\phi : X \to Y$ is a uniform homeomorphism and Y is reflexive. Then for all $(e_i)_1^k \in \{X\}_k$, integers $(a_i)_1^k$ and $\varepsilon > 0$, there exist permissible $(x_i)_1^k$ in X with $(x_i)_1^k \overset{1+\varepsilon}{\sim} (e_i)_1^k$ and permissible tuple $(h_i/\|h_i\|)_1^k$ in Y with $\|h_i\| \leq K|a_i|$ (K depends only on ϕ) such that

$$\left\| \phi\Big(\sum_{i=1}^k a_i x_i\Big) - \sum_{i=1}^k h_i \right\| \leq \varepsilon.$$