# The intrinsic geometry of topological groups

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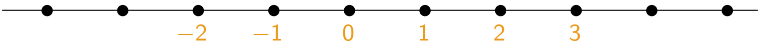
where $g \in \Gamma$ and $s \in \Sigma \setminus \{1\}$.

The resulting graph $\mathrm{Cayley}(\Gamma, \Sigma)$ is connected and hence $\Gamma$ is a metric space, when given the shortest-path metric $\rho_\Sigma$.

Consider first $(\mathbb{Z}, +)$ with generating set $\Sigma = \{-1, 1\}$.



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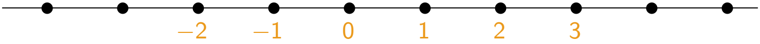
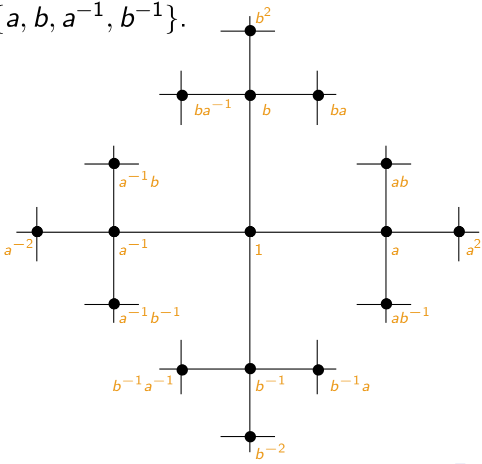



Similarly, let $\mathbb{F}_2$ be the free non-abelian group on generators $a, b$ and set $\Sigma = \{a, b, a^{-1}, b^{-1}\}$.



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$\rho_\Sigma$ is called the word metric induced by the generating set $\Sigma$.

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Taking $N$ to be the largest $n$ needed for the finitely many $s' \in \Sigma'$, one sees that

$$\rho_\Sigma \leqslant N \cdot \rho_{\Sigma'}.$$

By symmetry, we therefore see that there is a $K$ so that

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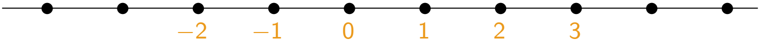

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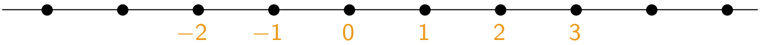
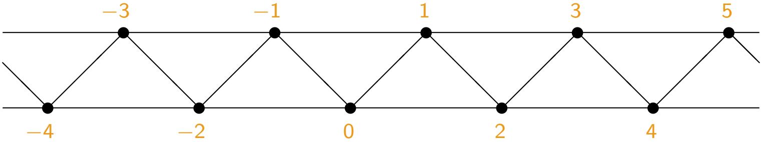



Whereas, with generating set $\Sigma=\{-2,-1,1,2\}$, we have



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And with only a minimal amount of care this can be modified to

> *if $G$ is locally compact metrisable group generated by a compact symmetric set $\Sigma$, then $G$ admits a compatible left-invariant metric $d$ quasi-isometric with the word metric $\rho_\Sigma$.*

## Definition

*A map* $F\colon (X,d) \to (Y,\partial)$ *between metric spaces is said to be a* *quasi-isometric embedding* *if there are constants* $K, C$ *so that*

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*Also, two metrics* $d$ *and* $\rho$ *on* $G$ *are quasi-isometric if the identity map*

$$\mathrm{id}\colon (G,d)\to(G,\rho)$$

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For familiarity, we may restrict the attention to Polish groups, i.e., separable and completely metrisable topological groups.

Polish groups encompass most topological transformation groups, e.g.,

$$\mathrm{Homeo}(M),\ \ \mathrm{Diff}^k(M),\ \ \mathrm{Isom}(X, \|\cdot\|).$$

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- By the existence of proper metrics, in locally compact groups, relative property (OB) coincides with relative compactness.
- Also, in the additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$, the relative property (OB) coincides with norm boundedness.

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**Lemma**

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1. *$A$ has property (OB) relative to $G$,*
2. *for every open $V \ni 1$ there are a finite subset $F \subseteq G$ and some $k \geqslant 1$ so that*
$$A \subseteq (FV)^k.$$

# Coarse geometry of Polish groups

Extending the definition of proper metrics on locally compact groups, we set

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As in the locally compact case, we see that a metrically proper metric is unique up to coarse equivalence.

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E.g., all locally compact second countable groups.

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*A metric space $(X, d)$ is said to be large scale geodesic if there is $K \geqslant 1$ so that, for all $x, y \in X$, there are $z_0 = x, z_1, z_2, \ldots, z_n = y$ satisfying*

1. $d(z_i, z_{i+1}) \leqslant K$,
2. $\sum_{i=0}^{n-1} d(z_i, z_{i+1}) \leqslant K \cdot d(x, y)$.

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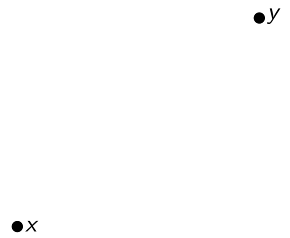

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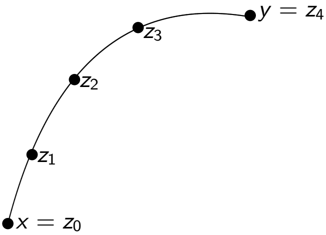

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Any choice of maximal metric on $G$ defines the same quasi-isometry type of $G$. So we can speak of the latter without refering to a choice of metric on $G$.

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*An continuous isometric action $G \curvearrowright (X, \partial)$ on a metric space is said to be metrically proper if, for all $x \in X$,*

$$\partial(g_n x, x) \to \infty \quad \text{whenever } g_n \to \infty.$$

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$$\partial(g_n x, x) \to \infty \quad \text{whenever } g_n \to \infty.$$

*Moreover, the action is cobounded if there is a set $A \subseteq X$ of finite $\partial$-diameter so that $X = G \cdot A$.*

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### Theorem (Milnor–Švarc)

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(a) *Then $G$ admits a maximal compatible left-invariant metric.*

(b) *Moreover, for every $x \in X$, the map*

$$g \in G \mapsto gx \in X$$

*is a quasi-isometry between $G$ and $(X, d)$.*

- Let $(X,+)$ be the underlying additive topological group of a Banach space $(X,\|\cdot\|)$. Then

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- Let $\mathbb{U}$ be the Urysohn metric space. Then

$$\mathrm{Isom}(\mathbb{U}) \simeq_{q.i.} \mathbb{U}.$$

- Let $X$ be one of $\ell^p$ or $L^p([0,1])$, $1 < p < \infty$. Then

$$\mathrm{Aff}(X) \simeq_{q.i.} X,$$

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In the last example, we know by work of W. B. Johnson, J. Lindenstrauss and G. Schechtman that two such spaces $X, Y$ are quasi-isometric only if either $X = Y$ or if $X, Y = \ell^2, L^2([0,1])$.

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So, apart from those cases, we conclude that

$$\mathrm{Aff}(X) \not\simeq \mathrm{Aff}(Y).$$

**Theorem**

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However, the identification of its actual quasi-isometry type remains a significant challenge.

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Moreover, if the action $\alpha\colon G \curvearrowright X$ is metrically proper, then $b\colon G \to X$ will be a coarse embedding of $G$ into $X$.

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- Hilbert spaces,
- Super-reflexive spaces,
- Reflexive spaces,
- General Banach spaces.

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**Theorem**

*Let $G$ be a locally (OB) Polish group. Then $G$ admits a metrically proper affine isometric action on a Banach space.*

Also, in the case of Hilbert spaces, we may elaborate a construction due to I. Aharoni, B. Maurey and B. S. Mityagin for the case of abelian groups and extended to locally compact amenable groups by Y. de Cornulier, R. Tessera and A. Valette.

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Whereas all amenable locally compact groups have the Haagerup property, this may fail for amenable locally (OB) Polish groups. $\mathrm{Isom}(\mathbb{U})$ and $c_0$ are counter-examples.

Stronger than amenability we have

### Definition

*A Polish group $G$ is* *approximately compact* *if there is a chain*

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Extending a previous construction due to V. Pestov, we have

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Again, $c_0$ (N. Kalton) is a counter-example in the Polish setting.