Low distortion embeddings between $C(K)$ spaces

joint work with Luis Sánchez González

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Université de Franche-Comté

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Maresias

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Definition (Lipschitz embedding/homeomorphism)

Let (M, d) be a metric space and $(X, \|\cdot\|)$ a Banach space. We denote $f : M \underset{D}{\hookrightarrow} X$ if

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

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Theorem

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In particular if $\alpha \neq \beta < \omega_1$ then, for any $n, m \in \mathbb{N}$, there is no Lipschitz homeomorphism $f : C([0, \omega^\alpha \cdot m]) \to C([0, \omega^\beta \cdot n])$ such that $\|f\|_{Lip} \left\|f^{-1}\right\|_{Lip} < 2$.

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Proposition

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Corollary

If $M_{\omega^\alpha} \underset{D}{\hookrightarrow} X$, $D < 2$, then $Sz(X) \geq \omega^{\alpha+1}$.

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there is an equivalent norm $|\cdot|$ on ℓ_1 such that $M \underset{1}{\hookrightarrow} (\ell_1, |\cdot|)$.

The unwieldy metric space M

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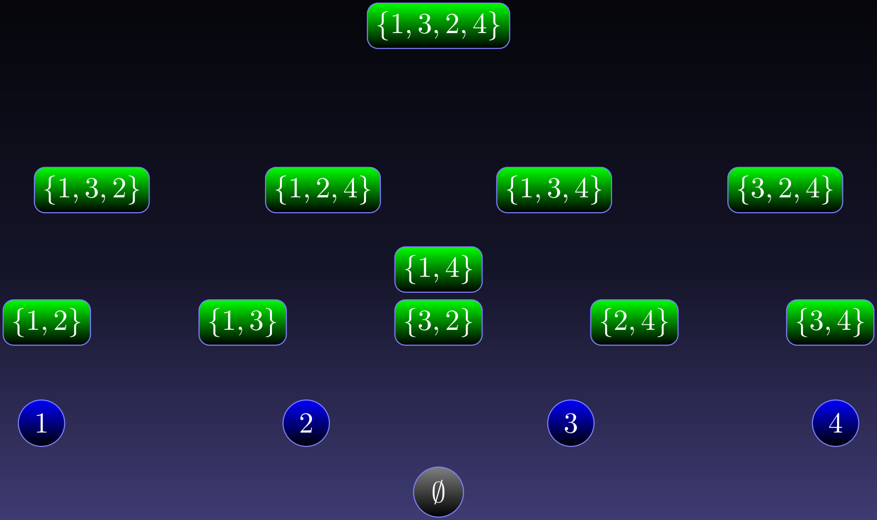

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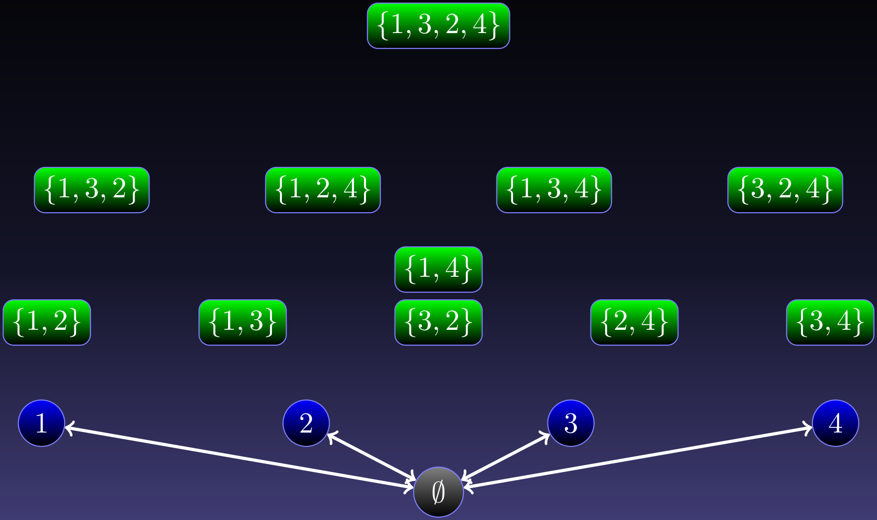

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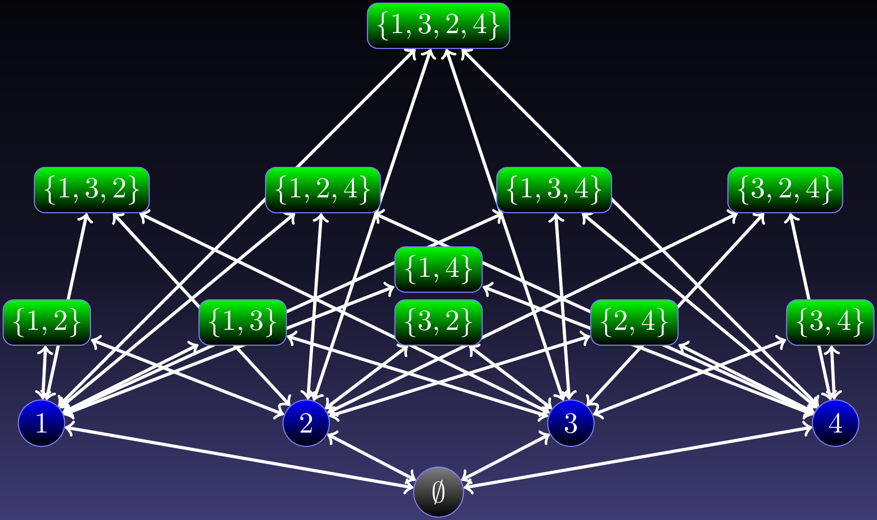

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$(x_{k_n})_n$ is not weakly Cauchy+Rosenthal's theorem $\Rightarrow \ell_1 \subset X$ $\square$

Left open

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- Is it true that $\ell_1 \underset{D}{\hookrightarrow} X$, $D < 2$, implies that $\ell_1 \subset X$?

Autumn 2014: Thematic trimester at the Université de Franche-Comté

“Geometric and noncommutative methods in functional analysis”