On the real polynomial Bohnenblust–Hille inequality

Daniel Pellegrino

Universidade Federal da Paraiba, Brazil

Maresias - SP
August, 2014

The multilinear Bohnenblust–Hille inequality

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1},\ldots,e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \|U\| \qquad (1)$$

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \left\| U \right\| \qquad (1)$$

for all m-linear forms $U : l_{\infty}^{N} \times \cdots \times l_{\infty}^{N} \to \mathbb{K}$ and every positive integer N.

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \|U\| \tag{1}$$

for all m-linear forms $U : l_\infty^N \times \cdots \times l_\infty^N \to \mathbb{K}$ and every positive integer N.

The exponent $\frac{2m}{m+1}$ is optimal...

The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \left\| U \right\| \qquad (1)$$

for all m-linear forms $U : l_\infty^N \times \cdots \times l_\infty^N \to \mathbb{K}$ and every positive integer N.

The exponent $\frac{2m}{m+1}$ is optimal... The best constant $C_{\mathbb{K},m}$ in this inequality will be denoted by $\mathrm{B}_{\mathbb{K},m}^{\mathrm{mult}}$.

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on l_∞^N,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{\|z\|_\infty \leq 1} |P(z)|$.

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on l_∞^N,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{\|z\|_\infty \leq 1} |P(z)|$.

The exponent $\frac{2m}{m+1}$ is optimal...

The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):

For any $m \geq 1$, there exists a constant $D_{\mathbb{K},m} \geq 1$ such that, for any $n \geq 1$, for any m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on l_∞^N,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{K},m} \|P\|_\infty,$$

where $\|P\|_\infty = \sup_{\|z\|_\infty \leq 1} |P(z)|$.

The exponent $\frac{2m}{m+1}$ is optimal...

The best constant $D_{\mathbb{K},m}$ in this inequality will be denoted by $\mathrm{B}_{\mathbb{K},m}^{\mathrm{pol}}$.

Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

1- To estimate the abscissae of convergence of Dirichlet series (this was the initial goal of Bohnenblust and Hille).

Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

1- To estimate the abscissae of convergence of Dirichlet series (this was the initial goal of Bohnenblust and Hille).

2- To estimate the Bohr radius of the n-dimensional polydisk.

Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

1- To estimate the abscissae of convergence of Dirichlet series (this was the initial goal of Bohnenblust and Hille).

2- To estimate the Bohr radius of the n-dimensional polydisk.

3- In Quantum Information Theory.

Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

1- To estimate the abscissae of convergence of Dirichlet series (this was the initial goal of Bohnenblust and Hille).

2- To estimate the Bohr radius of the *n*-dimensional polydisk.

3- In Quantum Information Theory.

It turns out that having good estimates of the constants $\mathrm{B}_{\mathbb{K},m}^{\mathrm{pol}}$ and $\mathrm{B}_{\mathbb{K},m}^{\mathrm{mult}}$ is crucial.

Estimates for the complex BH constants along the history

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

A.M. Davie (J. London Math. Soc., 1973)

$$\mathrm{qB}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\sqrt{2}\right)^{m-1}$$

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

A.M. Davie (J. London Math. Soc., 1973)

$$\mathrm{qB}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\sqrt{2}\right)^{m-1}$$

H. Queffelec (J. Analyse, 1995)

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$$

Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$$

A.M. Davie (J. London Math. Soc., 1973)

$$\mathrm{qB}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\sqrt{2}\right)^{m-1}$$

H. Queffelec (J. Analyse, 1995)

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$$

D. Nunez,D.P.,Serrano and Seoane (J. Functional Analysis, 2013)
....complicated recursive formula....but in any case

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} < (m-1)^{0.31}$$

Complex multilinear BH: estimates for the constants

	Optimal	2014	2013	1995	1978	1931
$B_{\mathbb{C},3}^{\mathrm{mult}} \leq$	?	1.2184	1.24	1.27	2	4.17
$B_{\mathbb{C},4}^{\mathrm{mult}} \leq$	?	1.2889	1.32	1.44	2.83	6.73
$B_{\mathbb{C},5}^{\mathrm{mult}} \leq$	?	1.3474	1.42	$\approx$???1.44	4	10.51
$B_{\mathbb{C},6}^{\mathrm{mult}} \leq$	?	1.3978	1.47	1.83	5.66	16.09
$B_{\mathbb{C},7}^{\mathrm{mult}} \leq$	?	1.4422	1.53	≈ 2.06	8	24.33
$B_{\mathbb{C},8}^{\mathrm{mult}} \leq$	?	1.4821	1.58	≈ 2.33	11.32	36.45
$B_{\mathbb{C},9}^{\mathrm{mult}} \leq$	?	1.5183	1.63	2.63	16	54.24
$B_{\mathbb{C},10}^{\mathrm{mult}} \leq$	?	1.5515	1.68	2.96	22.63	80.29
$B_{\mathbb{C},100}^{\mathrm{mult}} \leq$	?	2.5118	4.55	$1.56 \cdot 10^5$	$7.9 \cdot 10^{14}$	$8.14 \cdot 10^{15}$

Best known estimates

The best known (upper) formulas for the case of real and complex scalars, up to now, are:

(Bayart, D.P., Seoane, Advances in Mathematics 2014.

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{mult}} \leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}.$$

For real scalars and $m \geq 14$,

$$\mathrm{B}_{\mathbb{R},m}^{\mathrm{mult}} \leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}$$

and

$$\mathrm{B}_{\mathbb{R},m}^{\mathrm{mult}} \leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}.$$

for $2 \leq m \leq 13$.

Best known estimates

For instance, for real scalars,

$$\mathrm{B}_{\mathbb{R},10}^{\mathrm{mult}} \leq 2.6656$$

$$\mathrm{B}_{\mathbb{R},100}^{\mathrm{mult}} \leq 6.1493$$

Applications.....

.....In 2012, A. Montanaro, from the Centre for Quantum Information and Foundations, Cambridge University, has used our new estimates for the BH constants (case of real scalars) in Quantum Information Theory:

A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics, 2012.

Applications.....

.....In 2012, A. Montanaro, from the Centre for Quantum Information and Foundations, Cambridge University, has used our new estimates for the BH constants (case of real scalars) in Quantum Information Theory:

A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics, 2012.

In Montanaro´s terminology, our result is:

Applications.....

.....In 2012, A. Montanaro, from the Centre for Quantum Information and Foundations, Cambridge University, has used our new estimates for the BH constants (case of real scalars) in Quantum Information Theory:

A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics, 2012.

In Montanaro´s terminology, our result is:

Theorem. Let G be a k-player XOR game with n possible inputs per player. Then

$$\beta(G) = \Omega\left(k^{\frac{-3}{2}} n^{\frac{-(k-1)}{2}}\right).$$

Applications.....

.....In 2012, A. Montanaro, from the Centre for Quantum Information and Foundations, Cambridge University, has used our new estimates for the BH constants (case of real scalars) in Quantum Information Theory:

A. Montanaro, Some applications of hypercontractive inequalities in quantum information theory, J. Math. Physics, 2012.

In Montanaro´s terminology, our result is:

Theorem. Let G be a k-player XOR game with n possible inputs per player. Then

$$\beta(G) = \Omega\left(k^{\frac{-3}{2}} n^{\frac{-(k-1)}{2}}\right).$$

Please do not ask me what does it mean!

Lower estimates for BH multilinear constants: real case

Lower estimates for BH multilinear constants: real case

...if we look for lower estimates then, by finding adequate n-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

Lower estimates for BH multilinear constants: real case

...if we look for lower estimates then, by finding adequate n-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

$$
\begin{aligned}
\sqrt{2} \leq &\quad \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},2} &&\leq \sqrt{2}\\
1.587 \leq &\quad \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},3} &&\leq 1.6818\\
1.681 \leq &\quad \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},4} &&\leq 1.8877\\
1.741 \leq &\quad \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},5} &&\leq 2.0586\\
2^{1-\frac{1}{n}} \leq &\quad \mathrm{B}^{\mathrm{mult}}_{\mathbb{R},n} &&\leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}} < 1.3 m^{0.36482}.
\end{aligned}
$$

for $m \geq 14$. This last expression is dominated by (D. Diniz, G. Munoz, D.P, J. Seoane, Proc. Amer. Math. Soc., to appear)

How did we get these lower bounds?

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

Since the norm of T_2 is 2, from

$$\left(\sum_{i,j} \left| T_2 \left(e_i, e_j \right) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \left\| T_2 \right\|$$

How did we get these lower bounds?

Case $m = 2$:

Let

$$T_2 : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$$

be defined by

$$T_2(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

Since the norm of T_2 is 2, from

$$\left(\sum_{i,j} \left| T_2 \left(e_i, e_j \right) \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \left\| T_2 \right\|$$

we get

$$\mathrm{B}_{\mathbb{R},2}^{\mathrm{mult}} \geq 2^{1-\frac{1}{2}} = \sqrt{2}$$

Case $m = 3$:

Case $m = 3$:

Consider $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x, y, z) =$$

$$(z_1+z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right)+(z_1-z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right).$$

Case $m = 3$:

Consider $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x, y, z) =$$

$$(z_1+z_2)\,(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2)+(z_1-z_2)\,(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)\,.$$

Since $\| T_3\| = 4$ and

$$\left(\sum_{i,j,k} |\, T_3\,(e_i, e_j, e_k)|^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq \mathrm{B}_{\mathbb{R},3}^{\mathrm{mult}} \,\| T_3\|$$

Case $m = 3$:

Consider $T_3 : \ell_\infty^4 \times \ell_\infty^4 \times \ell_\infty^4 \to \mathbb{R}$ given by

$$T_3(x,y,z) =$$

$$(z_1+z_2)\left(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2\right)+(z_1-z_2)\left(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4\right).$$

Since $\|T_3\| = 4$ and

$$\left(\sum_{i,j,k} \left| T_3\left(e_i, e_j, e_k\right)\right|^{\frac{3}{2}}\right)^{\frac{2}{3}} \leq \mathrm{B}_{\mathbb{R},3}^{\mathrm{mult}} \|T_3\|$$

we get

$$\mathrm{B}_{\mathbb{R},3}^{\mathrm{mult}} \geq 2^{1-\frac{1}{3}}$$

General case

Using an induction argument, we obtain

$$\mathrm{B}_{\mathbb{R},n}^{\mathrm{mult}} \geq 2^{1-\frac{1}{n}}$$

General case

Using an induction argument, we obtain

$$\mathrm{B}_{\mathbb{R},n}^{\mathrm{mult}} \geq 2^{1-\frac{1}{n}}$$

This procedure is useless for the complex case....

The polynomial Bohnenblust–Hille inequality: estimates for the constants

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

Defant et al (Annals, 2011):

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

Defant et al (Annals, 2011): The polynomial BH inequality is hypercontractive.

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{m}(\sqrt{2})^{m-1}$$

The polynomial Bohnenblust–Hille inequality: estimates for the constants

Bohnenblust and Hille (Annals, 1931):

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{\frac{m}{2}}(m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

Defant et al (Annals, 2011): The polynomial BH inequality is hypercontractive.

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{m}(\sqrt{2})^{m-1}$$

Bayart, D.P and Seoane (Advances in Math 2014): For any $\varepsilon > 0$, there is a N such that, for any $m \geq N$,

$$\mathrm{B}_{\mathbb{C},m}^{\mathrm{pol}} \leq (1+\varepsilon)^m.$$

Application: the Bohr radius problem

Application: the Bohr radius problem

The Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z\in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in\mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

with

$$\mathbb{D}^n = \{(z_1, ..., z_n) : \max |z_j| < 1 \text{ for all } j\}.$$

Application: the Bohr radius problem

The Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z\in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in\mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

with

$$\mathbb{D}^n = \{(z_1, ..., z_n) : \max |z_j| < 1 \text{ for all } j\}.$$

The Bohr radius K_1 was studied and estimated by H. Bohr himself, and it was shown independently by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$. For $n \geq 2$, exact values of K_n are unknown.

Application: the Bohr radius problem

Application: the Bohr radius problem

Our subexponential estimate for the constants of the complex BH inequality was the key for the solution of the Bohr radius problem:

Application: the Bohr radius problem

Our subexponential estimate for the constants of the complex BH inequality was the key for the solution of the Bohr radius problem:

Theorem (Bayart, D.P, Seoane)

$$\lim_{m\to\infty} \frac{K_m}{\sqrt{\frac{\ln m}{m}}} = 1.$$

This finishes a problem that problem that numerous researchers have been chipping away at for more than fifteen years.

Back to real scalars

Next result shows that real scalars behaves differently from real scalars:

Back to real scalars

Next result shows that real scalars behaves differently from real scalars:

Theorem (Campos, Jimenez, Munoz, D.P and Seoane)

$$\mathrm{B}_{\mathbb{R},m}^{\mathrm{pol}} > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^m > (1.17)^m$$

for all positive integers $m > 1$.

Proof

Let m be an even integer. Consider the m-homogeneous polynomial

$$R_m(x_1,\ldots,x_m)=\left(x_1^2-x_2^2+x_1x_2\right)\left(x_3^2-x_4^2+x_3x_4\right)\cdots\left(x_{m-1}^2-x_m^2+x_{m-1}x_m\right).$$

Since $\|R_2\|=5/4$, it is simple to see that

$$\|R_m\|=(5/4)^{m/2}.$$

From the BH inequality for R_m we have

$$\left(\sum_{|\alpha|=m}|a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq D_{\mathbb{R},m}\,\|R_m\|,$$

that is,

$$D_{\mathbb{R},m}\geq\frac{\left(3^{\frac{m}{2}}\right)^{\frac{m+1}{2m}}}{\left(\frac{5}{4}\right)^{\frac{m}{2}}}\geq\frac{\left(\sqrt{3}\right)^{\frac{m+1}{2}}}{\left(\frac{5}{4}\right)^{\frac{m}{2}}}>\left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^{m}.$$

Proof

The case m is odd is similar. Keeping the previous notation, consider the m homogeneous polynomial

$$R_m(x_1, ..., x_{2m})$$

$$= (x_{2m} + x_{2m-1}) R_{m-1}(x_1, ..., x_{m-1}) + (x_{2m} - x_{2m-1}) R_{m-1}(x_m, ..., x_{2m-2})$$

and we get the same estimate.

The case of real scalars

In fact we have

Theorem (Campos, Jimenez, Munoz, D.P, Seoane)

$$\limsup_m \left(B_{\mathbb{R},m}^{pol}\right)^{1/m} = 2.$$

Proof

If $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ is an m-homogeneous polynomial, by a result of Visser if we consider the same polynomial $P_{\mathbb{C}} : l_\infty(\mathbb{C}) \to \mathbb{C}$ we have

Proof

If $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ is an m-homogeneous polynomial, by a result of Visser if we consider the same polynomial $P_\mathbb{C} : l_\infty(\mathbb{C}) \to \mathbb{C}$ we have

$$\|P_\mathbb{C}\| \leq 2^{m-1} \|P\| .$$

Proof

If $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ is an m-homogeneous polynomial, by a result of Visser if we consider the same polynomial $P_\mathbb{C} : l_\infty(\mathbb{C}) \to \mathbb{C}$ we have

$$\|P_\mathbb{C}\| \leq 2^{m-1} \|P\| .$$

So, for a real polynomial $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ given by $P = \sum_{|\alpha|=m} a_\alpha z^\alpha$, we consider $P_\mathbb{C}$ and we easily get from our estimate for complex scalars (and big m),

$$\begin{aligned} \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} &\leq (1+\varepsilon)^m \|P_\mathbb{C}\| \\ &\leq (1+\varepsilon)^m 2^{m-1} \|P\| \\ &\leq (2+\delta)^m \|P\| \end{aligned}$$

and we conclude that

$$\limsup_m \left(B_m^{pol}\right)^{1/m} \leq 2.$$

Proof

The other inequality is a little bit more technical.

References

This talk contains results from the following papers:

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, Journal of Functional Analysis (**2012**)

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, Journal of Functional Analysis (**2012**)
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., in press.

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, Journal of Functional Analysis (**2012**)
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., in press.
- D. Nuñez, D. Pellegrino and J.B. Seoane, On the Bohnenblust-Hille inequality and a variant to Littlewood's 4/3 inequality, J. Functional Analysis (**2013**).

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, Journal of Functional Analysis (**2012**)
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., in press.
- D. Nuñez, D. Pellegrino and J.B. Seoane, On the Bohnenblust-Hille inequality and a variant to Littlewood's 4/3 inequality, J. Functional Analysis (**2013**).
- D. Nuñez, D. Pellegrino and J.B. Seoane, D. M. Serrano-Rodriguez, There exist multilinear Bohnenblust-Hille constants $(C_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty}(C_{n+1} - C_n) = 0$, J. Functional Analysis (**2013**).

References

This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$, Advances in Math 2014.
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, The asymptotic growth of the constants in the Bohnenblust–Hille inequality is optimal, Journal of Functional Analysis (**2012**)
- D. Diniz, G.A. Muñoz, D.P. and J.B. Seoane, Lower bounds for the constants in the Bohnenblust–Hille inequality: the case of real scalars, Proc. Amer. Math. Soc., in press.
- D. Nuñez, D. Pellegrino and J.B. Seoane, On the Bohnenblust-Hille inequality and a variant to Littlewood's 4/3 inequality, J. Functional Analysis (**2013**).
- D. Nuñez, D. Pellegrino and J.B. Seoane, D. M. Serrano-Rodriguez, There exist multilinear Bohnenblust-Hille constants $(C_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty}(C_{n+1} - C_n) = 0$, J. Functional Analysis (**2013**).
- N. Albuquerque, F. Bayart, D. Pellegrino and J.B. Seoane, Sharp generalizations of the multilinear Bohnenblust–Hille inequality, J. Functional Analysis, in press.

References

...and preprints:

J.R. Campos, D. Nunez-Alarcon, D. Pellegrino, J.B. Seoane-Sepulveda and D. M. Serrano-Rodriguez, The best known upper bounds for the real BH inequality are not optimal, preprint.