On spectra of measures

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Definitions

$M(\mathbb{T})$ – the Banach algebra of complex-valued, Borel regular measures on the circle group with the usual convolution as multiplication.
For $\mu \in M(\mathbb{T})$ we define the spectrum $\sigma(\mu)$ of μ as the set

$$\sigma(\mu) = \{\lambda \in \mathbb{C} : \mu - \lambda\delta_0 \text{ is not invertible}\}.$$

It follows from the general theory of commutative Banach algebras that the spectrum of a measure is an image of its Gelfand transform and it is non-empty compact subset of the complex plane. The alternative definition of the spectrum goes as follows: with every $\mu \in M(\mathbb{T})$ we can associate an operator $T_\mu : L^1(\mathbb{T}) \mapsto L^1(\mathbb{T})$ by the formula $T_\mu(f) = \mu * f$ and then $\sigma(\mu) = \sigma(T_\mu)$.
For $\mu \in M(\mathbb{T})$ we also define the n-th Fourier-Stieltjes coefficient:

$$\widehat{\mu}(n) = \int_{\mathbb{T}} e^{-int} d\mu(t).$$

One can also easily check that $\widehat{\mu}(\mathbb{Z}) = \sigma_p(T_\mu)$.

Formulation of the problem

Main problem

It is obvious that for every $\mu \in M(\mathbb{T})$ we have $\overline{\widehat{\mu}(\mathbb{Z})} \subset \sigma(\mu)$. But when do we have $\overline{\widehat{\mu}(\mathbb{Z})} = \sigma(\mu)$?

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Natural spectrum

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Examples

Absolutely continuous and purely discrete measures have a natural spectrum.

The Wiener-Pitt phenomenon

There exists a measure $\mu \in M(\mathbb{T})$ for which $\overline{\widehat{\mu}(\mathbb{Z})} \neq \sigma(\mu)$.

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The Proof of the existence of the Wiener-Pitt phenomenon by C. C. Graham

Consider the usual Riesz product:

$$\mu = \prod_{k=1}^{\infty} (1 + \cos(3^k t)) \text{ understood as a weak}^* \text{ limit of finite products.}$$

Then

$$\overline{\widehat{\mu}(\mathbb{Z})} = \{0\} \cup \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty} \cup \{1\}.$$

The proof continued

The formula for the Fourier-Stieltjes coefficients for the Riesz product is

$$\widehat{\mu}\left(\sum_{k=1}^{n} \varepsilon_k 3^k\right) = \prod_{k=1}^{n} \left(\frac{1}{2}\right)^{|\varepsilon_k|} \text{ where } \varepsilon_k \in \{-1, 0, 1\} \text{ and}$$

$\widehat{\mu}(m) = 0$ if m is not expressible in the above form.
Now, let us take two disjoint open sets $A, B \subset \mathbb{C}$ such that $\overline{\widehat{\mu}(\mathbb{Z})} \subset A \cup B$ and $\frac{1}{2} \in B$. Assume that $\sigma(\mu) = \overline{\widehat{\mu}(\mathbb{Z})}$, then the function f defined on $A \cup B$ by the formula $f = 1$ on B and $f = 0$ on A is holomorphic on $A \cup B$ and hence we can apply functional calculus obtaining a measure $\nu := f(\mu)$. By the properties of the functional calculus we have

$$\widehat{\nu}(\pm 3^k) = 1 \text{ and } \widehat{\nu}(n) = 0 \text{ otherwise .}$$

This result contradicts the classical Helson's theorem (finitely-valued Fourier-Stieltjes sequence has to be periodic with only finitely many exceptions).

Since the set $\widehat{\mu}(\mathbb{Z})$ is the only part of the spectrum of μ which is easily computable, we asked if it is possible to decide whether a measure $\mu \in M(\mathbb{T})$ has a natural spectrum examining the set $\widehat{\mu}(\mathbb{Z})$.

The notion of Wiener-Pitt sets

We say that a compact set $A \subset \mathbb{C}$ is a Wiener-Pitt set, if every measure $\mu \in M(\mathbb{T})$ such that $\widehat{\mu}(\mathbb{Z}) \subset A$ has a natural spectrum.

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Example

Every finite set is a Wiener-Pitt set.
Let $A = \{a_1, \ldots, a_n\}$ and assume that $\mu \in M(\mathbb{T})$ is such that $\widehat{\mu}(\mathbb{Z}) = \{a_1, \ldots, a_k\}$, $k \leq n$. Then

$$(\mu - a_1\delta_0) * (\mu - a_2\delta_0) * \ldots * (\mu - a_k\delta_0) = 0 \text{ so}$$

$$(\varphi(\mu) - a_1) \cdot (\varphi(\mu) - a_2) \cdot \ldots \cdot (\varphi(\mu) - a_k) = 0 \text{ for every } \varphi \in \triangle(M(\mathbb{T})).$$

Since the spectrum of a measure is the image of its Gelfand transform the proof is finished.

Main theorems

Our theorems went in direction of constructing infinite Wiener-Pitt sets.

The case of continuous measures

Theorem (Ohrysko, Wojciechowski)

There exists an open set $U \subset \mathbb{C}$ with $0 \in \overline{U}$ such that every continuous measure μ with $\widehat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$ has a natural spectrum.

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The general case

Theorem (Ohrysko, Wojciechowski)

There exists a set K homeomorphic to the Cantor set such that $0 \in K$ and every measure μ with $\widehat{\mu}(\mathbb{Z}) \subset K$ has a natural spectrum.

In fact, the set U from Theorem 1 has the following form

$$U = \bigcup_{n\in\mathbb{N}} (s_n \cdot A + B(0, r_n)) \text{ where } A = \{-1, 1\}$$

where for $n = \sum_i a_i 2^i$, $a_i \in \{0, 1\}$ we put $s_{n+1} = \prod_i \varepsilon_i^{a_i}$.
($\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ are suitably chosen sequences).
Moreover, we constructed a singular measure with Fourier-Stieltjes coefficients included in this set. It is an instance of generalized Riesz products and the formula is as follows:

$$\mu = \prod_{k=1}^{\infty} (1 - w_k) \text{ where } w_k = \varepsilon_k P_{n_k}(m_k t) e^{i r_k t} + \varepsilon_k \overline{P_{n_k}(m_k t)} e^{-i r_k t}$$

and the polynomials P_n are Rudin-Shapiro polynomials.

Ingredients of the proof I

Theorem (Bożejko, Pełczyński; Bourgain)

Let $\Lambda \subset \mathbb{N}$ be a finite set with $\#\Lambda = k$. Then for every $\varepsilon > 0$ there exists $f \in \mathscr{P}$ such that

1. $\widehat{f}(n) = 1$ *for* $n \in \Lambda$.
2. $||f||_{L^1(\mathbb{T})} \leq 1 + \varepsilon$
3. $\#\{n \in \mathbb{N} : \widehat{f}(n) \neq 0\} \leq \left(\frac{\alpha}{\varepsilon}\right)^{2k}$ *for some* $\alpha > 0$.

Ingredients of the proof II

Theorem (McGehee, Pigno, Smith; Konyagin)

For every $f \in \mathscr{P}$ of the form

$$f(t) = \sum_{k=1}^{N} c_k e^{in_k t},$$

where n_k is the sequence of increasing integers and $|c_k| \geq 1$, $1 \leq k \leq N$, we have

$$||f||_{L^1(\mathbb{T})} > L \ln N,$$

where the constant $L > 0$ does not depend on N.

Ingredients of the proof III

Definition

Let $\mathscr{C}$ denote the set of measures with a natural spectrum with Fourier-Stieltjes coefficients from c_0, i.e.

$$\mathscr{C} = \{\mu \in M_0(\mathbb{T}) : \sigma(\mu) = \overline{\widehat{\mu}(\mathbb{Z})} = \widehat{\mu}(\mathbb{Z}) \cup \{0\}\}.$$

Theorem (Zafran)

The following hold true:

1. *if* $h \in \mathfrak{M}(M_0(\mathbb{T})) \setminus \mathbb{Z}$, *then* $h(\mu) = 0$ *for* $\mu \in \mathscr{C}$.
2. $\mathscr{C}$ *is a closed ideal in* $M_0(\mathbb{T})$.
3. $\triangle(\mathscr{C}) = \mathbb{Z}$.

Open problems

Assume that $\mu \in M(\mathbb{T})$ has a natural spectrum. Does it follow that $\mu * \delta_\tau$ $(\tau \in \mathbb{T})$ have a natural spectrum?
Does there exist an infinite Wiener-Pitt set for every locally compact abelian group?
For which sequences $\{a_n\}_{n=1}^{\infty}$ converging to zero any measure $\mu \in M_0(\mathbb{T})$ with $\widehat{\mu}(\mathbb{Z}) \subset \{a_n\}_{n=1}^{\infty} \cup \{0\}$ has a natural spectrum?
For which non-commutative (infinite) discrete groups the Wiener-Pitt phenomenon occurs?

Properties of the Gelfand space

1. The cardinality of $\triangle(M(\mathbb{T}))$ is $2^{\mathfrak{c}}$.
2. It is not a separable topological space.
3. It contains uncountably many pairwise disjoint copies of $\beta\mathbb{Z}$.
4. It has uncountable first Čech cohomology group with integral coefficients.
5. 'Measures glue ultrafilters'.

These results can be found on arxiv: "On the Gelfand space of the measure algebra on the circle group" (preprint).

Inhabitants of $\triangle(M(\mathbb{T}))$

First part

Elements of $\mathbb{Z}$ are 

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Second part

Elements of $\overline{\mathbb{Z}} \setminus \mathbb{Z}$ are 

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Elements of $\mathbb{Z}$ are 

Second part

Elements of $\overline{\mathbb{Z}} \setminus \mathbb{Z}$ are 

Third part

Elements of $\triangle(M(\mathbb{T})) \setminus \overline{\mathbb{Z}}$ are 

Thank you for your attention!