Around Approximate Fixed Point Property (AFPP)

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Motivations

Hausdorff, 1935

Let C be a nonempty compact convex subset of a locally convex space (LCTVS) X and let $f : C \longrightarrow C$ be a continuous map. Then f has a fixed point in C.

Generalising Brouwer fixed point theorem ($X = \mathbb{R}^N$) and Schauder fixed point theorem (X=Banach space).
Now true for every topological vector space (Cauty, 2010)

Question

What is the situation if we need a common fixed point theorem for more than one function?

Answer

Boyce, 1969 and Huneke, 1969

There exist continuous functions f and g which map the unit interval $[0,1]$ onto itself and commute under functional composition but have no common fixed point. i.e no point $x \in [0,1]$ such that $f(x) = x = g(x)$.

Corollary

The Schauder fixed point theorem can't not be extend for more than one function.

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Note

If a common fixed point theorem were to hold, there should be further restrictions beyond commutativity of the family of maps.

Definition

1. A flow is a pair (G, X) where G is a topological group acting continuously on X.
2. A compact flow is a flow (G, K) where K is a compact.
3. An affine flow is a flow (G, Q) where Q is a convex subset of a LCTVS E and for each $g \in G$ the map $Q \ni x \longmapsto g.x \in Q$ is affine.
4. The flow (G, Q) is distal if $\lim_\alpha s_\alpha.x = \lim_\alpha s_\alpha.y$ for some net s_α in G, then $x = y$.
5. The flow (G, Q) is equicontinuous if for each neighborhood U of 0, there is neighborhood V of 0 such that $x - y \in V$ imply $s.x - s.y \in U$ for each $s \in S$.

Reformulation of the question

Under what conditions does an compact affine flow (G,Q) admit a common F.P?

Amenability

Definition

A topological group G has the **Fixed Point Property (FPP)** if every affine compact flow (G, X) has a common fixed point $x \in X$ i.e $g.x = x$ for each $g \in G$.

Definition

A topological group G is **amenable** if it admit an invariant mean on $RUCB(G)$.

Where:
$RUCB(G)$=Right Uniformly Continuous Bounded functions $f : G \longrightarrow \mathbb{C}$.

Some amenable groups

- Finite groups
- Abelian groups
- Nilpotent group
- Solvable group
- Compact groups

Warning

Warning

Many authors use the phrase amenable group to mean a group which is amenable in its discrete topology. The danger of this is that many theorems concerning amenable discrete groups do not generalize in the ways one might expect.

Some Polish amenable groups

- $Aut\,(\mathbb{Q}, \leq)$ the group of all order-preserving bijections of $\mathbb{Q}$, with the topology of simple convergence.
- The unitary group $\mathcal{U}(\ell^2)$, equipped with strong operator topology.
- The infinite symetric group S_∞, with the topology of simple convergence.
- The group $\mathcal{J}(k)$ of all formal power series in a variable x that have the form $f(x) = x + \alpha_1 x^2 + \alpha_2 x^3 + ...,\ \ \alpha_n \in k$. Where k is a commutative unital ring.

Some non-amenable groups

- The free group $\mathbb{F}_2$ of two generators with discrete topology.
- The group $Aut\,(X,\mu)$ of all measure-preserving automorphisms of a standard Borel measure space (X,μ), equipped with the uniform topolgy $(d(\tau,\sigma)=\mu\{x\in X:\ \tau(x)\neq\sigma(x)\})$ is non-amenable. (Giordano and Pestov 2002)

Some answers

Let (G, Q) be a compact affine flow. Then G admits a common F.P. in Q in the following case:.

1. G is abelian (Markov and Kakutani,)
2. G is amenable (Day,)
3. The flow (G, Q) is distal (Hahn).
4. The flow (G, Q) is equicontinuous (Kakutani)
5. There is a nonempty compact G-invariant subset K such that (G, K) is distal. (Furstenberg).

In the same spirit

Folklore

If K is a nonempty compact convex subset of a Banach space, then every nonexpansive map of K into K has a fixed point.

Note: Another history if replace compact by closed bounded.

De Marr, 1963

Let B be a Banach space and let K be a nonempty compact convex subset of B. If $\mathfrak{F}$ is a nonempty commutative family of contraction mappings of K into itself, then $\mathfrak{F}$ has a common fixed point in K.

W. Takahashi, 1969

Let B be a Banach space and let K be a nonempty compact convex subset of B and If S is an amenable semigroup of nonexpansive mapping of K into K, then it has a common fixed point in K.

Note: In this case there is no need to further restrictions contrary to the Schauder case.

Approximate fixed point

Another important and current branch of fixed point theory is the study of the approximate fixed point sequence.

Definition

Let C be a nonempty convex subset of a topological vector space X. An approximate fixed point sequence for a map $f : C \longrightarrow C$ is a sequence (x_n) in C so that $x_n - f(x_n) \longrightarrow 0$.

Definition

Let X be a Banach space. A Nonempty, Bounded, Closed, Convex (NBCC) set $C \subseteq X$ is said to has the weak-AFPP if for any continuous map $f : C \longrightarrow C$ there is a sequence(u_k) in C so that $u_k - f(u_k) \longrightarrow 0$ weakly.

Barroso, Kalenda and Rebouças, 2013

Let X be a topological vector space, $C \subset X$ a nonempty bounded convex set, and let $f : C \longrightarrow C$ an affine selfmap, then the mapping f has an approximate fixed point sequence.

Kalenda, 2011

X has the weak AFPP iff $\ell_1 \not\subseteq X$

Question

Under what conditions does an bounded affine flow (G, Q) admit a common approximate fixed point sequence?

Definition

A topological group G has the **Approximate Fixed Point Property (AFPP)** if every bounded affine flow (G, Q) admit an approximate fixed point sequence That is a sequence $(x_n) \subseteq Q$ which is approximative fixed for every translation $\tau_\gamma : Q \ni x \longmapsto \gamma x \in Q$.

Theorem

The following conditions are equivalents for a discrete group or a locally compact group G:

1. *G is amenable*
2. *G has the AFPP*

Idea of the proof: Discrete case

Følner

A discrete group G is amenable if and only if it satisfies the *Følner's condition*: For every finite $F \subseteq G$ and $\varepsilon > 0$, there is a finite set $\Phi \subseteq G$ such that for each $g \in F$,

$$|g\Phi \triangle \Phi| < \varepsilon |\Phi|$$

Proof.

1. By Følner condition, construct a Følner net: that is a net of non-empty finite subsets $(\Phi_i)_{i \in I} \subset G$ such that

$$\frac{|\gamma\Phi_i \triangle \Phi_i|}{|\Phi_i|} \longrightarrow 0 \quad \forall \gamma \in G$$

2. Fix some $x \in Q$ and define $x_i = \dfrac{1}{|\Phi_i|} \sum_{g \in \Phi_i} gx$

□

General case

Theorem

The infinite symetric group S_∞ equipped with is natural polish topology does not have the AFPP.

Idea of the proof

1. Take $E = \ell^1(\mathbb{N})$ and $Q = prob(\mathbb{N})$ the subset consist of all Borel probability measures on $\mathbb{N}$.
2. If the natural action of S_∞ on Q have an approximate fixed point sequence, then the free group $\mathbb{F}_2$ is amenable.

Thank to

Reiter's condition

Let p be any real number such that $1 \leq p \leq \infty$. A locally compact group G is amenable iff For any compact set $C \subseteq G$ and $\varepsilon > 0$, There exists $f \in \{h \in L^p(G) : \ h \geq 0, \ \|h\|_p = 1\}$ such that: $\|g.f - f\| < \varepsilon$ for all $g \in C$.

Reflexive case

Theorem

The following propositions are equivalents for a Polish group G:

1. *G is amenable*
2. *G have the AFPP for every bounded convex subset of a reflexive locally convex space.*

Thank to:

M. Megrelishvili

Let $(V, \|.\|)$ be an Asplund Banach space and let $\pi : G \times V \longrightarrow V$ be a continuous linear action of a topological group G on V, then the dual action $\pi^\star$ is continuous.

Future work

1. Try to link distality or equicontinuity of the flow with the AFPP
2. Do the same for others fixed point theorems

N.P. Brown and N. Ozawa, 2008

Amenability of a group admits the largest known number of equivalent definitions: $10^{10^{10}}$.

Up to date this number is $10^{10^{10}} + N$ where $N \geq$ the number of criteria obtain in this talk.