Maximal left ideals of operators acting on a Banach space

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Let $E = X_{\mathrm{AH}} \oplus \ell_\infty$. We identify operators T on E with (2×2)-matrices

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Theorem (Argyros–Haydon 2011). *There is a Banach space* X_{AH} *such that:*

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A separable example

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Open problems

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Key references

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