Products of free spaces and applications

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Spaces of Lipschitz functions

Let (M, d) be a metric space, $0 \in M$.

Notation

$$Lip_0(M) := \{f : M \to \mathbb{R} | f \text{ is Lipschitz, } f(0) = 0\}$$

is a Banach space when equipped with the norm

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

A predual for $Lip_0(M)$

For each $x \in M$, consider the evaluation functional $\delta_x \in Lip_0(M)^*$ por $\delta_x f := f(x)$.

Definition/Proposition

$$\mathcal{F}(M) := \overline{span}\{\delta_x | x \in M\}$$

is the *free space over* M, and it is an isometric predual to $Lip_0(M)$.

- **Geometric interpretation:** μ, ν finitely supported probabilities $\Rightarrow \|\mu - \nu\|_{\mathcal{F}}$ is the *earthmover distance* between μ and ν.

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Relationship between M and $\mathcal{F}(M)$

Linear interpretation property

$\forall\ L : M \to N$ Lipschitz with $L(0_M) = 0_N$ $\exists!$ $\hat{L} : \mathcal{F}(M) \to \mathcal{F}(N)$ *linear* such that te following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad L \quad} & N \\ \Big\downarrow \delta^M & & \Big\downarrow \delta^N \\ \mathcal{F}(M) & \xrightarrow{\quad \hat{L} \quad} & \mathcal{F}(N) \end{array}$$

• In particular, $M \overset{L}{\sim} N \Rightarrow \mathcal{F}(M) \simeq \mathcal{F}(N)$. The converse does not hold in general.

• (Godefroy, Kalton 2003) If X is Banach and $\lambda \geq 1$, X is λ-BAP $\Leftrightarrow \mathcal{F}(X)$ is λ-BAP.

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The structure of the free spaces is still a big mystery

• (Godard 2010) $\mathcal{F}(M)$ is isometric to a subspace of $L_1 \Leftrightarrow M$ is a subset of an $\mathbb{R}$-tree;

• (Naor, Schechtman 2007) $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1;

• Let $(X, \|\cdot\|)$ be a finite dimensional Banach space. Then (Godefroy, Kalton 2003) $\mathcal{F}(X)$ has MAP and (Hájek, Pernecká 2013) $\mathcal{F}(X)$ admits a Schauder basis;

• **Problem:** (posed by Hájek, Pernecká) $F \subset \mathbb{R}^n \Rightarrow \mathcal{F}(F)$ has Schauder basis?

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Products of free spaces

Main Result

Let X be a Banach space. Then $\mathcal{F}(X) \simeq \left(\sum_{n=1}^{\infty} \mathcal{F}(X)\right)_{\ell_1}$.

Recall: Let M be a metric space, $N \subset M$. N is a *Lipschitz retract* of M if there is a Lipschitz function $L : M \to N$ (called *Lipschitz retraction*) such that $L|_N = Id$. M is an *absolute Lipschitz retract* if it is a Lipschitz retract of any metric space containing it.

Consequence 1: nonlinear Pełczyński's method for free spaces

Let X be a Banach space and M be a metric space, and suppose that X and M admit Lipschitz retracts N_1 and N_2, respectively, such that X is Lipschitz equivalent to N_2 and M is Lipschitz equivalent to N_1. Then $\mathcal{F}(X) \simeq \mathcal{F}(M)$.

Proof: Linear interpretation property + Main Result + classic Pełczyński's method applied to the free spaces.

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Consequence 2: free space of balls

Let X be a Banach space. Then $\mathcal{F}(B_X) \simeq \mathcal{F}(X)$.

Proof: an adaptation of the proof of the main result.

Consequence 3: about $\mathcal{F}(c_0)$

Let M be a separable metric space which is an absolute Lipschitz retract and $F \subset M$ a Lipschitz retract of M such that $B_{c_0} \overset{L}{\sim} F$. Then $\mathcal{F}(M) \simeq \mathcal{F}(c_0)$. In particular, if K is an infinite compact metric space, then $\mathcal{F}(C(K)) \simeq \mathcal{F}(c_0)$.

Proof: all separable metric spaces are Lipschitz equivalent to subsets of c_0 (Aharoni 1974) + linear interpretation property + main result + classic Pełczyński's method.

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Ingredient 1 to prove that $\mathcal{F}(X) \simeq \left(\sum_{n=1}^{\infty} \mathcal{F}(X)\right)_{\ell_1}$: linear extensions of Lipschitz functions

Definition

Given a pointed metric space $(M, d, 0)$ and a subset F containing 0, let us denote by $Ext_0(F, M)$ the set of all extensions $E : Lip_0(F) \to Lip_0(M)$ which are **linear** and continuous. Let $Ext_0^{pt}(F, M)$ be the subset of $Ext_0(F, M)$ consisting of all pointwise-to-pointwise continuous elements.

• (Brudnyi, Brudnyi 2007) There exists a 2-dimensional Riemannian manifold M and a subset F such that $Ext_0(F, M) = \emptyset$.
• (Banach space example) Let $X \subset c_0$ be a subspace failing AP. Then $\mathcal{F}(X)$ is not complemented in $\mathcal{F}(c_0)$, thus $Ext_0^{pt}(X, c_0) = \emptyset$.
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Ingredient 2: Metric quotients and a decomposition result

Definition: metric quotient

Let (M, d) be a metric space, $F \subset M$ be closed and nonempty, and let $\sim_F$ the equivalence relation on M which identifies all elements of F. Then

$$\tilde{d}(\tilde{x}, \tilde{y}) := \min\{d(x, y), d(x, F) + d(y, F)\}, \tilde{x}, \tilde{y} \in M/\sim_F$$

is a distance on $M/\sim_F$, and $(M/\sim_F, \tilde{d})$ is called the quotient metric space of M by $\sim_F$, which we denote by M/F.

- $Lip_0(M/F) \cong \{f \in Lip_0(M) : f|_F = constant\}$.

Lemma (quotient decomposition)

Let $(M, d, 0)$ be a pointed metric space and F be a subset containing 0, and suppose that there exists $E \in Ext_0^{pt}(F, M)$. Then $\mathcal{F}(M) \simeq \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F)$.

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Ingredient 3: Kalton's approximation results

Let $(M, d, 0)$ be a pointed metric space, and denote $B_r := B_r(0)$.

K1

Given $r_1, \dots, r_n, s_1, \dots, s_n \in \mathbb{Z}, r_1 < s_1 < r_2 < \dots < s_n$ and $\gamma_k \in \mathcal{F}(B_{2^{s_k}} \setminus B_{2^{r_k}})$ and writing $\theta := \min_{k=1,\dots,n-1}\{r_{k+1} - s_k\}$, then

$$\|\gamma_1 + \dots + \gamma_n\|_{\mathcal{F}} \geq \frac{2^\theta - 1}{2^\theta + 1} \sum_{k=1}^{n} \|\gamma_k\|_{\mathcal{F}}.$$

Ingredient 3: Kalton's approximation results

K2

Consider, for each $k \in \mathbb{Z}$, the linear operator $T_k : \mathcal{F}(M) \to \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})$ defined by

$$T_k\delta_x := \begin{cases} 0, & \text{if } x \in B_{2^{k-1}}; \\ (\log_2 d(x,0) - k + 1)\delta_x, & \text{if } x \in B_{2^k} \setminus B_{2^{k-1}}; \\ (k + 1 - \log_2 d(x,0))\delta_x, & \text{if } x \in B_{2^{k+1}} \setminus B_{2^k}; \\ 0, & \text{if } x \notin B_{2^{k+1}}. \end{cases}$$

Then, for each $\gamma \in \mathcal{F}(M)$, we have that $\gamma = \sum_{k\in\mathbb{Z}} T_k\gamma$ unconditionally and

$$\sum_{k\in\mathbb{Z}} \|T_k\gamma\|_{\mathcal{F}} \leq 72\|\gamma\|_{\mathcal{F}}.$$

Proof that X Banach $\Rightarrow \mathcal{F}(X) \simeq \left(\sum_{n=1}^{\infty} \mathcal{F}(X)\right)_{\ell_1}$

- First, note that, for each $k \in \mathbb{Z}$,

$\mathcal{F}(B_{2^{k+1}} \setminus B_{2^k}) \cong \mathcal{F}(B_2 \setminus B_1) \simeq \mathcal{F}(B_4 \setminus B_1) \cong \mathcal{F}(B_{2^{k+1}} \setminus B_{2^{k-1}})$.

Strategy: Show that $\mathcal{F}(X) \overset{c}{\hookrightarrow} \left(\sum_{n=1}^{\infty} \mathcal{F}(B_2 \setminus B_1)\right)_{\ell_1}$ and that $\mathcal{F}(X) \overset{c}{\hookleftarrow} \left(\sum_{n=1}^{\infty} \mathcal{F}(B_2 \setminus B_1)\right)_{\ell_1}$, then apply Pełczyński's method.

- ($\overset{c}{\hookrightarrow}$) Define T and S as follows:

$$\begin{array}{ccccc} \mathcal{F}(X) & \overset{T}{\hookrightarrow} & \left(\sum_{n=1}^{\infty} \mathcal{F}(B_{2^{k+1}}) \setminus \mathcal{F}(B_{2^{k-1}})\right)_{\ell_1} & \overset{S}{\twoheadrightarrow} & \mathcal{F}(X) \\ & & (\gamma_k) & \mapsto & \sum_{k \in \mathbb{Z}} \gamma_k \\ \gamma & \mapsto & (T_k \gamma) & & \end{array}$$

Then $T \circ S$ is a projection onto $T(\mathcal{F}(X)) \simeq \mathcal{F}(X)$.

- ($\overset{c}{\hookleftarrow}$) The speaker will explain. $\square$

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Strategy: Show that $\mathcal{F}(X) \overset{c}{\hookrightarrow} \left(\sum_{n=1}^{\infty} \mathcal{F}(B_2 \setminus B_1)\right)_{\ell_1}$ and that $\mathcal{F}(X) \overset{c}{\hookleftarrow} \left(\sum_{n=1}^{\infty} \mathcal{F}(B_2 \setminus B_1)\right)_{\ell_1}$, then apply Pełczyński's method.

•$(\overset{c}{\hookrightarrow})$ Define T and S as follows:

$$\begin{array}{ccccc} \mathcal{F}(X) & \overset{T}{\hookrightarrow} & \left(\sum_{n=1}^{\infty} \mathcal{F}(B_{2^{k+1}}) \setminus \mathcal{F}(B_{2^{k-1}})\right)_{\ell_1} & \overset{S}{\twoheadrightarrow} & \mathcal{F}(X) \\ & & (\gamma_k) & \mapsto & \sum_{k \in \mathbb{Z}} \gamma_k \\ \gamma & \mapsto & (T_k \gamma) & & \end{array}$$

Then $T \circ S$ is a projection onto $T(\mathcal{F}(X)) \simeq \mathcal{F}(X)$.

•$(\overset{c}{\hookleftarrow})$ The speaker will explain. □

An application

Theorem (free spaces over compact riemannian manifolds)

Let M be a compact metric space such that each $x \in M$ admits a neighborhood which is bi-Lipschitz embeddable in $\mathbb{R}^n$. Then there is a complemented copy of $\mathcal{F}(M)$ in $\mathcal{F}(\mathbb{R}^n)$.
If moreover the unit ball of $\mathbb{R}^n$ is bi-Lipschitz equivalent to a Lipschitz retract of M, then $\mathcal{F}(M) \simeq \mathcal{F}(\mathbb{R}^n)$. In particular, the Lipschitz-free space over any n-dimensional compact Riemannian manifold equipped with its geodesic metric is isomorphic to $\mathcal{F}(\mathbb{R}^n)$.

For the proof we make use of the following:

Lang, Plaut 2001 (bi-Lipchitz embeddability into $\mathbb{R}^n$)

Let M be a compact metric space such that each point of M admits a neighborhood which is bi-Lipschitz embeddable in $\mathbb{R}^n$. Then M is bi-Lipschitz embeddable in $\mathbb{R}^n$.

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Muito obrigado!