Twisted sums of Banach spaces generated by complex interpolation

Manuel González

Departamento de Matemáticas
Universidad de Cantabria
Santander, Spain

First Brazilian Workshop in Geometry of Banach Spaces
Maresias, São Paulo, 25-29 August 2014

Joint work with Jesús M. F. Castillo and Valentin Ferenczi.

Exact sequences of Banach spaces

Let Y and Z be Banach spaces. An exact sequence is a sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0,$$

j, q continuous operators, $\mathrm{Ker}\, j = \{0\}$, $\mathrm{Ran}\, j = \mathrm{Ker}\, q$, and $\mathrm{Ran}\, q = Z$.

- $j(Y)$ is a closed subspace of X and $X/j(Y) \equiv Z$.
- X is a F-space; sometimes not (equivalent to) a Banach space.

The exact sequence is trivial if $\mathrm{Ran} j$ is complemented: $X \equiv Y \times Z$.

A twisted sum of Y and Z is a non-trivial exact sequence.
(i.e., the space X in a twisted sum of Y and Z).

A twisted sum of Y and Z is called singular if q is strictly singular.
($q|_M$ is an isomorphism for no inf. dim. subspace M of X).

Construction of twisted sums

Let Y and Z be (infinite dimensional) Banach spaces.

A quasi-linear map from Z to Y is a map $F : Z \to Y_0, \quad Y_0 \supset Y$, with $F(\lambda u) = \lambda F(u),\ F(u+v) - F(u) - F(v) \in Y\ \ \forall \lambda \in \mathbb{K}; u, v \in Z$, and $\|F(u+v) - F(u) - F(v)\|_Y \leq M\|u+v\|_Z$ for some $M > 0$.

A quasi-linear map $F : Z \to Y$ induces an exact sequence

$$0 \to Y \xrightarrow{j} Y \oplus_F Z \xrightarrow{q} Z \to 0,$$

where $\quad Y \oplus_F Z := \{(y, z) \in Y_0 \times Z : y - F(z) \in Y\} \quad$ endowed with the quasi-norm $\quad \|(y, z)\|_F = \|y - F(z)\|_Y + \|z\|_Z$.

The embedding is $j(y) = (y, 0)$ while the quotient map is $q(y, z) = z$.

Each exact sequence can be obtained by means of a quasi-linear map. F is called singular if q is strictly singular (i.e., $Y \oplus_F Z$ singular).

We can identify exact sequences and quasi-linear maps.

An example: the Kalton-Peck map $\mathcal{K}(\cdot)$

$$\mathcal{K} : x = (x_n) \in \ell_2 \to \left(-x_n \log \frac{|x_n|}{\|x\|_2}\right) \in S \quad \text{for } x \neq 0, \quad \text{and } \mathcal{K}(0) = 0.$$

is a quasi-linear map from ℓ_2 to ℓ_2, with S the space of all sequences.

$$0 \to \ell_2 \xrightarrow{j} Z_2 := \ell_2 \oplus_{\mathcal{K}} \ell_2 \xrightarrow{q} \ell_2 \to 0.$$

$$Z_2 := \left\{ ((y_n), (x_n)) \in S \times \ell_2 : \sum_{n=1}^{\infty} |x_n|^2 + \left| y_n + x_n \log \frac{|x_n|}{\|x\|_2} \right|^2 < \infty \right\}.$$

- Z_2 is a singular twisted sum of ℓ_2 with itself: q is strictly singular.
- Z_2 contains no complemented copies of ℓ_2.

Complex interpolation (two spaces)

(X_0, X_1) compatible pair of complex Banach spaces.

$\mathbb{S} := \{\lambda \in \mathbb{C} : 0 < Re\,\lambda < 1\}$ unit strip.

$\mathcal{H}(\overline{\mathbb{S}})$: Bounded continuous functions $g : \overline{\mathbb{S}} \to X_0 + X_1$
with g analytic on $\mathbb{S}$, $g(0 + it) \in X_0$ and $g(1 + it) \in X_1$;
endowed with the supremum norm $\|\cdot\|_\infty$.

Fix $0 < \theta < 1$. For $n = 0, 1, 2, \ldots$,

$$\delta_\theta^{(n)} : g \in \mathcal{H}(\overline{\mathbb{S}}) \to g^{(n)}(\theta) \in X_0 + X_1; \quad \text{defines a bounded operator.}$$

We write $\delta_\theta = \delta_\theta^{(0)}$ and $\delta'_\theta = \delta_\theta^{(1)}$.

Complex interpolation spaces: $X_\theta := \{g(\theta) : g \in \mathcal{H}(\overline{\mathbb{S}})\} \equiv \frac{\mathcal{H}(\overline{\mathbb{S}})}{\ker \delta_\theta}$.

$$\|x\|_\theta := \inf\{\|g\|_\infty : g \in \mathcal{H}(\overline{\mathbb{S}}), g(\theta) = x\}$$

Complex interpolation (family of spaces)

$\{X_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}$ compatible family of complex Banach spaces

$\Sigma(X_{j,t})$ denote the algebraic sum of these spaces.

$\mathcal{H}(X_{j,t})$: Bounded continuous functions $g : \overline{\mathbb{S}} \to \Sigma(X_{j,t})$, analytic on $\mathbb{S}$, and satisfying $g(it) \in X_{(0,t)}$ and $g(it+1) \in X_{(1,t)}$ for $t \in \mathbb{R}$, endowed with $\|g\|_\infty = \sup\{\|g(j+it)\|_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}$.

Fix $\theta \in \mathbb{S}$. For $n = 0, 1, 2, \ldots$,

$\delta_\theta^{(n)} : g \in \mathcal{H}(X_{j,t}) \to g^{(n)}(\theta) \in \Sigma(X_{j,t})$ are bounded operators.

Complex interpolation spaces: $X_\theta := \{g(\theta) : g \in \mathcal{H}(X_{j,t})\} \equiv \frac{\mathcal{H}(X_{j,t})}{\ker \delta_\theta}$.

Complex interpolation and quasi-linear maps

We consider the quotient map $\delta_\theta : g \in \mathcal{H}(\bar{\mathbb{S}}) \to g(\theta) \in X_\theta$.

We fix a homogeneous bounded selection $B_\theta : X_\theta \to \mathcal{H}(\bar{\mathbb{S}})$ of δ_θ.
It satisfies $\delta_\theta \circ B_\theta = I_{X_\theta}$.

Then $\Omega_\theta := \delta'_\theta \circ B_\theta : x \in X_\theta \to B_\theta(x)'(\theta) \in X_0 + X_1$ defines a quasi-linear map from X_θ to X_θ and

$$X_\theta \oplus_{\Omega_\theta} X_\theta = \Big\{ \Big(g'(\theta), g(\theta) \Big) : g \in \mathcal{H}(\bar{\mathbb{S}}) \Big\} \equiv \mathcal{H}(\bar{\mathbb{S}}) / \big(\mathrm{Ker}\delta_\theta \cap \mathrm{Ker}\delta'_\theta \big).$$

QUESTIONS. Given the exact sequence

$$0 \to X_\theta \xrightarrow{j_\theta} X_\theta \oplus_{\Omega_\theta} X_\theta \xrightarrow{q_\theta} X_\theta \to 0,$$

- when is $X_\theta \oplus_{\Omega_\theta} X_\theta$ a twisted sum?
- when is q_θ strictly singular?
- which spaces appear as $\ell_2 \oplus_{\Omega_\theta} \ell_2$ (twisted Hilbert spaces)?

Singularity criterion for a pair with unconditional basis

For two functions $f, g : \mathbb{N} \to \mathbb{R}$ we write $f \sim g$
if $0 < \liminf f(n)/g(n) \leq \limsup f(n)/g(n) < +\infty$.

$A_X(n) := \sup\{\|x_1 + \ldots + x_n\| \;:\; \|x_i\| \leq 1,\; n < x_1 < \ldots < x_n\}$.

Proposition

Let (X_0, X_1) be a pair of spaces with a common 1-unconditional basis and $A_{X_0} \not\sim A_{X_1}$, and let $0 < \theta < 1$.
Suppose $A_{X_0}^{1-\theta} A_{X_1}^{\theta} \sim A_{X_\theta} \sim A_Y$ for all subspaces $Y \subset X_\theta$.
Then Ω_θ is singular.

EXAMPLE: X_j reflexive, asymptotically ℓ_{p_j}, $p_0 \neq p_1$, with uncond. basis.

NOTE: $(X, X^*)_{1/2} \equiv \ell_2$ when X is reflexive with uncond. basis.

In this way we get a family $\ell_2 \oplus_{\Omega_\theta^i} \ell_2$ $(i \in \mathbb{R})$ of pairwise non-isomorphic twisted Hilbert spaces.

Singularity criterion for a pair of Köthe function spaces

$M_X(n) := \sup\{\|x_1 + \ldots + x_n\| \ : \ \|x_i\| \leq 1, \ (x_i) \text{ disjoint in } X\}$.

Ω_θ disjointly singular: the restriction of Ω_θ to a subspace of X_θ generated by a disjoint sequence is never trivial.

Proposition

Let (X_0, X_1) be an admissible pair of Köthe function spaces with $M_{X_0} \not\sim M_{X_1}$, and $0 < \theta < 1$. Suppose $M_{X_0}^{1-\theta} M_{X_1}^{\theta} \sim M_{X_\theta} \sim M_Y$ for each $Y \subset X_\theta$ generated by a disjoint sequence, and X_θ reflexive. Then Ω_θ is disjointly singular; hence non-trivial.

Corollary

Let X be a reflexive, p-convex Köthe function space with $p > 1$. Assume $M_X \sim M_{[x_n]}$ for every disjoint sequence $(x_n) \subset X$. Then the Kalton-Peck map $\mathcal{K}(f) = f \log \frac{|f|}{\|f\|}$ is disjointly singular on X.

Singularity criterion for a family of spaces with a basis

Proposition (MON)

Let $\{X_{(j,t)} : j = 0, 1; t \in \mathbb{R}\}$ be an admissible family of spaces with a common 1-monotone basis.
Let $1 \leq p_0 \neq p_1 \leq +\infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $0 < \theta < 1$.

Assume the spaces $X_{j,t}$ satisfy an asymptotic upper ℓ_{p_j}-estimate with uniform constant, and for every block-subspace W of X_θ, there exist a constant C and for each n, a C-unconditional finite block-sequence $n < y_1 < \ldots < y_n$ in B_W such that $\|y_1 + \cdots + y_n\| \geq C^{-1} n^{1/p}$ and $[y_1, \cdots, y_n]$ is C-complemented in X_θ.

Then Ω_θ is singular.

Twisting Ferenczi's space $\mathcal{F}_1$

For each $t \in \mathbb{R}$, take $X_{(1,t)} = \ell_q$ with $1 < q < \infty$, and $X_{(0,t)}$ a GM-like space (varies with t) with 1-monotone basis.

Fix $\theta \in \mathbb{S}$. Then

$$\mathcal{F}_1 = \{x \in \Sigma(X_{j,t}) : x = g(\theta) \text{ for some } g \in \mathcal{H}(X_{j,t})\} \equiv \mathcal{H}(X_{j,t}) / \ker \delta_\theta.$$

is a uniformly convex H.I. Banach space (Ferenczi 1997).

Theorem

The spaces in the construction of $\mathcal{F}_1$ satisfy the conditions of Proposition (MON). So Ω_θ gives a singular twisted sum

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 := \mathcal{F}_1 \oplus_{\Omega_\theta} \mathcal{F}_1 \xrightarrow{\pi_{2,1}} \mathcal{F}_1 \longrightarrow 0.$$

Corollary

Since $\pi_{2,1}$ is strictly singular, the space $\mathcal{F}_2$ is H.I.

Iterated twisting of $\mathcal{F}_1$

Recall that $\mathcal{F}_2 = \{(g'(\theta), g(\theta)) \,:\, g \in \mathcal{H}(X_{j,t})\}$.

Given $g \in \mathcal{H}(X_{j,t})$ and $k \in \mathbb{N}$, we denote $\hat{g}[k] := g^{(k-1)}(\theta)/(k-1)!$.

For $n \geq 3$ we define:

$$\mathcal{F}_n := \{(\hat{g}[n], \dots, \hat{g}[2], \hat{g}[1]) \,:\, g \in \mathcal{H}(X_{j,t})\} \equiv \mathcal{H}(X_{j,t}) / \bigcap_{k=0}^{n-1} \ker \delta_\theta^{(k)}.$$

Proposition

Let $m, n \in \mathbb{N}$ with $m > n$.

1. $\pi_{m,n} : (x_m, \dots, x_n, \dots, x_1) \in \mathcal{F}_m \to (x_n, \dots, x_1) \in \mathcal{F}_n$ *is surjective.*
2. $i_{n,m}(x_n, \dots, x_1) \in \mathcal{F}_n \to (x_n, \dots, x_1, 0, \dots, 0) \in \mathcal{F}_m$ *is an isomorphic embedding with* $Ran(i_{n,m}) = Ker(\pi_{m,m-n})$.
3. *The operator* $\pi_{m,n}$ *is strictly singular.*

Iterated twisting of $\mathcal{F}_1$ (II)

Corollary

For $m, n \in \mathbb{N}$ with $m > n$, the sequence

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{i_{n,m}} \mathcal{F}_m \xrightarrow{\pi_{m,m-n}} \mathcal{F}_{m-n} \longrightarrow 0$$

is exact and singular.
Hence all the spaces $\mathcal{F}_m$ are H.I.

Proposition

Let $l, m, n \in \mathbb{N}$ with $l > n$. Then the diagonal push-out sequence

$$0 \longrightarrow \mathcal{F}_l \xrightarrow{i} \mathcal{F}_n \oplus \mathcal{F}_{l+m} \xrightarrow{\pi} \mathcal{F}_{m+n} \longrightarrow 0,$$

where $i(x) = (-\pi_{l,n}\, x,\ i_{l,l+m}\, x)$ *and* $\pi(y, z) = i_{n,m+n}\, y + \pi_{l+m,m+n}\, z$,

is a twisted sum (nontrivial exact sequence) which is not H.I.

Thank you
for your attention.