Representations of ideals in Banach spaces

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joint work with

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First Brazilian Workshop in Geometry of Banach Spaces

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Classical summable ideals

Definition

Let $h : \omega \to [0, \infty)$ be a sequence such that $\sum_{n\in\omega} h(n) = \infty$. Then the ***summable ideal associated to*** h is

$$\mathcal{I}_h = \left\{ A \subseteq \omega : \sum_{n\in A} h(n) < \infty \right\} \quad \text{(an } F_\sigma \text{ P-ideal).}$$

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Why $h : \omega \to [0, \infty)$? What if $h : \omega \to \mathbb{R}$?
We have to change the definition: $A \in \mathcal{I}'_h$ if the sum of $(h(n))_{n\in A}$ is ***unconditionally convergent***, that is, the net

$$\sum h \restriction A = \left\{ s_h(F) = \sum_{n\in F} h(n) : F \in [A]^{<\omega} \right\} \text{ is convergent.}$$

But we still have the same family of ideals: $A \in \mathcal{I}'_h$ iff $A \in \mathcal{I}_{|h|}$...

Generalized summable ideals

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Definition

Let G be a Polish Abelian group and $h : \omega \to G$ such that $\sum h$ is not convergent. Then the ***generalized summable ideal associated to** G **and** h* is

$$\mathcal{I}_h^G = \left\{ A \subseteq \omega : \sum h \restriction A \text{ is convergent} \right\}.$$

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$$\mathcal{I}_h^G = \left\{ A \subseteq \omega : \sum h \restriction A \ \text{ is convergent} \right\}.$$

We say that an ideal $\mathcal{J}$ on ω is ***representable in G*** if there is an $h : \omega \to G$ such that $\mathcal{J} = \mathcal{I}_h^G$. If **C** is a class of groups then $\mathcal{J}$ is ***C-representable*** if it is representable in a $G \in$ **C**.

Examples

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Example

If $(G_k)_{k\in\omega}$ is a sequence of non-trivial discrete Abelian groups, then $\mathcal{J}$ is representable in $\prod_{k\in\omega} G_k$ iff $\mathcal{J}$ is representable in $\mathbb{Z}_2^\omega$ iff there is a family $\{X_k\colon k\in\omega\}\subseteq[\omega]^\omega$ such that

$$\mathcal{J}=\left\{A\subseteq\omega\colon \forall\, k\in\omega\ |A\cap X_k|<\omega\right\}.$$

Examples

Example

The ***density zero ideal***

$$\mathcal{Z} = \left\{ A \subseteq \omega : \frac{|A \cap n|}{n} \to 0 \right\} = \left\{ A \subseteq \omega : \frac{|A \cap [2^n, 2^{n+1})|}{2^n} \to 0 \right\}$$

(an $F_{\sigma\delta}$ P-ideal) is representable in c_0:

Examples

$$
\begin{aligned}
h(0) &= (0, 0, \ 0\ , \ 0\ , \ 0\ , \dots)\\
h(1) &= (0, 1, \ 0\ , \ 0\ , \ 0\ , \dots)\\
h(2) &= (0, 0, 1/2, \ 0\ , \ 0\ , \dots)\\
h(3) &= (0, 0, 1/2, \ 0\ , \ 0\ , \dots)\\
h(4) &= (0, 0, \ 0\ , 1/4, \ 0\ , \dots)\\
h(5) &= (0, 0, \ 0\ , 1/4, \ 0\ , \dots)\\
h(6) &= (0, 0, \ 0\ , 1/4, \ 0\ , \dots)\\
h(7) &= (0, 0, \ 0\ , 1/4, \ 0\ , \dots)\\
&\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned}
$$

If $A \subseteq \omega$ then $\sum h \restriction A$ is convergent $\iff$

$$\sum_{n \in A} h(n) = \left(0, \frac{|A \cap [2,4)|}{2}, \frac{|A \cap [4,8)|}{4}, \dots\right) \in c_0 \iff A \in \mathcal{Z}.$$

Why are we doing this?

(i) Our approach reveals some “geometric” properties of ideals and therefore it can be helpful in classifying certain classes of ideals.

(ii) These methods can be useful in providing new interesting examples of non-pathological analytic P-ideals (see later).

(iii) Representability of certain ideals in a Banach space can be seen as a combinatorial property of the space itself and this may lead us to develop new methods in the theory of Banach spaces.

Polish- and Banach-representability

Theorem

$\mathcal{J}$ is Polish-representable iff $\mathcal{J}$ is an analytic P-ideal.

Theorem

$\mathcal{J}$ is Banach-representable iff $\mathcal{J}$ is a non-path. analytic P-ideal.

Analytic P-ideals

A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is

- a ***submeasure*** (on ω) if $\varphi(\emptyset) = 0$, φ is monotonic, subadditive, and $\varphi(\{n\}) < \infty$ for every $n \in \omega$;
- ***lower semicontinuous*** (lsc, in short) if $\varphi(X) = \lim_{n\to\infty} \varphi(X \cap n)$ for each $X \subseteq \omega$.

If φ is an lsc submeasure then let

$$\mathrm{Fin}(\varphi) = \left\{A \subseteq \omega : \varphi(A) < \infty\right\} \quad \text{(an } F_\sigma \text{ ideal).}$$

$$\mathrm{Exh}(\varphi) = \left\{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\right\} \quad \text{(an } F_{\sigma\delta} \text{ P-ideal).}$$

Analytic P-ideals

Theorem (Mazur, Solecki)

Let $\mathcal{J}$ be an ideal on ω. Then the following are equivalent:

(i) $\mathcal{J}$ is an analytic *P*-ideal.

(ii) $\mathcal{J} = \mathrm{Exh}(\varphi)$ for some (finite) lsc submeasure φ.

(iii) There is a Polish group topology on $\mathcal{J}$ (with respect to $\triangle$) such that the Borel structure of this topology coincides with the Borel structure inherited from $\mathcal{P}(\omega)$.

Furthermore, $\mathcal{J}$ is an F_σ P-ideal iff $\mathcal{J} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ for some lsc submeasure φ.

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- (Generalized) Density ideals: Let $\vec{\mu} = (\mu_n)_{n\in\omega}$ be a sequence of (sub)measures on ω with pairwise disjoint finite supports with $\limsup_{n\to\infty} \mu_n(\omega) > 0$. Then the ***(generalized) density ideal associated to*** $\vec{\mu}$ is

$$\mathcal{Z}_{\vec{\mu}} = \{A \subseteq \omega : \mu_n(A) \to 0\}.$$

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- ***The trace of the null ideal:***

$$\mathrm{tr}(\mathcal{N}) = \{A \subseteq 2^{<\omega} : \lambda\{f \in 2^\omega : \exists^\infty n\, f \restriction n \in A\} = 0\}.$$

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- ***Tsirelson ideals.***

Non-pathological ideals

A submeasure φ is ***non-pathological*** if for every $A \subseteq \omega$

$$\varphi(A) = \sup\left\{\mu(A) : \mu \text{ is a measure on } \mathcal{P}(\omega) \text{ and } \mu \leq \varphi\right\}.$$

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Theorem (Hrušák)

An analytic P-ideal $\mathcal{J}$ is non-pathological iff $\mathcal{J} \restriction X \leq_K \mathcal{Z}$ (i.e. there is an $f : \omega \to X$ such that $f^{-1}[A] \in \mathcal{Z}$ for every $A \in \mathcal{J}$) for every $X \in \mathcal{P}(\omega) \setminus \mathcal{J}$.

Representability in c_0

Proposition

An ideal $\mathcal{J}$ is representable in c_0 iff there is a family $\{\mu_k : k \in \omega\}$ of measures on ω such that $\mathcal{J} = \mathrm{Exh}(\varphi)$ where $\varphi = \sup\{\mu_k : k \in \omega\}$ and $\{k : n \in \mathrm{supp}(\mu_k)\}$ is finite for each n.

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Density ideals are representable in c_0.

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We show that $\mathrm{tr}(\mathcal{N})$, Farah's ideal, and Tsirelson ideals are not representable in c_0.

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If $\mathcal{J}$ is representable in c_0 and ***totally bounded*** (that is, if $\mathcal{J} = \mathrm{Exh}(\varphi)$ then $\varphi(\omega) < \infty$), then $\mathcal{J}$ is a generalized density ideal.

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Corollary

$\mathrm{tr}(\mathcal{N})$ is not representable in c_0.

Proof:
1. $\mathrm{tr}(\mathcal{N})$ is not a generalized density ideal (because e.g. it is ***summable-like***).
2. Hrusak+Hernandez-Hernandez: $\mathrm{tr}(\mathcal{N})$ is totally bounded (because e.g. $\mathfrak{s}(\mathrm{tr}(\mathcal{N})) = \omega$).

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Farah's ideal and Tsirelson ideals are not representable in c_0.

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Proof (sketch): Our first, still unfinished (but 99% working☺) idea was to build blocks of the form $(0,\dots,0,\varepsilon_0/a_n,\dots,\varepsilon_{n-1}/a_n,0,\dots)$ where $\varepsilon_i \in \{\pm 1\}$ and a_n is an appropriate sequence tending to ∞. Problem: If we add all 2^n possible new elements to our family, then we should understand the following constant:

$$\inf\left\{\frac{\max\{\|\sum F\|:\emptyset\neq F\subseteq S\}}{\sum\{\|v\|:v\in S\}} : \emptyset\neq S\in[X\setminus\{\mathbf{0}\}]^{<\omega}\right\}.$$

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Piotr's refinement: At the nth stage we add "Rademacher-like" vectors only and then we can apply Khintchine's inequality etc.

Questions

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General version: Is there any reasonable characterization of $\mathbb{R}^\omega$-, or c_0-, or ℓ_1-representable ideals?
Reasonable versions:

- Which ideals can be covered by a summable ideal?
- Are all ℓ_1-representable ideals F_σ?

Question

Assume that all non-pathological analytic P-ideals are representable in a Banach space X. Does it imply that X is universal for the class of all separable Banach spaces?

Question

What can we say about ideals representable in (completions of) weak topologies?

Thank you for your attention!